8: RELATIONS AND FUNCTIONS

Relations

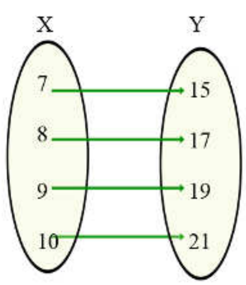
In the real world, there are several situations that involve relationships between two sets. We often use the term 'relation' to describe these relationships. When we speak of a relation in mathematics, we refer to a well-defined relationship between two sets of numbers.

In mathematics, a relation between two sets exists when there is a clear rule that defines the relationship from the members of one set, called the **domain** to the members of the other set, called the **co-domain**.

Representing relations

An **Arrow Diagram** is often used to represent a relation. The diagram below, shows the sets:
X = {7, 8, 9, 10} and Y = {15, 17, 19, 21}.

In this mapping, the members are related by the rule, $y = 2x+1$, where x are members of X. The arrows connect members of X to its corresponding member in Y. The rule between the members of X and the members of Y must hold for all possible pairs that are connected.



The relation can also be represented as a set of **ordered pairs**:

(7, 15), (8, 17), (9, 19), (10, 21).

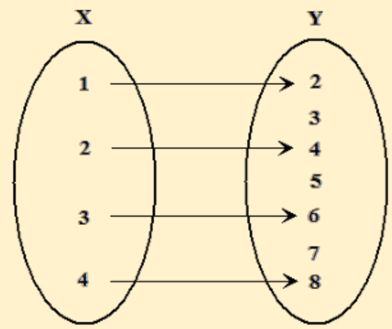
We can also refer to the members of the set X as the **input** and the members of set Y as the **output**. The direction of the arrows is always from the input to the output. The corresponding y value in Y is described as the ***image*** of the x value in X.

If we plot ordered pairs on a Cartesian Plane, we obtain a graph of a relation. We have met such graphs in our study of coordinate geometry and quadratic functions. Hence, **graphs** are also used to represent relations.

Range and Co-domain

Let us define a relation from a set of integers,
X = {1, 2, 3, 4} to the set of integers,
Y = {2, 3, 4, 5, 6, 7, 8} as $x \to 2x$.
The arrow diagram is shown for this relation below. Notice that the members 3, 5 and 7 of the set Y are not outputs. The entire set, Y, is called the ***co-domain***. The subset Y consisting of {2, 4, 6, 8} is called the ***range***.

Range and Codomain



The range is defined as those members of the co-domain that are 'used', that is, they are connected to some member of the domain. They are also the output values or the images of the input values. The entire set Y is the co-domain.

In the above example if we define the set Y as a set of even numbers {2, 4, 6, 8}, then we will have a situation where the range will be equal to the codomain. The members 3, 5 and 7 will no longer be members of Y. So, it may be that the range is equal to the co-domain in certain situations. But range and co-domain are not synonymous terms.

Example 1

A relation is represented by the ordered pairs:
(1, 5), (2, 7), (3, 9), (4, ?).

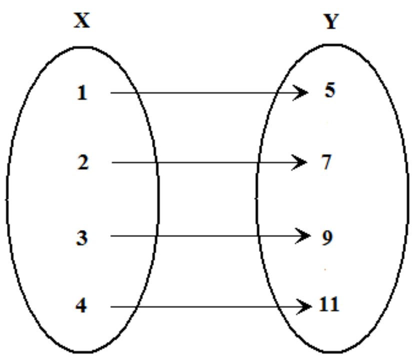
a. State the rule for the relation.
b. What is the image of 4?
c. What is the input for an output of 23?
d. Draw an Arrow Diagram to represent the relation.

Solution

a. By inspection, the rule for the relation is $2x+3$.
b. The image of 4 is calculated by substituting $x=4$ in $2x+3$.
$2(4)+3=11$. The image of 4 is 11.

c. The input when the output is 23 is found by solving for x when $2x+3=23$.
$2x+3=23,\Rightarrow 2x=20, x=10$.
The input is 10 for an output of 23.

d. The Arrow diagram which illustrates this relation is shown below.



