

25. LINEAR PROGRAMMING

TERMINOLOGY

Linear Programming is a technique used in mathematics to solve problems involving maximising and minimising linear functions when several constraints are to be considered. It is usually associated with optimising costs in production.

The general processes for solving linear programming problems involves the graphing of a set of inequalities, called the constraints, obtaining a region that is common to all (the feasible region) and then applying the fundamental theorem of linear programming to determine the optimum solution.

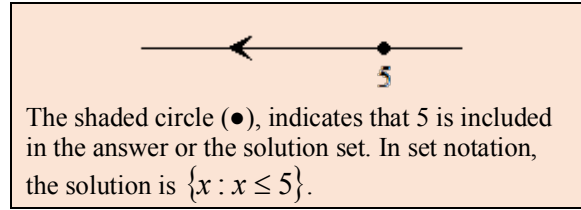
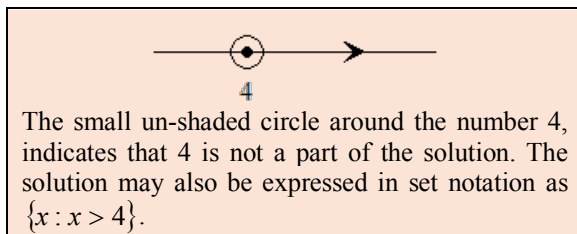
It is therefore important to understand the meaning of the solution of linear inequations before preceding to the study linear programming.

Equalities and Inequalities in one variable

The solution of an equation (or equality) in one variable is different from the solution of an inequality in one variable. When we solve an equality in one variable or equation, *there is only one solution*. In the case of an inequality or inequation in one variable, *there is an infinite number of solutions*.

Solution of an equation	Solution of an inequation	
$3x + 1 = 13$	$3x + 1 > 13$	$3x - 2 \leq 13$
$3x = 13 - 1$	$3x > 13 - 1$	$3x \leq 13 + 2$
$3x = 12$	$3x > 12$	$3x \leq 15$
$x = 4$	$x > 4$	$x \leq 5$
Only one solution	An infinite number of solutions	

When the solution of an inequation in one variable is a real number, the solution or set of solutions may be illustrated on a number line.



Note if the unknown is an integer, we can use the set builder notation or a list to represent the solution.

$x > 4$	$x = \{5, 6, 7, \dots\}$ $\{x : x > 4, x \in Z\}$
$x \leq 5$	$x = \{\dots, 3, 4, 5\}$ $\{x : x \leq 5, x \in Z\}$

Linear inequations on the Cartesian Plane

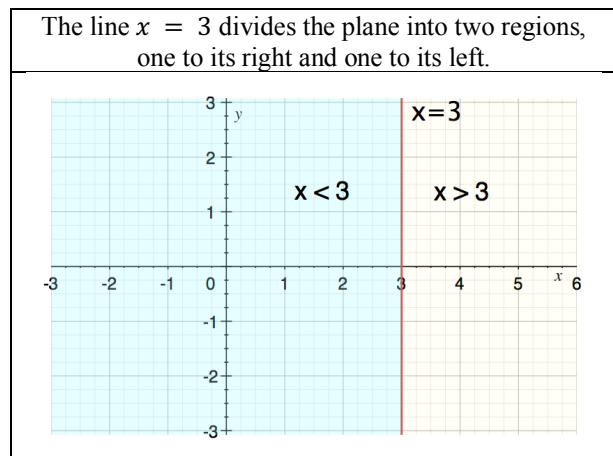
We will first examine the special cases when the straight line is vertical, or horizontal.

Vertical Lines

A vertical line has a general equation, $x = a$ and cuts the horizontal axis at a . For example, when $a = 3$, the line cuts the x -axis at 3 and all points on the line have the x coordinate of 3, such as $(3, -3)$, $(3, -2)$, $(3, -1)$, $(3, 0)$, $(3, 1)$, $(3, 2)$, $(3, 3)$, ...

The region to the right side of the vertical line, $x = 3$ is the region $x > 3$. Each point in this region has an x -coordinate which is greater than 3.

