1. COORDINATE GEOMETRY OF A STRAIGHT LINE

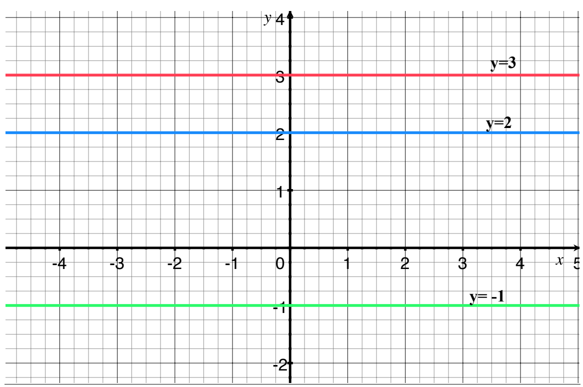
Lines on the Cartesian Plane

A straight line, drawn on a Cartesian Plane can be described by an equation. Such equations have a general form and vary depending on where the line cuts the axes and its degree of slope.

Horizontal Lines

Horizontal lines are all parallel to the x-axis. Hence, their equations resemble the equation of the x-axis. We recall that the equation of the x-axis is $y = 0$. This is because all points on this axis have a y-coordinate of zero, regardless of their different x-coordinates.

So, the horizontal line which cuts the y-axis at 2 has equation $y = 2$. Also, the horizontal line that cuts the y-axis at -1 has equation $y = -1$, and so on.

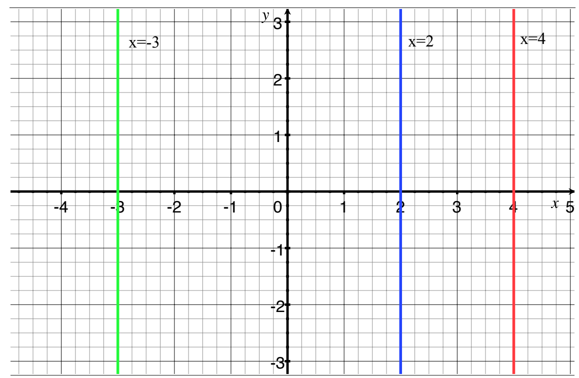


In general, the equation of a horizontal line is $y = a$ (where a is a constant). This line would cut the vertical axis at a and all points on the line have a y-coordinate of a. Different points, though, have different x-coordinates.

Vertical Lines

Vertical lines are all parallel to the y-axis. Hence, their equations resemble the equation of the y-axis. We recall that the equation of the y-axis is $x = 0$. This is because all points on this axis have an x-coordinate of zero, regardless of their different y-coordinates.

So, the vertical line which cuts the x-axis at 2 has equation $x = 2$. Also, the vertical line that cuts the x-axis at -3 has equation $x = -3$ and so on.



In general, the equation of a vertical line is $x = b$ (where b is a constant). This line cuts the horizontal axis at b and all points on the line have an x-coordinate of b. Different points, though, have different y-coordinates.

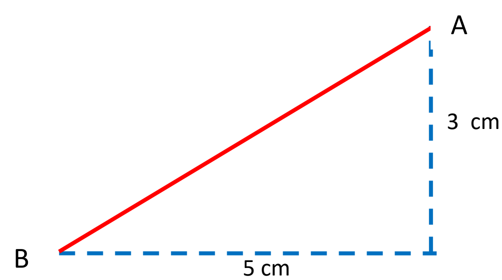
The gradient of a straight line

To determine the equation of a line other than a horizontal or a vertical line, we must know how to measure its gradient. The gradient of a line is a measure of its slope or steepness. It is defined as a ratio of its vertical displacement or *Rise*, to its horizontal displacement or *Run*.

To determine the magnitude of these displacements, we can use:

- direct measurement OR
- count the number of units on the vertical and horizontal lines when the line is drawn on the Cartesian plane or a grid.

In the diagram shown, we are given measurements.

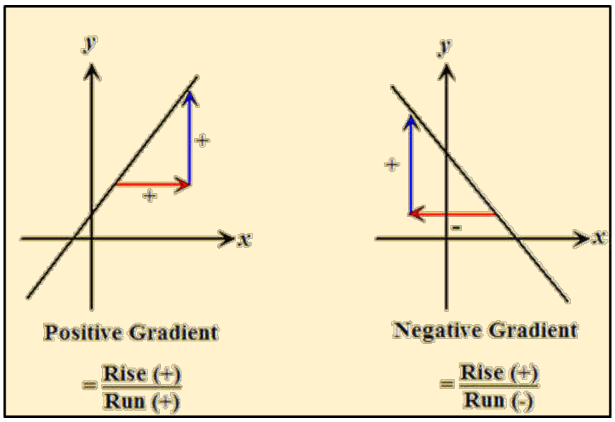


Hence, we can determine the gradient as follows:

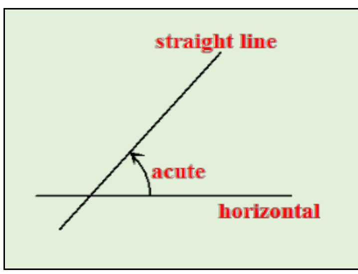
$$Gradient = \frac{Vertical\ Displacement}{Horizontal\ Displacement} = \frac{3}{5}$$

Positive and negative gradient

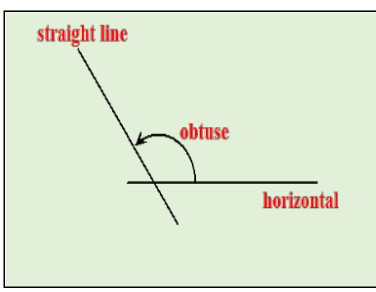
The gradient of a line can be positive or negative, depending on the direction of its vertical and horizontal displacements. These differences are best shown on the Cartesian Plane. A positive gradient is obtained when both displacements, are in the same direction. A negative gradient is obtained when the signs of the displacements are in opposite directions.



Positive and negative gradients can also be recognised by observing the angle, ***measured in an anticlockwise direction***, between the line and any horizontal that intersects the line. It is often convenient to use the x-axis as the horizontal line.



Lines that are inclined at an acute angle to the horizontal, have positive gradients.



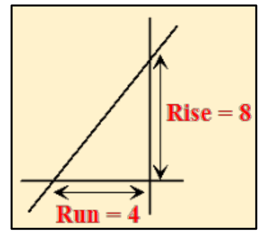
Lines that are inclined at an obtuse angle to the horizontal, have negative gradients.

Calculating gradient

The gradient of a straight line can be found when the following information is given:

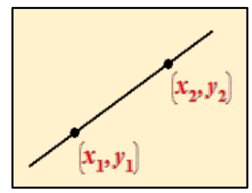
(i) By reading off, the vertical and horizontal displacements from a given line, drawn on a grid, we can obtain this ratio directly.

$$\text{Gradient} = \frac{\text{Rise}}{\text{Run}} = \frac{8}{4} = 2$$



(ii) Given the coordinates of any two points on the line and using,

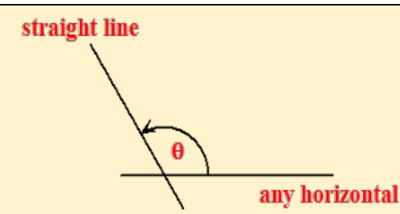
$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1}.$$



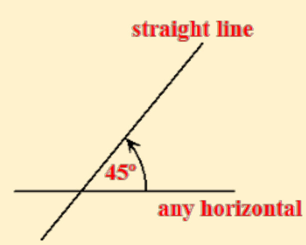
Example, $A = (1, 2)$ and $B = (3, 6)$

$$\text{Gradient} = \frac{6-2}{3-1} = \frac{4}{2} = 2$$

(iii) Calculating the tangent of the angle, θ, which the line makes with any horizontal and where θ is measured in an anticlockwise direction.



$$\text{Gradient} = \tan\theta$$



$$\text{Gradient} = \tan 45° = 1$$

(iv) Obtaining the gradient, m from the general equation of a straight line, $y = mx + c$,

For example, $y + 2x + 1 = 0$.
We re-write the equation in this form
$y = -2x - 1$ where $m = -2$
Hence, the gradient is -2

Gradient of horizontal and vertical lines

Horizontal lines are said to have no steepness. For any 'Run', the 'Rise' is always zero. So, the gradient of a horizontal line is $\frac{0}{Run}$ = zero.

Vertical lines are said to have maximum steepness. For any 'Rise', the 'Run' is always zero. The gradient of a vertical line is $\frac{Rise}{0} \rightarrow \infty$, which is undefined.

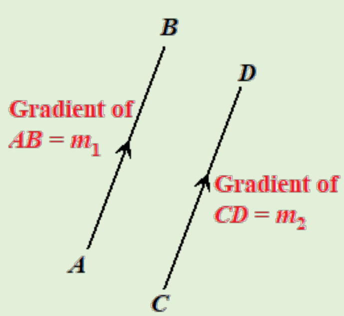
We can verify these results using the definition, gradient = tan θ.

For horizontal lines, $\theta = 0^0$ and $\tan 0^0 = 0$.

For vertical lines, $\theta = 90^0$ and $\tan 90^0$ is undefined.

Parallel Lines

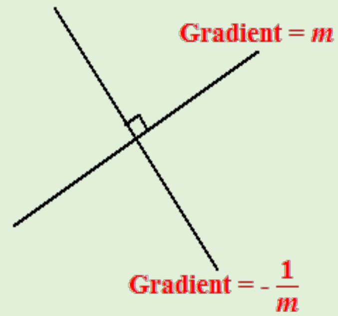
Parallel lines have the same slope and so must have the same gradient. The converse is also true. That is, lines that have the same gradients are parallel. We can use this fact to prove that any two straight lines are parallel or not.