Functions and relations

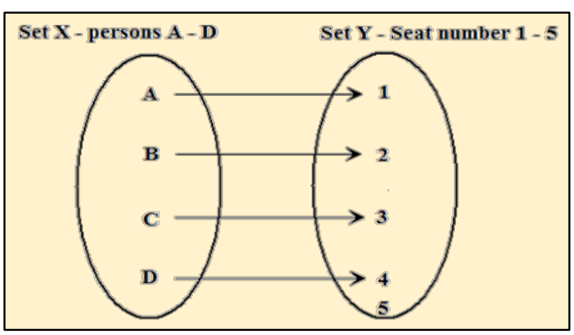
A **function** is a relation for which **each** value from the domain is associated with **exactly one** value from the codomain. Functions can be either **one to one** or **many to one.**

Relations can be **one to one**, **many to one, one to many** or **many to many.** The set of functions is, therefore, a subset of the set of relations.

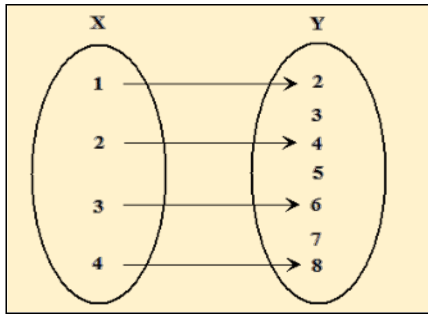
One to one functions

Consider a function in which elements of a set X are mapped onto elements of a set Y. The function is **one to one** if, every input has one and only one output. Consequently, no two elements in the set X have the same output in Y.

We can think of passengers on a bus as the set X and the seats as the set Y. If only one person sits on a seat, then this is a one to one mapping. Note, however, that there can be empty seats on the bus, but every person has a unique seat. In this example which is illustrated below, seat number 5 is shown to be unoccupied.



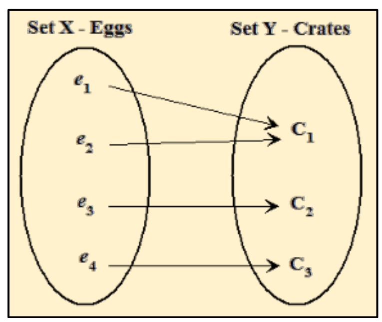
One to one functions are extremely common in the real world. For example, if the cost of an orange is \$2 and we map the number of oranges (X) onto the cost (Y), then each member of X will be assigned to one and only one member of Y.
The arrow diagram below shows an example of this one-one function. Each quantity can only be assigned one cost.



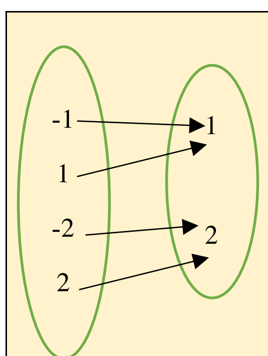
The set X represents the number of oranges. The set Y represents the cost of the oranges.

Many to one functions

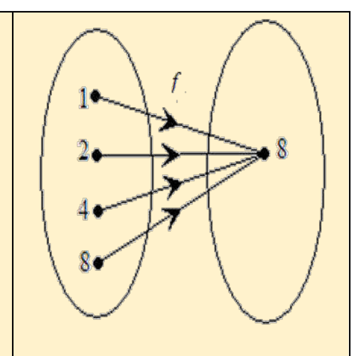
A function is said to be **many to one** if, for at least one element of Y, there corresponds more than one element of X. We may think about packing eggs in crates, where the set X consists of eggs and the set Y consists of crates. Many eggs can be packed in the same crate, but the same egg cannot be packed into two different crates.



The examples below show many to one functions.



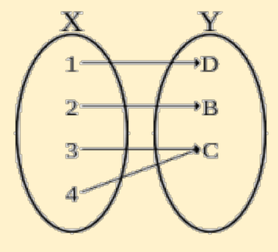
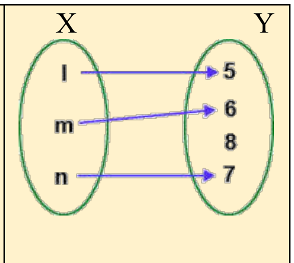
$f(x)=x^2$



f = 'is a factor of'