The region to the left side of the line $x = 3$ is the region $x < 3$. Each point in this region has an x -coordinate which is less than 3.



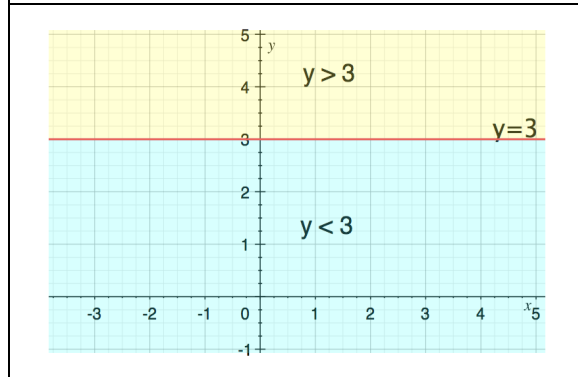
Horizontal Lines

A horizontal line has a general equation, $y = b$ and cuts the vertical axis at b . For example, when $b = 3$, the line cuts the y -axis at 3 and all points on the line $y = 3$ have the y coordinate of 3, such as $(-3, 3)$, $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$, $(2, 3)$, $(3, 3)$, ...

The region above the horizontal line, $y = 3$ is the region $y > 3$ since each point in this region has a y -coordinate which is greater than 3.

The region below the line $y = 3$ is the region $y < 3$ since each point in this region has a y -coordinate which is less than 3.

The line $y = 3$ divides the plane into two regions, one above and one below.



Lines whose equations are of the form $y = mx + c$

When a line of the form $y = mx + c$ is drawn on a Cartesian Plane, it divides the plane into two areas or regions. One region satisfies $y > mx + c$ and the other satisfies $y < mx + c$.

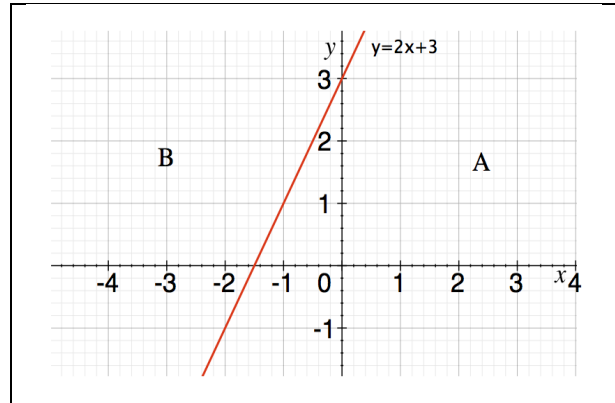
To represent regions associated with the four types of linear inequalities $y > mx + c$, $y < mx + c$, $y \geq mx + c$ and $y \leq mx + c$ we must first draw the linear equations and then decide on the region that contains the solution.

To represent the region that satisfies $y \leq 2x + 3$, we carry out the following steps:

1. Create a table of values to include at least two points on the line $y = 2x + 3$. Choose any two x -values and substitute them in the equation to obtain the corresponding y -values.

x	0	-1
y	3	1

2. Plot the points and join them to form a straight line. The line may be extended in either direction to any desired length.
3. Locate the two regions on both sides of the line. One is the region $y > 2x + 3$ and the other is the region, $y < 2x + 3$. We may label the regions A and B for easy reference.



In locating the desired region in step 3, we choose a point either in region A or in B and test to determine if the point lies in the region.

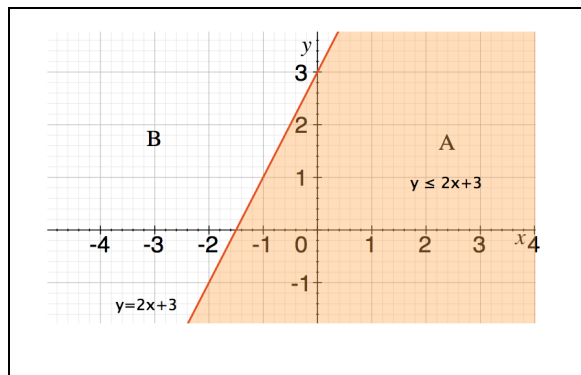
If the inequality is satisfied then this region chosen is the desired region. If not, then the other region satisfies the inequality. The point $(0, 0)$ is a convenient point to test when the graph does not pass through the origin.

