16. INTEGRATION

Defining Integration

Integration is defined as the reverse process of differentiation. When we differentiate a function, the result is called the differential. When we integrate a function, the result is called the integral or the antiderivative.

Notation

The symbol for integration is $\int$ and the term or expression that is to be integrated is called the integrand. If the integrand is, say, a function of x, then we must integrate with respect to x and this is written as,

$\int f(x)dx$

The integrand can be a function in any variable and so we can have integrands such as

$\int f(r)dr \qquad \int g(t)dt \qquad \int h(\theta)d\theta$

Law of integration

To derive the law of integration, we are interested in deriving a rule for the reverse process of differentiation. Recall the rule for differentiation:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

To differentiate x^n, multiply by n and decrease the power n by 1.
To integrate x^n, increase the power n by 1 and divide by the new power, n+1.

The law of integration can now be formally stated.

$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$, where C is the constant of integration and $n \neq -1$.

The constant of Integration

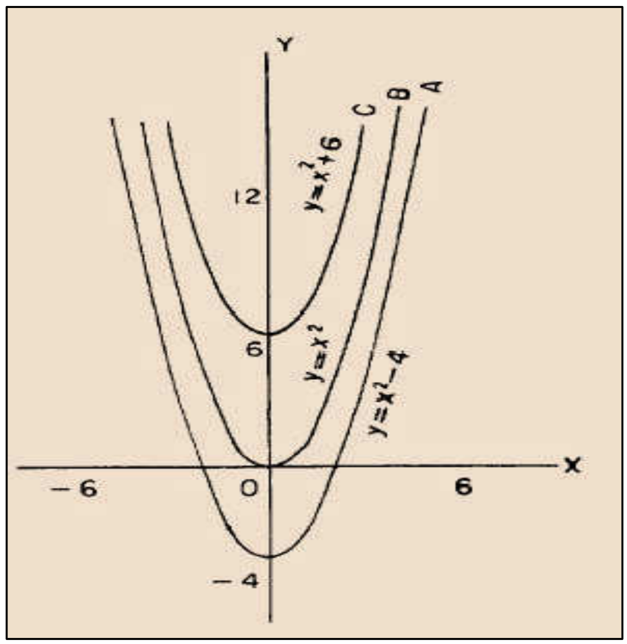
When we differentiate any constant, the result is zero. Consider the following family of functions and their derivatives.

$$\frac{d}{dx}(x^2) = 2x$$

$$\frac{d}{dx}(x^2+6) = 2x$$

$$\frac{d}{dx}(x^2-4) = 2x$$

The integral of $2x$ can be any of the functions:
$x^2 + 0$
$x^2 + 6$,
$x^2 - 4$ or in general
$x^2 +$ some constant.
In fact, the integral of $2x$ can be represented by a family of curves and not a definite curve. Let us examine the graphs of these curves.



We can deduce that
$\int 2x \, dx = x^2 + C$, where C is a constant to be determined.
Since the result of this integral is dependent on the value of C, we refer to integrals such as these as indefinite integrals. At a later stage, we explain how to evaluate this constant in a given situation.

We noted earlier that the differential of a constant is zero, we can also conclude that the integral of zero is a constant.

The integral x^{-1}

The fundamental law of integration is not applicable to the integrand of x^{-1}. This is the one case when the rule, shown above, cannot be applied, that is, the exception to the law.

If we tried to apply the law to, $\int x^{-1}$, we would obtain $\frac{x^{-1+1}}{-1+1}+C=\frac{x^0}{0}+C=\frac{1}{0}+C$.

This result is undefined or tends to infinity. So, we can see that the law is inapplicable for $n=-1$.

When we have to integrate $\frac{1}{x}$ with respect to x, we obtain the following result.

$\int \frac{1}{x}\,dx=\ln|x|+C$, where C is a constant.

Since we can only find the log of a positive value, x has a modulus sign. Note also that *ln* is really log_e.

Integration of x^n and ax^n

Just as in differentiation, if an expression is not in the form that is conformable to the use of the law of integration, we use algebraic techniques to express it in such a form.