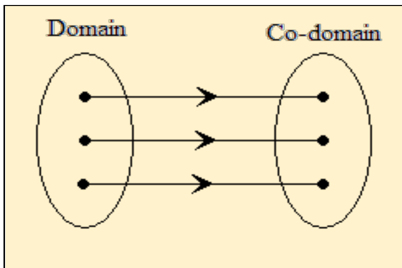
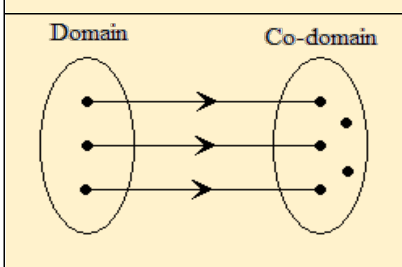
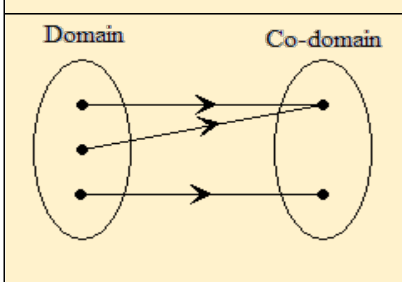
Onto Functions

A function $f: X \to Y$ is onto if for every y in Y, there is an x in X, such that $f(x) = y$. Hence, every output has an input, which makes the range equal to the co-domain.

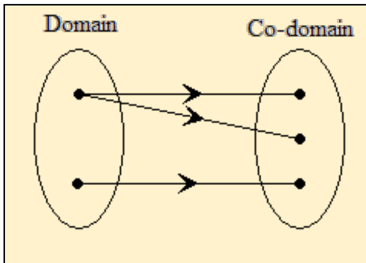
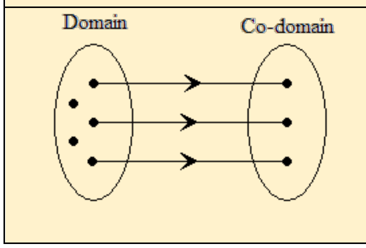
	
An **onto function** since all outputs have inputs. Range = Co-domain	A function that is **NOT onto**, there is no input for the output, 8. Range ≠ Co-domain

Examples of functions

The arrow diagrams below illustrate mappings that are examples and non-examples of functions. In a function, each element of the domain must be mapped onto one element of the co-domain. Hence, if one **arrow**, only, leaves **EACH** element of the domain, then the mapping is a function.

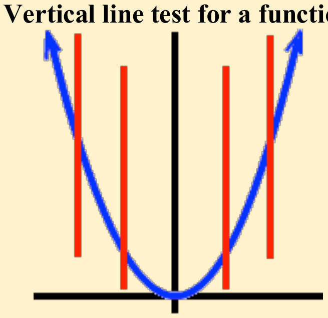
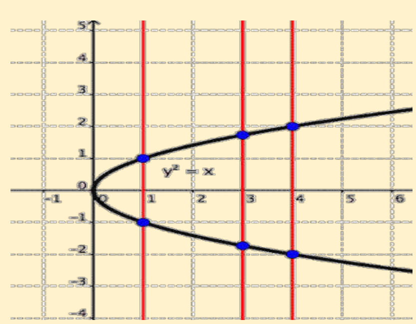
	A function that is both one-to-one and onto. Every output has an input so the range and the codomain are the same.
	A function that is one-to-one but not onto. The range is not the same as the codomain.
	A many to one and onto function. The range is the same as the codomain.

Non-examples

	This is not a function. One member of the domain is mapped onto two members of the co-domain.
	This is not a function. There are two members of the domain not mapped onto any member of the co-domain.

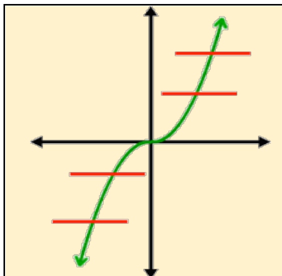
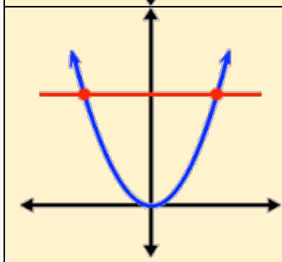
Vertical Line Test for a function

The vertical line test may be used to determine if a graph represents a function. If a vertical line cuts the graph at most once then the graph represents a function. This test works because a function has only one output for each input.

Vertical line test for a function

This graph represents a function since the vertical lines cut the curve once, regardless of its position. This indicates there are no one-to-many relationships.

This graph does **NOT** represent a function, since the vertical line cuts the curve more than once, indicating that there is more than one output for one input.

Horizontal Line Test for a One to One Function

If a horizontal line intersects a graph of a function at most once, then the graph represents a one-to-one function.