Testing points in each region	
Region A	Region B
Point $(0, 0)$	Point $(-1, 2)$
$y < 2x + 3$	$y < 2x + 3$
$0 < 2(0) + 3$	$2 < 2(-1) + 3$
$0 < 0 + 3$	$2 < -2 + 3$
$0 < 3$	$2 < 1$
True	False

Our test indicates that the point $(0, 0)$ which lies in region A, satisfies the inequality. Note all other points in the region A would also have satisfied the inequality, $y \leq 2x + 3$.

Our test of the point $(-1, 2)$ in region B revealed that it did not satisfy the inequality. Similarly, all other points chosen in region B would not have satisfied the inequality.

The shaded region shows the solution of $y \leq 2x + 3$.



Conventions in drawing lines

In the convention followed in this chapter, the solution of inequalities of the form $y \geq mx + c$ and $y \leq mx + c$ are drawn as solid lines to indicate that such lines include all points on the line $y = mx + c$.

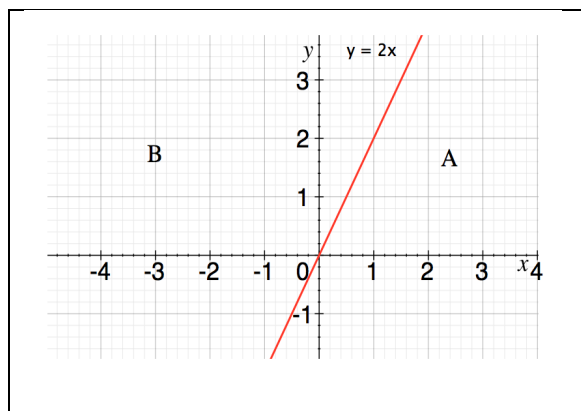
The solution of the inequalities of the form $y > mx + c$ and $y < mx + c$ are drawn as broken lines to indicate that such lines do not include the points on the line $y = mx + c$.

We shall now apply the above steps and convention to identify the region $y > 2x$.

1. Create a table of values for the line $y = 2x$

x	0	2
y	0	4

2. Plot the points and draw the line $y = 2x$.

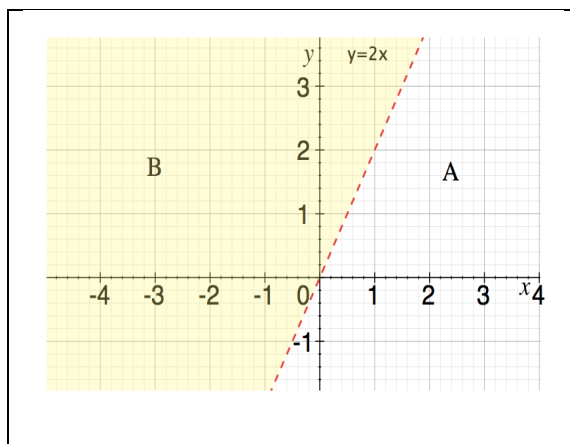


3. Locate the two regions on both sides of the line and label them as A and B. Since the point $(0, 0)$

lies on the line we cannot choose it for testing. In this case, we choose points on the x -axis as these facilitate computation.

Region A. Point $(2, 0)$ $y > 2x$ $0 > 2(2)$ $0 > 4$ False	Region B Point $(-2, 0)$ $y > 2x$ $0 > 2(-2)$ $0 > -4$ True
---	--

The solution of $y > 2x$ is shown below. Points in the region labelled as B will belong to the solution set. We refer to this region as the wanted region. Note that points on the line do not belong to the solution set, this is illustrated by a broken line using the convention stated above.