Example 1

$$\int \frac{1}{x^3}\,dx$$

Solution

$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx$ The law of integration can now be applied.

$$\int x^{-3}\,dx=\frac{x^{-3+1}}{-3+1}+C=\frac{x^{-2}}{-2}+C$$

$$=-\frac{1}{2x^2}+C \quad \text{(where } C \text{ is a constant)}$$

Example 2

$$\int \sqrt{x}\,dx$$

Solution

$$\int \sqrt{x}\,dx=\int x^{\frac{1}{2}}\,dx=\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$=\frac{2}{3}\sqrt{x^3}+C \quad \text{(where } C \text{ is a constant)}$$

The integral of ax^n

When we have a coefficient of x^n other than 1, we can move it to outside the integral sign and integrate as usual. However, the coefficient need not necessarily be moved to attain the desired result. For example, we may choose any one of the following ways to handle coefficients.

$\int 6x^3\,dx=6\int x^3\,dx$ $=6.\frac{x^4}{4}+C$ $=3\frac{x^4}{2}+C$	$\int 6x^3\,dx=\frac{6x^{3+1}}{3+1}+C$ $=\frac{6x^4}{4}+C$ $=\frac{3x^4}{2}+C$

The integral of ax^n is defined as follows:

$\int ax^n\,dx = a\int x^n\,dx$, where a is a constant

Example 3

$$\int \frac{3}{4\sqrt{x}}\,dx$$

Solution

$$\int \frac{3}{4\sqrt{x}}\,dx=\int \frac{3}{4x^{\frac{1}{2}}}\,dx$$

$$=\int \frac{3}{4}x^{-\frac{1}{2}}\,dx=\frac{3}{4}\int x^{-\frac{1}{2}}dx$$

$$=\frac{3}{4}.\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=\left(\frac{3}{4}\times\frac{2}{1}\right)x^{\frac{1}{2}}+C$$

$$=\frac{3}{2}\sqrt{x}+C \quad \text{(where } C \text{ is a constant)}$$

Integrating a sum

When the integrand is composed of more than one term, it must be written within brackets. Each term is to be integrated individually and then the constant of integration is added on in the final result.

$$\int\left(5x^2+8x\right)dx\equiv\int 5x^2\,dx+\int 8x\,dx$$

$$=\frac{5x^3}{3}+\frac{8x^2}{2}+C=\frac{5x^3}{3}+4x^2+C$$

(where C is constant)

The integral of a constant, *k*

To evaluate $\int k\ dx$. We write $kx^0 = k$ because $x^0 = 1$. So,

$$\int 4\ dx = \int 4x^0\ dx$$

$$= 4\frac{x^{0+1}}{0+1} + C = 4x + C \quad (\text{where } C \text{ is a constant})$$

Example 4

$$\int -\frac{1}{2}\ dx$$

Solution

$$\int -\frac{1}{2}\ dx = \int -\frac{1}{2}x^0\ dx$$

$$= -\frac{1}{2}\frac{x^{0+1}}{0+1} + C = -\frac{1}{2}x + C \quad (\text{where } C \text{ is a constant})$$

We can now deduce, that when A is a constant, then

$$\int A\ dx = Ax + C \quad (C \text{ is a constant})$$

Integral of products and quotients

We do not have a product rule or a quotient for integration. In simple cases involving these situations, we apply the distributive law to remove the brackets and convert the expression to a sum of terms.

Example 5

$$\int x^2(3x-4)\ dx$$

Solution

First, we expand the integrand to get

$$\int x^2(3x-4)\ dx = \int \left(3x^3 - 4x^2\right)\ dx$$

Now we integrate each term separately:

$$\int x^2(3x-4)\ dx = \int \left(3x^3 - 4x^2\right)\ dx$$

$$= \frac{3x^4}{4} - \frac{4x^3}{3} + C \quad (\text{where } C \text{ is a constant})$$

Example 6

$$\int (3x-1)(x+4)\ dx$$

Solution

We expand the integrand to get:

$$\int (3x-1)(x+4)\ dx = \int \left(3x^2 + 11x - 4\right)\ dx$$

We proceed to integrate each term separately.

$$\int (3x-1)(x+4)\ dx = \int \left(3x^2 + 11x - 4\right)\ dx$$

$$= \frac{3x^3}{3} + \frac{11x^2}{2} - 4x + C$$

$$= x^3 + \frac{11}{2}x^2 - 4x + C \quad (\text{where } C \text{ is a constant})$$

Example 7

$$\int \left(\frac{4x^3 - 3x}{x}\right)\ dx$$

Solution

We divide each term by x, to get:

$$\int \left(\frac{4x^3 - 3x}{x}\right)\ dx \equiv \int \left(\frac{4x^3}{x} - \frac{3x}{x}\right)\ dx$$

$$= \int \left(4x^2 - 3\right)\ dx$$

$$= \frac{4x^3}{3} - 3x + C \quad (\text{where } C \text{ is a constant})$$

The integral of $(ax+b)^n$

To find $\int (2x+3)^n\ dx$, when $n = 2$, we may choose to expand as was done in the examples above.