	**One-one** The graph represents a one to one function since the horizontal lines cut the curve once. Note, the graph also must pass the vertical line test.
	**Not one-one** Although this function passes the vertical line test and represents a function, it does **not** represent a one-to-one function since the horizontal lines cut the curve more than once.

Example 2

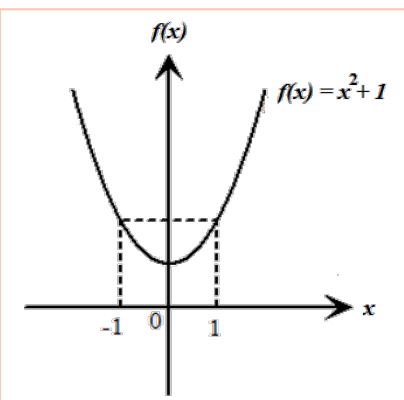
Show that $f(x)=x^2+1$ is not one-to-one for $x\in R$.

Solution

$f(1)=(1)^2+1=2$

$f(-1)=(-1)^2+1=2$

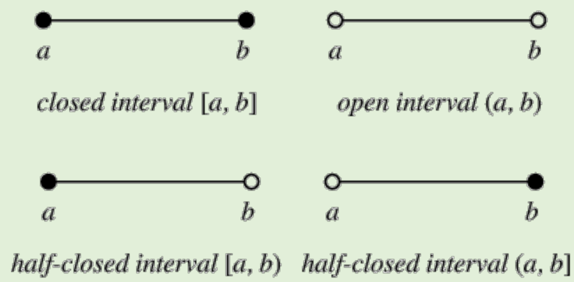
There are two values of x mapping onto the same value of $f(x)$.Therefore $f(x)=x^2+1$ is not one-to-one for $x\in R$. It is sufficient to say that the rule does not hold if only one example can be found that contradicts the rule for a one-to-one function.



Describing the range of a function

The set of numbers on a number line is infinite with a range from negative infinity to positive infinity. In order to describe a subset of real numbers, we must be familiar with **interval notation**.

The set of all numbers between two endpoints is called an interval.
A closed interval includes the endpoints.
An open interval does not include the endpoints.
A half-open interval includes one of the endpoints.



We have already defined the range of a function as the set of members of the codomain that are used. We may refer to these members as output values or images. For a function from a set A to a set B, the subset of B which contains all the elements of A under f is called the **range**.

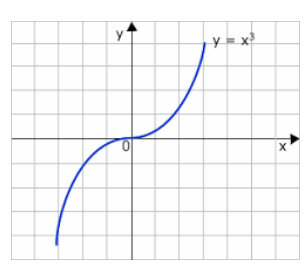
When the domain is a set of real numbers, it is not possible to list all the members of the range. So, we use the interval notation to describe the range. Sometimes it is necessary to sketch a function in order to determine and then describe its range.

Example 3

Find the range of $f(x)=x^3$, where x is a real number.

Solution

The function $f(x)$ may take any value on the set of real numbers from ∞ to -∞., A sketch of the graph is shown below.

	This is an open interval since there is no lower boundary or no upper boundary for $f(x)$ The range of $f(x)$ is $-\infty\le f(x)\le\infty$.

Example 4

Find the range of $f(x)=x^2-4x+3$ (x is a real number).

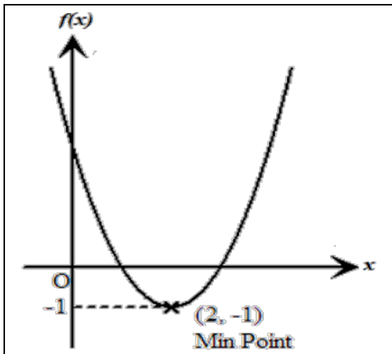
Solution

The minimum value is first obtained by completing the square,

$f(x)=(x-2)^2-4+3=(x-2)^2-1$

The minimum value is -1, and this occurs at $x = 2$.

The graph shows that the range of f is $f(x) > -1$.



This is a half-open interval since it has a lower boundary of -1 but no upper boundary. The function may take any value from -1 up to ∞.

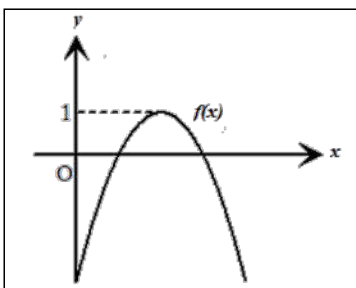
Example 5

Find the range of $f(x)=4x-3-x^2$, where x is a real number.