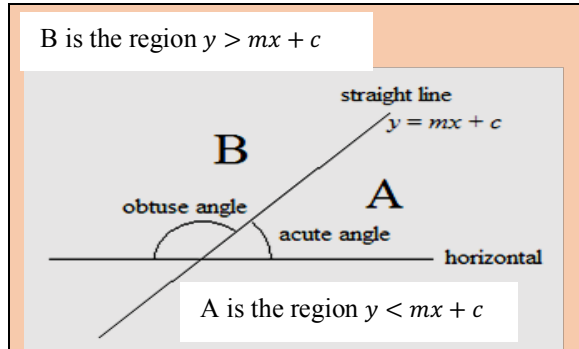
Note that we do not need to test a point in both of the regions. We simply test one point in one region. If the inequality is satisfied, then our choice was correct. If not, the other region will satisfy the inequality.

Alternative method to identify the region that satisfies a linear inequality

A rather simple method that always works in identifying the wanted region in solving an inequality is explained below.

1. Draw the line which divides the $x - y$ plane into two regions, say A and B.
2. Draw any horizontal to cut the line. We might find the x -axis a convenient horizontal.

3. The region that contains the **acute** angle is the 'less than' region and the region that holds the **obtuse** angle is the 'greater than' region.



Example 1

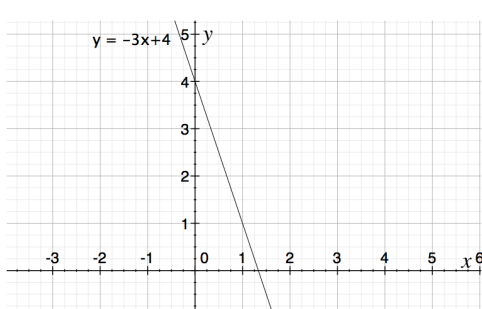
Identify the region, $y \geq -3x + 4$

Solution

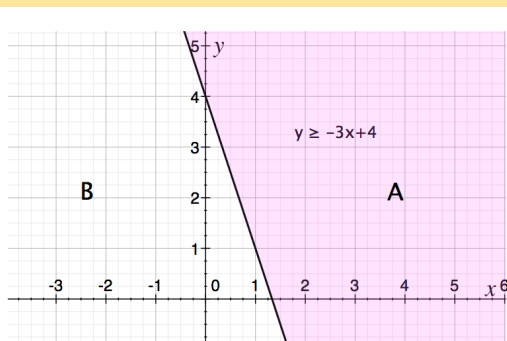
1. Create a table of values for $y = -3x + 4$

x	0	1
y	4	1

2. Draw the line $y = -3x + 4$



3. Locate the wanted region by examining the angle between the straight line and any horizontal line. In this case, we use the x - axis. The region, A holds the **obtuse** angle and is the 'greater than' region.



Solution of A System of Inequalities

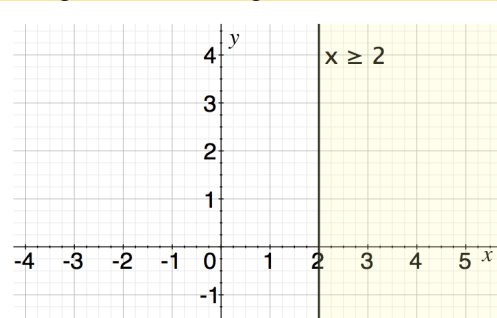
Sometimes, we may have to identify the common region that is satisfied by more than one inequation. In such cases, we draw the graphs and shade each region separately. Then, we locate the region that is the intersection of all the solution sets.

Example 2

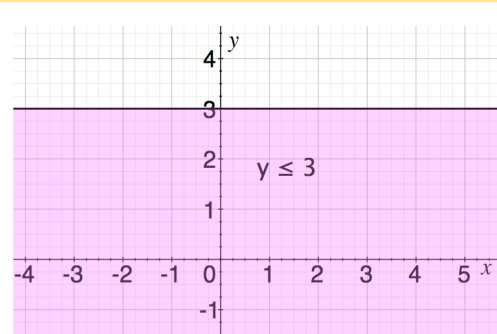
Shade the region that is the solution of $x \geq 2$ and $y \leq 3$.