As n becomes larger, say 3 or greater, the expansion will result in many terms and integration can become tedious. In such cases, the method of substitution is very helpful.

$$\int (ax+b)^n\ dx$$

Let $t = ax + b \quad \frac{dt}{dx} = a \qquad dx = \frac{dt}{a}$

$$\therefore \int (ax+b)^n\ dx \equiv \int t^n . \frac{dt}{a} = \frac{t^{n+1}}{(n+1)a} + C$$

Re-substituting for t, we get the following rule:

$$\int (ax+b)^n\ dx = \frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (\text{where } C \text{ is a constant})$$

We can apply the above rule in evaluating the integral of functions of the form $(ax+b)^n$ as illustrated in the following examples.

Example 8

$$\int (2x-1)^{10}\, dx$$

Solution

In this example, $a = 2$, $b = -1$, and $n = 10$

$$\therefore \int (2x-1)^{10}\, dx = \frac{(2x-1)^{11}}{22} + C \quad \text{(where } C \text{ is a constant)}$$

Example 9

$$\int \sqrt{4x-1}\, dx$$

Solution

$$\int \sqrt{4x-1}\, dx \equiv \int (4x-1)^{\frac{1}{2}}\, dx$$

In this example, $a = 4,\ b = -1$ **and** $n = \frac{1}{2}$.

$$\therefore \int \sqrt{4x-1}\, dx = \frac{(4x-1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\times 4} + C = \frac{(4x-1)^{\frac{3}{2}}}{6} + C$$

$$= \frac{1}{6}\sqrt{(4x-1)^3} + C \quad \text{(where } C \text{ is a constant)}$$

Example 10

$$\int \frac{1}{\sqrt[3]{2x-1}}\, dx$$

Solution

$$\int \frac{1}{\sqrt[3]{2x-1}}\, dx \equiv \int (2x-1)^{-\frac{1}{3}}\, dx$$

Using the formula, $a = 2, b = -1$ and $n = -\frac{1}{3}$, we have

$$\int \frac{1}{\sqrt[3]{2x-1}}\, dx = \frac{(2x+1)^{-\frac{1}{3}+1}}{\left(-\frac{1}{3}+1\right)\times 2} + C = \frac{(2x+1)^{\frac{2}{3}}}{\frac{4}{3}} + C$$

$$= \frac{3\left(\sqrt[3]{(2x+1)^2}\right)}{4} + C \quad \text{(where } C \text{ is a constant)}$$

Using integration to find the equation of a curve

We can use integration to find the equation of a curve when we know its gradient function and one point on the curve. The point will help us to find the numerical value of the constant of integration.

Example 16

The gradient function, $\frac{dy}{dx}$, of a curve is given by $6x^2 + 6x - 5$ Find the equation of the curve if the point $(2, 12)$ lies on the curve.

Solution

The equation of the curve will be

$$y = \int (6x^2 + 6x - 5)\, dx$$

$$y = \frac{6x^3}{3} + \frac{6x^2}{2} - 5x + C \quad \text{(where } C \text{ is a constant)}$$

$$y = 2x^3 + 3x^2 - 5x + C$$

Since $(2, 12)$ lies on the curve, then the value of C, can be found by substituting $x = 2$ and $y = 12$

$$12 = 2(2)^3 + 3(2)^2 - 5(2) + C = 16 + 12 - 10 + C$$

$$12 = 18 + C$$

$$C = -6$$

$\therefore$ The equation of the curve is, $y = 2x^3 + 3x^2 - 5x - 6$.

Indefinite and definite integrals

All the integration examples done so far are examples of what is referred to as indefinite integrals. The results were written in terms of the variable together with the constant of integration added on.

For example, an integral such as $\int x^n\, dx$ = $\frac{x^{n+1}}{n+1} + C$ is said to be of indefinite value.