Solution

By completing the square, $f(x)=-(x-2)^2+1$ The maximum value is 1 and this occurs at $x = 2$.

The graph shows that the range of $f(x)$ is $f(x)\le 1$.



This is a half-open interval since it has an upper boundary of 1 but no lower boundary, i.e $f(x)$ may take any value from 1 to $-\infty$.

Example 6

Find the range of $f(x)=x^2+3$ for $0\le x\le 4$.

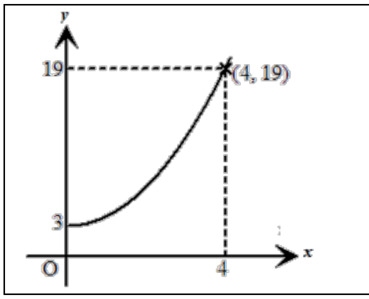
Solution

This function has a restricted domain.

Substituting the values $x = 0$ and $x = 4$ in the function, $f(x)$, we have:

$f(0)=3, f(4)=19$ and the graph is increasing (rising only) for this domain.

Therefore, the range is $3\le f(x)\le 19$



This is a closed interval since $f(x)$ has a lower boundary and an upper boundary. The function may take any value from 3 to 19.

Example 7

Find the range of $f(x)=x^2-4x-5$ for $-1\le x\le 8$.

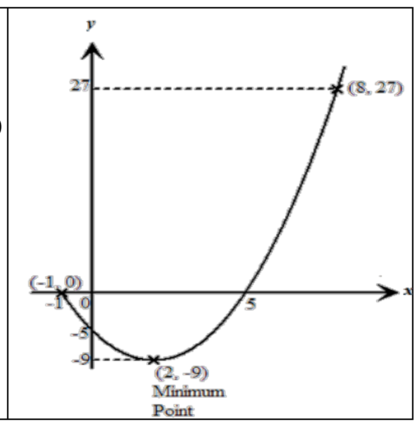
Solution

Now $f(x)=(x-2)^2-4-5=(x-2)^2-9$

Hence the graph has a minimum point at $(2, -9)$.
Taking into consideration, the given domain, $-1\le x\le 8$, we compute the values of $f(x)$ at the end points of the interval.
When $x = -1, f(x) = 0$
When $x = 8, f(x) = 27$

We should note that in this case that the range is NOT to be the interval $0\le f(x)\le 27$ because the y-coordinate of the minimum point is -9. A sketch illustrates this point.

Let $y = 0$,
$x^2-4x-5=0$
$(x+1)(x-5)=0$
The curve cuts the x-axis at $(-1, 0)$ and $(5, 0)$.

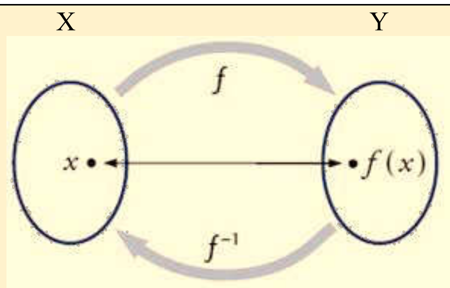


The range of $f(x)=x^2-4x-5$ for $-1\le x\le 8$ is $-9\le f(x)\le 27$.

So, we must be mindful that the range for a particular domain does not necessarily correspond to the y values at the endpoints of the domain. We cannot assume that the end values of a domain will necessarily correspond to the upper and lower ends of the range. A sketch is usually helpful so as to avoid costly mistakes.

The inverse of a function

In a function, each value from the domain, X, is associated with **exactly one** value from the codomain, Y, by a given rule. We denote this by $f:x\to f(x)$. The inverse of *f*, denoted by $f^{-1}(x)$ is the rule that maps members of Y back onto X. We denote this by $f^{-1}(x):f(x)\to x$. This is illustrated in the diagram below.



In an inverse function, the range of *f* becomes the domain of f^{-1} and the domain of *f* becomes the range of f^{-1}.

Conditions for a function to have an inverse

The function, *f* from X to Y maps each member of X to a member of Y. So too, the function, f^{-1} from Y to X maps each member of Y to a member of X. This implies that there must be a one to one correspondence between the members of X and Y. In addition, *f* must also be an onto function, since all the Y values must be associated with an X value.

The inverse of a function, *f*, exists if and only if *f* is both one to one and onto.