Solution

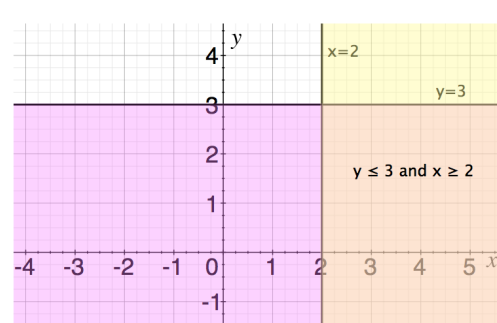
1. Draw the lines $x = 2$ and shade the wanted region. It is to the right of the line $x = 2$.



2. Shade and identify, $y \leq 3$. It is below $y = 3$.



3. The common region is the rectangular region in which both colours overlap.



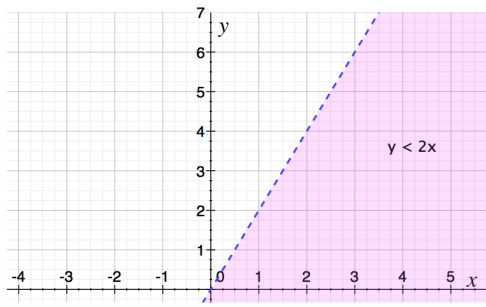
Example 3

Shade the region that is satisfied by $y < 2x$ and $x < 3$.

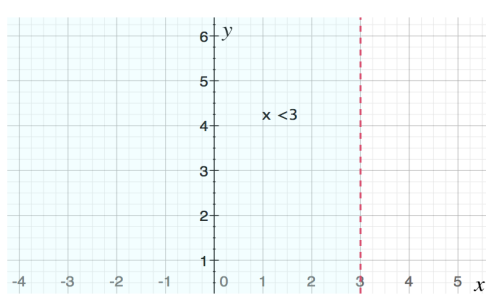
Solution

Draw the line $y = 2x$. We use broken lines because the inequality does not include equal signs.

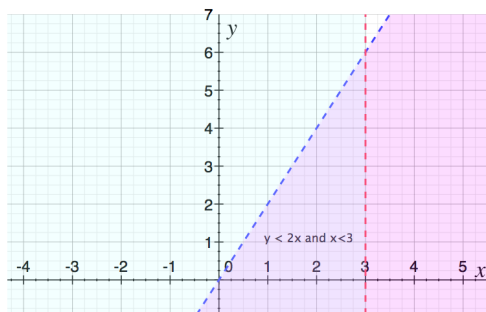
Identify the wanted region, $y < 2x$. Since the sign is 'less than', we look for the acute angle formed by the horizontal and the line $y = 2x$. The acute angle is to the right of the line $y = 2x$ and this is the region $y < 2x$.



Then we draw the line $x = 3$ and shade and identify, $x < 3$. This region is to the left of the line $x = 3$.



The common region is the triangular region in which both colours overlap. This region satisfies both $x < 3$ and $y < 2x$. Note also that points on both lines bordering the region do not belong to the solution.



Example 4

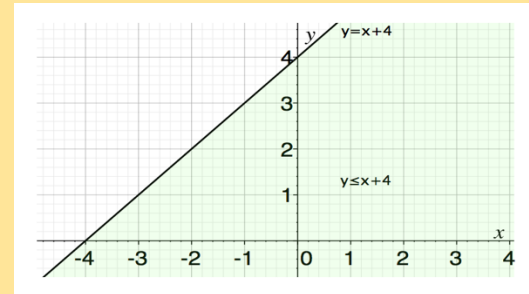
Shade and identify the region that is satisfied by $y \leq x + 4$ and $y \leq -x + 1$.

Solution

The line $y = x + 4$

x	0	-4
y	4	0

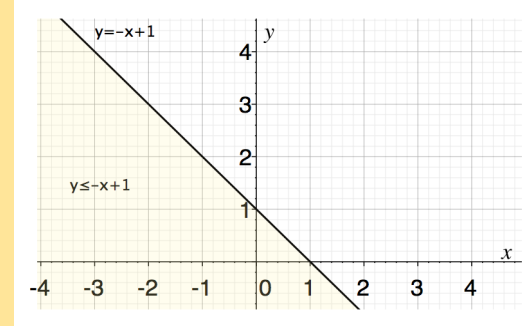
The region $y \leq x + 4$ is shown.