As we shall explain in a later chapter on the meaning of integration, indefinite integrals cannot be evaluated and are non-numerical in nature.

We can, however, evaluate a definite integral but we must be given the limits that define the integral. We do so by using the constants, *a* and *b*, called the upper limit and the lower limit respectively.

When an integral is written as, $\int_b^a x^n \, dx$, it can be evaluated to give a numerical value and is said to be a definite integral.

In $\int_1^3 x^n \, dx$, 3 is the upper limit and 1 is the lower limit.

To evaluate $\int_a^b x^n \, dx$, the procedure is as follows:

1. We first find the indefinite integral and which is an expression in *x*.
2. Write the integral using square brackets and place the limits, written in the same order, on the outside of the outer square bracket.
3. Substitute $x = b$ the upper limit, in the integral.
4. Substitute $x = a$ the lower limit, in the integral.
5. Subtract the value obtained in step 4 from the value obtained in step 3. The numerical result is the definite integral.

Example 11

Evaluate $\int_1^2 x^2 \, dx$.

Solution

$$\int_1^2 x^2 \, dx = \left[\frac{x^3}{3} + C\right]_1^2 \quad \text{(where } C \text{ is a constant)}$$

$$= \left(\frac{(2)^3}{3} + C\right) - \left(\frac{(1)^3}{3} + C\right)$$

$$= \left(2\frac{2}{3} + C\right) - \left(\frac{1}{3} + C\right) = 2\frac{1}{3}$$

Notice that the constant of integration cancels off in a definite integral and so it may be excluded in the evaluation. It is considered good practice, though, to write it in the step immediately following the integration and omit it afterwards.

Example 12

Evaluate $\int_{-1}^1 (2x-3) \, dx$.

Solution

$$\int_{-1}^1 (2x-3) \, dx = \left[\frac{2x^2}{2} - 3x + C\right]_{-1}^1$$

$$= \left[x^2 - 3x + C\right]_{-1}^1$$

$$= \left\{(1)^2 - 3(-1)\right\} - \left\{(-1)^2 - 3(-1)\right\}$$

$$= (1-3) - (1+3) = -2 - 4 = -6$$

Example 13

Evaluate $\int_{-1}^3 (2x-1)^4 \, dx$

Solution

Using the formula, $a = 2$, $b = -1$ and $n = 4$

$$\int (2x-1)^4 \, dx = \frac{(2x-1)^5}{10} + C$$

$$\int_{-1}^3 (2x-1)^4 \, dx = \left[\frac{(2x-1)^5}{10}\right]_{-1}^3$$

$$= \frac{(2(3)-1)^5}{10} - \frac{(2(-1)-1)^5}{10}$$

$$= \frac{(5)^5}{10} - \frac{(-3)^5}{10} = \frac{3125}{10} + \frac{243}{10}$$

$$= \frac{3368}{10} = 336\frac{4}{5} \text{ units}$$

Example 14

Evaluate $\int_0^2 \sqrt{4x+1} \, dx$.

Solution

$$\int \sqrt{4x+1} \, dx \equiv \int (4x+1)^{\frac{1}{2}} \, dx$$

$$\therefore \int \sqrt{4x+1} \, dx = \frac{(4x+1)^{\frac{1}{2}+1}}{4\left(\frac{1}{2}+1\right)} + C$$

$$= \frac{(4x+1)^{\frac{3}{2}}}{6} + C = \frac{\sqrt{(4x+1)^3}}{6} + C$$

$$\int_0^2 \sqrt{4x+1}\, dx = \left[\frac{\sqrt{(4x+1)^3}}{6}\right]_0^2 = \left(\frac{\sqrt{(9)^3}}{6}\right) - \left(\frac{\sqrt{(1)^3}}{6}\right)$$

$$= \frac{27}{6} - \frac{1}{6} = \frac{26}{6} = 4\frac{1}{3} \text{ units}$$

Rules involving definite integrals

If $f(x)$ is continuous (the graph runs continuously) in the interval $[a, c]$, the following rules are used when evaluation definite integrals.

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$$

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

(where b lies between a and c)

The examples, shown below, give an illustration of these rules.