To determine the inverse of a function

Consider the function $f(x)=3x+2$, where x is a real number. When $x=4$, $f(4)=3(4)+2=14$

We note that $f(x)$ maps 4 onto 14.
The inverse function, written as $f^{-1}(x)$ is that function which maps 14 onto 4.

To determine f^{-1}, the inverse of *f*, we need a rule for f^{-1}. For simple linear functions, we can readily obtain the rule by reversing the forward process. For example, the inverse of $f(x)=3x+2$, is obtained as shown below.

Forward	Reverse
×3 then +2	−2 then ÷3

Verifying, $f^{-1}(14)=\dfrac{14-2}{3}=4$

For more complex functions, we need to apply the following procedures.

Step 1: Rewrite the function letting $y=f(x)$

So $f(x)=3x+2$ is written as $y=3x+2$

Step 2: Make x the subject by basic algebra

$$\frac{y-2}{3}=x$$

Step 3: Replace y by x to get f^{-1}

$$f^{-1}(x)=\frac{x-2}{3}$$

Example 8

Find the inverse of the function $f(x)=\frac{3x-1}{x+2}$.

Solution

Let $y=\frac{3x-1}{x+2}$

Next we make x the subject.

$3x-1=y(x+2)$

$3x-1=xy+2y$

$3x-xy=1+2y$

$x(3-y)=1+2y$

$x=\frac{1+2y}{3-y}$

Replacing y by x on the right side to obtain

$f^{1}(x)=\frac{1+2x}{3-x}$

Self-Inverse

In some cases, the inverse of a function is exactly the same as the function. For example, if $f(x) = \frac{3}{x}$, then $f^{-1}(x) = \frac{3}{x}$. Such a function is called self-inverse.

So, $f(x) = \frac{3}{x}$ is an example of a self-inverse function.

Undefined values

When a function is expressed in the form $f(x) = \frac{A}{B}$, the function is undefined or not-defined when $B = 0$. This is because, $\frac{A}{0} \to \infty$.

If we have to state the value of x for which $f(x) = \frac{3x+1}{2x-4}$ is undefined, we equate the denominator of $f(x)$ to zero and solve for x.

When $2x - 4 = 0, x = 2$.
Hence, $f(x)$ is undefined at $x = 2$.
When a function is undefined at a point, say $x = k$, there is no corresponding value for y at $x = k$, that the function does not exist for this value of x.
Example, $f(x) = \frac{3x+1}{(x-1)(x+3)}$ will not exist at $x = 1$ and at $x = -3$.

The Graph of a Function and its Inverse

If a function, $f(x)$ maps the element a onto b, then its inverse $f^{-1}x$ will map b onto a. This means that any point (a, b) on the graph of f will be mapped onto (b, a) on the graph of f^{-1} .
Geometrically speaking, the inverse function is seen as a reflection of $f(x)$ in the line $y = x$. This is because a reflection in the line $y = x$ maps a point (a, b) onto the image (b, a). Hence, if $f(x)$ is known, then we may easily sketch the graph of $f^{-1}x$.

$$f \xrightarrow{\text{Reflection in the line } y=x} f^{-1}$$

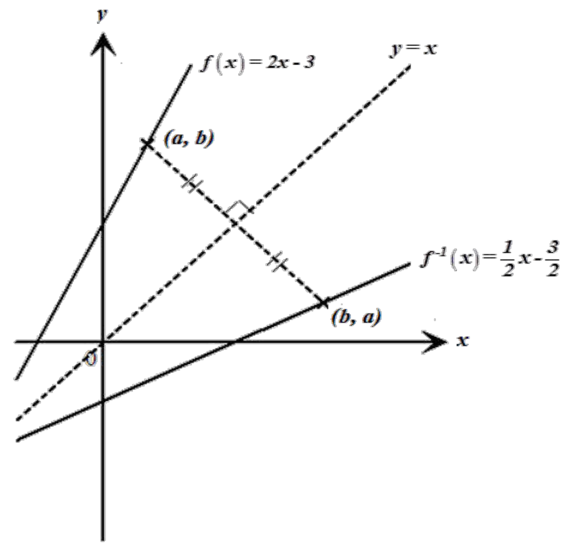
Example 9

On the same diagram, show the graph of $f(x) = 2x + 3$ and the graph of $f^{-1}(x)$.