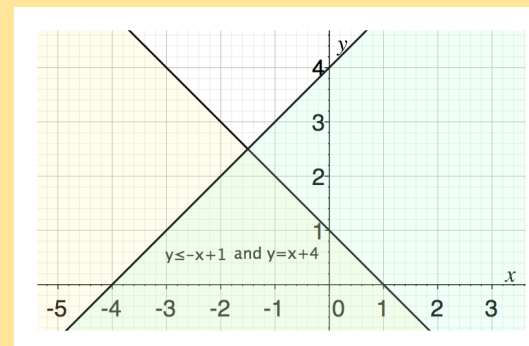
Draw the line $y = -x + 1$

x	0	1
y	1	0

The region is shown.



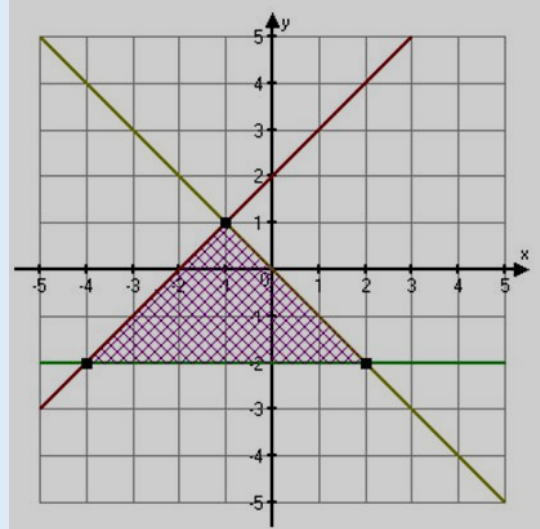
The common region, which satisfies both of the inequations is shown.



Therefore, the open triangular region overlapping both shaded areas shown above is satisfied by both inequations.

Example 5

Describe the shaded region in the diagram below.



Solution

For the horizontal line, $y = -2$, the shaded area is above the line. This region is therefore $y \geq -2$.

For the line $y = -x$, the shaded area lies in the region with the acute angle. This region is therefore $y \leq -x$

For the line $y = x + 2$, the shaded area lies in the region with the acute angle. This region is therefore $y \leq x + 2$

The three inequalities that define the shaded region are:

$$y \geq -2$$

$$y \leq -x$$

$$y \leq x + 2$$

Converting Worded Problems into Linear Inequalities

Sometimes a question may require us to convert a worded statement into a linear inequality. In writing inequalities, it is convenient to have y on the left side of an inequality. This makes it easier to draw the lines when solving the linear programming problem.

Consider the worded statements in the table below and their corresponding inequalities.

Worded statements	Inequality
The number, x , of roses in a bouquet must be less than twice the number, y , of lilies.	$x < 2y$ OR $y > \frac{x}{2}$
The number, c , of cars that pass on a certain road is at least twice the number, t , of trucks that pass the same road.	$c \geq 2t$ OR $t \leq \frac{c}{2}$
In a mixture of ice cream, the number, s , grams of sugar must not be more than the number, c , grams of coconut powder.	$s \nlessgtr c$ $s \leq c$
In a parking lot, the number of spaces, c , allocated for cars must not be less than the number, b , spaces allotted for vans.	$c \nlessgtr b$ $c \geq b$
In a motel, the number, l , of large rooms must be at least the same as the number, s , of the number of small rooms	$l \geq s$
In the intake of new recruits of cadets, the number, m , of males must not be more than twice the number, f , of females.	$m \nlessgtr 2f$ $m \leq 2f$

Linear Programming

The general process for solving linear-programming exercises is to graph the inequalities (called the "constraints") and to identify the region on a Cartesian plane that satisfies all the constraints (called the "feasibility region").

Having identified the region containing all possible solutions, we may now wish to find out what pair of x and y values will lead to maximum or minimum profits. The equation that is used to represent the profit is called the profit or optimisation equation.

The fundamental theorem of linear programming gives us a law that is useful in determining the maximum or minimum value of this profit equation. We will now present this law.