We can illustrate that $\int_1^2 3x\, dx = -\int_2^1 3x\, dx$

$$\int_1^2 3x\, dx = \left[\frac{3x^2}{2} + C\right]_1^2$$

$$= \left(\frac{3(2)^2}{2}\right) - \left(\frac{3(1)^2}{2}\right) = 6 - \frac{3}{2} = 4\frac{1}{2}$$

$$\int_2^1 3x\, dx = \left[\frac{3x^2}{2} + C\right]_2^1$$

$$= \left(\frac{3(1)^2}{2}\right) - \left(\frac{3(2)^2}{2}\right) = \frac{3}{2} - 6 = -4\frac{1}{2}$$

Similarly, $\int_1^2 3x\, dx + \int_2^4 3x\, dx = \int_1^4 3x\, dx$

$$\int_1^2 3x \;\; dx + \int_2^4 3x \;\; dx = [\frac{3x^2}{2}]\,{}_1^2 + [\frac{3x^2}{2}]\,{}_2^4$$

$$= (\frac{3(2)^2}{2} - \frac{3(1)^2}{2}) + (\frac{3(4)^2}{2} - \frac{3(2)^2}{2})$$

$$= (6 - \frac{3}{2}) + (24 - 6) = 4\frac{1}{2} + 18 = 22\frac{1}{2}$$

$$\int_1^4 3x\, dx = \left[\frac{3x^2}{2}\right]_1^4 = \frac{3(4)^2}{2} - \frac{3(1)^2}{2} = 24 - \frac{3}{2} = 22\frac{1}{2}$$

The integral of trigonometric functions

We already know the differential of the basic trigonometric functions, so we can state their integrals as follows.

$$\int \cos x\, dx = \sin x + C$$

$$\int \sin x\, dx = -\cos x + C$$

where C is a constant

Example 6

$$\int \cos 2x\, dx$$

Solution

Let $t = 2x,\ \frac{dt}{dx} = 2$

$$\therefore \int \cos 2x\, dx = \int \cos t\, \frac{dt}{2} = \frac{\sin t}{2} + C \ [C \text{ is a constant}]$$

$$\int \cos 2x\, dx = \frac{1}{2}\sin 2x + C$$

Example 7

$$\int \sin 3x\, dx$$

Solution

Let $t = 3x,\ \frac{dt}{dx} = 3$

$$\therefore \int \sin 3x\, dx = \int \sin t\, \frac{dt}{3} = -\frac{\cos t}{3} + C$$

$$\int \sin 3x\, dx = -\frac{1}{3}\cos 3x + C$$

We can, therefore, state the following:

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax + C$$

Example 8

Evaluate $\int_0^{\frac{\pi}{2}} \sin \frac{1}{2}x\, dx$.

Solution

Let $t = \frac{1}{2}x$, $\frac{dt}{dx} = \frac{1}{2}$

$\therefore \int \sin\frac{1}{2}x\ dx = \int \sin t\ .\ 2dt$

$= -(\cos t)2 + C = -2\cos\frac{1}{2}x + C$

$\int_0^{\frac{\pi}{2}} \sin\frac{1}{2}x\ dx = \left[-2\cos\frac{1}{2}x\right]_0^{\frac{\pi}{2}} = \left(-2\cos\frac{\pi}{4}\right) - (-2\cos 0)$

$= \left(-2\times\frac{1}{\sqrt{2}}\right) - (-2\times 1) = 2 - \frac{2}{\sqrt{2}} = 2 - \frac{2}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}} = 2 - \sqrt{2}$

Example 9

Evaluate, $\int_1^2 \sin(3\pi x)\ dx$.

Solution

Let $t = 3\pi x$ $\qquad \frac{dt}{dx} = 3\pi$

$\therefore \int \sin(3\pi x)\ dx = \int \sin t\ \frac{dt}{3\pi}$

$= \frac{-\cos t}{3\pi} + C$

$= -\frac{\cos 3\pi x}{3\pi} + C$

$\int_1^2 \sin(3\pi x)\ dx = \left[-\frac{\cos 3\pi x}{3\pi}\right]_1^2$

$= \left(-\frac{\cos 3\pi(2)}{3\pi}\right) - \left(-\frac{\cos 3\pi(1)}{3\pi}\right)$

$= \frac{1}{3\pi}(-\cos 6\pi + \cos 3\pi)$

$= \frac{1}{3\pi}(-(1) + (-1)) = -\frac{2}{3\pi}$ (exact)