Solution

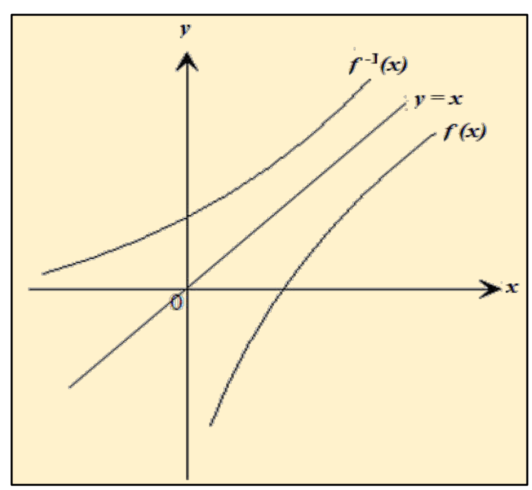
First, draw the graph of $y = 2x + 3$ by plotting a pair of points.
Next, draw the line $y = x$ (shown as a broken line below).

Then reflect the line $y = 2x + 3$ in the line $y = x$. This is simply done (in this case because it is a straight line) by reversing the coordinates of any two points on $y = 2x + 3$ and joining these points to form the line of the inverse.



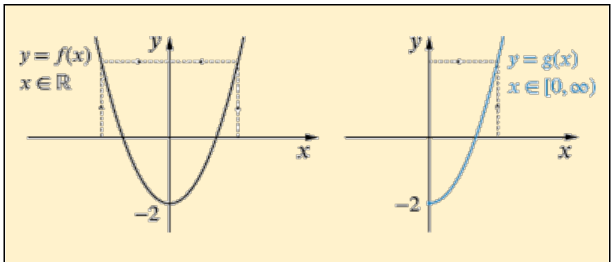
The inverse of non-linear functions

If a non-linear function has an inverse, then the above principle for a linear function is still applicable. We can draw the graph of the inverse function, f^{-1} by reflecting the function f in the line $y = x$.
The diagram below shows the graph of a non-linear function and the graph of its inverse. The line $y = x$ will be an axis of symmetry for the function, f and its inverse, f^{-1} .



In general, curves such as quadratics do not have an inverse for $x \in R$, since they are clearly not one-to-

one. However, if we restrict the domain of a quadratic function, an inverse may exist for the restricted interval.
If we draw a line of symmetry on a quadratic graph and define the domain to the right or to the left of the line, we will have a one-one and onto function.
This is illustrated in the diagrams below for the quadratic function $y = x^2 - 2$ with the axis of symmetry, $x = 0$.



$f(x)$ is not one-one if we look at the entire domain $(-\infty, \infty)$.	If we restrict the domain to, say, the interval $[0, \infty)$, then the function is one-one.

The equation of the axis of symmetry of a quadratic function has the general form $x = -\frac{b}{2a}$.
For any quadratic function (maximum or minimum),

- when $x \geq -\dfrac{b}{2a}$, the function is both one to one and onto and hence has an inverse for this restricted domain.
- when $x \leq -\dfrac{b}{2a}$, the function is both one to one and onto and hence has an inverse for this restricted domain.

Composite functions

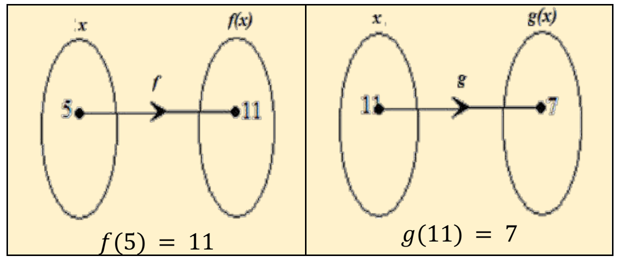
Sometimes it is necessary to obtain a single function that is equivalent to two functions. We refer to the single function as the composite of the two separate functions.

A **composite function** is a combination of two functions, where we apply the first function and the output is used as the input into the second function. In combining two functions, we must note that the order is important.