The Fundamental Theorem of Linear Programming

It has been proven that, for a system of linear inequalities, the maximum and minimum values of the profit or optimisation equation will always be on the corners (vertices) of the feasibility region; the region that satisfies all the inequalities.

The coordinates of the corners of this feasibility region are identified and each corner point is tested in the equation for which we are trying to find the highest or the lowest value.

The three inequalities are the constraints. The area of the plane that is satisfied by all three inequalities is now the feasibility or feasible region.

Summary of Steps in Solving a Linear Programming Problem

1. Read the constraints and write down the system of linear inequalities.
2. Sketch the system of linear inequalities to obtain the feasible region.
3. Identify each corner point of the feasible region.
4. Obtain the profit equation, it is usually in the form, $z = ax + by$.
5. Substitute each corner point of the feasible region into the profit equation.
6. Choose the point yielding the largest value or smallest value depending on whether the problem is a maximisation or minimisation problem. This is called the optimal solution.

Example 6

Natasha wishes to buy x oranges and y mangoes. Her bag can only hold 6 fruits. She must buy at least 2 mangoes. The number of mangoes is less than or equal to twice the number of oranges.

- a. Write three inequalities to represent this information.
- b. Represent the three inequalities on a graph.
- c. Shade on your graph the region which represents the solution set for the three inequalities.
- d. Natasha makes a profit of \$1 on an orange and a profit of \$2 on a mango. Calculate (i) her maximum profit and (ii) her minimum profit.

Solution -Part a

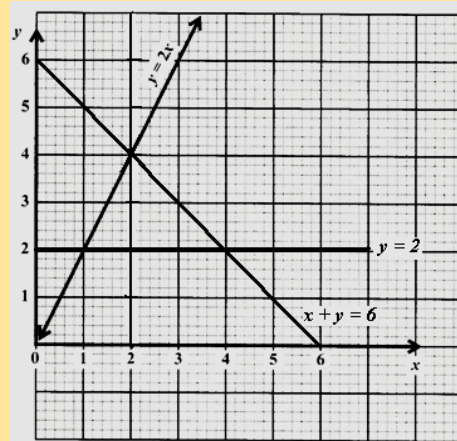
Number of oranges = x
Number of mangoes = y

The total number of fruits must not exceed 6. $x + y \leq 6$

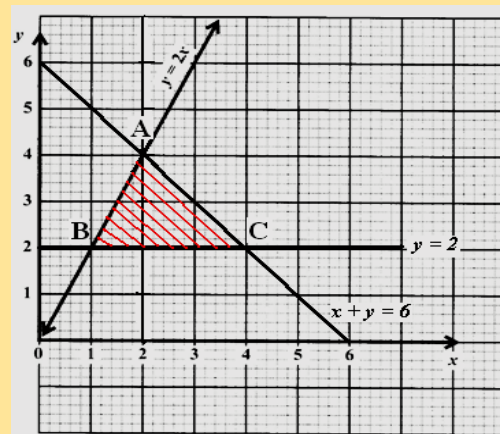
She must buy at least 2 mangoes $y \geq 2$

The number of mangoes is less than or equal to twice the number of oranges $y \leq 2x$

(b) To represent the three inequalities on a graph, we draw the three lines x



(c) For $y \geq 2$, shade the region above the line. For $x + y \leq 6$, shade the region below the line (angle made with the horizontal is acute). For $y \leq 2x$, shade the region to the right of the line (angle made with the horizontal is acute).



Solution – Part d

d. She makes a profit of \$1 on an orange and \$2 on a mango profit expression is:

$$x + 2y$$

To obtain the maximum profit, we apply the theorem of linear programming and test the vertices of the region:

For the point A (2, 4), the profit is
 $(2) + 2(4) = \$10$

For the point B(1, 2), the profit is
 $1 + 2(2) = \$5$

For the point C(4, 2), the profit is
 $4 + 2(2) = \$8$

Her maximum profit is \$10 if she sells 2 oranges and 4 mangoes.

Her minimum profit is \$5 if she sells 1 orange and 2 mangoes.