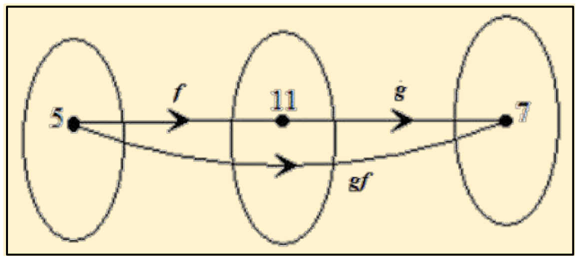
Consider the following functions:

$f(x) = 2x + 1$ and $g(x) = \dfrac{x+3}{2}$

If we obtain the image of, say 5, in $f(x)$.
$f(5) = 2(5) + 1 = 11$
Then we use this image as the input in $g(x)$.
$g(11) = \dfrac{11+3}{2} = \dfrac{14}{2} = 7$.
Our result after applying these two functions in the order, f followed by g, is 7. Illustrating this process in two separate diagrams, we have:



The single function that maps 5 onto the final image 7 is written as $gf(x)$, where $gf(5) = 7$.



Notation for a composite function

The function gf is called a **composite** function, which is equivalent to a combination of $f(x)$ and then $g(x)$. We can say that $gf(x)$ is the combined function of f and g when f is performed first and then g after.

We may also say that $gf(x)$ is f followed by g. and $fg(x)$ is g followed by f.

The commutative property does not hold when combining two functions to form the composite function. So that,

$$gf(x) \neq fg(x)$$

Deriving the Composite Function

To obtain an expression for the composite function $gf(x)$ where g and f are defined as:

$f(x) = 2x + 1$ and $g(x) = \dfrac{x+3}{2}$

We may think of $gf(x)$ as $g[f(x)]$, read as "g of $f(x)$".

$$gf(x) = g[f(x)]$$
$$= g[2x+1]$$
$$= \frac{(2x+1)+3}{2}$$
$$= \frac{2x+4}{2}$$

Replacing x in $g(x)$ by $2x + 1$

We can also derive the composite function $fg(x)$ as follows.

$$fg(x) = f[g(x)]$$
$$= f\left[\frac{x+3}{2}\right]$$
$$= 2\left(\frac{x+3}{2}\right)+1$$
$$= x+3+1$$
$$= x+4$$

Replacing x in $f(x)$ by $\frac{x+3}{2}$

Redefining inverse functions

If f and g are two functions such that
$f[g(x)] = x$, for every x in the domain of g
$g[f(x)] = x$, for every x in the domain of f
Then function g is said to be the inverse of the function f.

Example 10

Use the definition of inverse functions to determine if the functions f and g are inverses of each other, where

$f(x) = 8x - 7$ and $g(x) = \frac{x+7}{8}$

Solution

$$f[g(x)] = f\left(\frac{x+7}{8}\right)$$
$$= 8\left(\frac{x+7}{8}\right) - 7$$
$$= x + 7 - 7$$
$$= x$$

$$g[f(x)] = g(8x-7)$$
$$= \frac{(8x-7)+7}{8}$$
$$= \frac{8x}{8}$$
$$= x$$

Therefore $f[g(x) = g[f(x)]$ and f and g are inverses of each other.

Example 11

$f(x) = x+2$ and $g(x) = \sqrt{x}$, where $x \in R$.
Determine the composite function $gf(x)$. Hence state its domain.

Solution

$f(x) = x+2 \qquad g(x) = \sqrt{x}$

Let us first examine the domain and range of f.
The domain of f is any real number, R.
The range of f is the domain or input for g.

To obtain the domain for g, consider the composite function, $gf(x)$.

$\therefore gf(x) = \sqrt{x+2}$

For $gf(x)$ to exist, the square root of $(x+2)$ must be positive $\forall$ real values of x.
$\therefore x+2 \geq 0$ that is $x \geq -2$.

The domain of $f(x)$ is R and the domain of $g(x)$ is $x \geq -2$. Hence,
$gf(x)$ has a domain which is the intersection set of these two domains and which is $x \geq -2$.

Example 12

Given $f(x) = \sqrt{x}$ and $g(x) = x+5$
Write expressions for
$f^{-1}(x)$, $g^{-1}(x)$, $g^2(x)$, gf^{-1}.

Solution

$$f(x) = \sqrt{x}$$
Let $y = \sqrt{x}$
$$x = \sqrt{y}$$
$$y = x^2$$
$$f^{-1}(x) = x^2$$

$$g(x) = x+5$$
Let
$$y = x+5$$
$$x = y+5$$
$$y = x-5$$
$$g^{-1}(x) = x-5$$

$$g^2(x) = g[g(x)]$$
$$= g(x+5)$$
$$= (x+5)+5$$
$$= x+10$$

$$gf^{-1}(x) = g[f^{-1}(x)]$$
$$= g(x^2)$$
$$= x^2 + 5$$