If m_1 and m_2 are the gradients of two lines that are parallel, then
$m_1 = m_2$.

Perpendicular Lines

The product of the gradients of perpendicular lines is equal to -1. The converse is also true. That is, if the product of the gradients of any two lines is -1, then the lines are perpendicular to each other. We can use this fact to prove that and two straight lines are perpendicular or not.

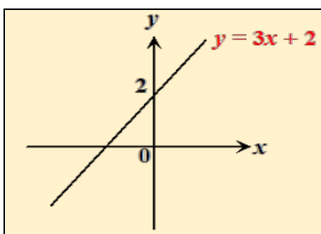


If m_1 and m_2 are the gradients of two perpendicular lines, then

$$m_1 \times m_2 = -1$$

General form of the equation of a straight line

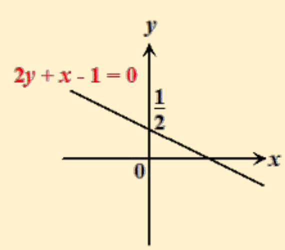
All straight lines have a general equation of the form, $y = mx + c$, where m is the gradient and c is the intercept on the vertical axis.



$y = 3x + 2$

This line has gradient 3 and cuts the y-axis at 2.

Sometimes we may have to rearrange the equation to obtain this general form.



$$2y + x - 1 = 0$$
$$2y = -x + 1$$
$$y = -\frac{1}{2}x + \frac{1}{2}$$

This line has gradient $-\frac{1}{2}$ and cuts the y-axis at $\frac{1}{2}$.

To determine the equation of a straight line, we require

(i) The coordinates of one point, (x_1, y_1) on the line.

(ii) The gradient, m of the line.

(iii) Then we use the equation $\frac{y - y_1}{x - x_1} = m$

Example 1

Determine the equation of the straight line that passes through the point (– 1, 4) and has gradient of – 2.

Solution

Using $\frac{y-y_1}{x-x_1}=m$, we have

$$\frac{y-4}{x-(-1)}=-2$$
$$y-4=-2(x+1)$$
$$y=-2x+2$$

Alternative Method

Using $y=mx+c$, we can substitute $x=-1, y=4$ and $m=-2$ to solve for c.

$$4=-2(-1)+c$$
$$4=2+c$$
$$c=2$$
$$y=-2x+2$$

Example 2

Find the equation of the line joining (2,1) to (3,4).

Solution

The gradient is first calculated using:

gradient $=\frac{y_2-y_1}{x_2-x_1}$, where

$(x_1,y_1)=(2,-1)$ and $(x_2,y_2)=(3,4)$

$$m=\frac{4-(-1)}{3-2}=\frac{5}{1}=5$$

Then, we proceed to find the equation using

$\frac{y-y_1}{x-x_1}=m$ and either of the two given points

$$\frac{y-(-1)}{x-2}=5$$
$$y+1=5(x-2)$$
$$y+1=5x-10$$
$$y=5x-11$$

Had we used the other point, the equation would have been the same.

Example 3

Determine whether the lines $y = 2x + 3$ and $2y + x - 1 = 0$ are parallel or perpendicular.

Solution

The line $y=2x+3$ has a gradient of 2.

The line $2y+x-1=0$ has a gradient of $-\frac{1}{2}$.

Since,

$$2\times-\frac{1}{2}=-1$$

The lines are perpendicular to each other.

Example 4

Find the equation of the line passing through O and perpendicular to the line $3y=4x+2$.

Solution

By expressing the equation $3y=4x+2$ in the form

$$y=\frac{4}{3}x+\frac{2}{3}$$

The gradient of the given line is deduced to be $\frac{4}{3}$.

∴ the gradient of the line perpendicular to the given line is $=-\frac{3}{4}$ (since the product of the gradients of perpendicular lines is -1)

We can find the equation using $\frac{y-y_1}{x-x_1}=m$ and take

$$(x_1,y_1)=(0,0)$$
$$\frac{y-0}{x-0}=-\frac{3}{4}$$
$$4y=-3x$$
$$y=-\frac{3}{4}x$$

Example 5

Find the equation of the line passing through $(1,-1)$ and parallel to $y=3x-1$.

Solution

By comparison with the general form of an equation, $y=mx+c$ the gradient of $y=3x-1$ is deduced as 3. Hence the gradient of the required line is also 3 since parallel lines have equal gradients.

We can find the equation using $\frac{y-y_1}{x-x_1}=m$ and

$$(x_1,y_1)=(1,-1)$$

$$\frac{y-(-1)}{x-1}=3$$
$$y+1=3(x-1)$$
$$y+1=3x-3$$
$$y=3x-4$$

Example 6

Find the equation of the straight line that passes through the point $(2, -2)$ and inclined at $45°$ to the horizontal axis.

Solution

The gradient is deduced to be $\tan 45^0 = 1$.

We can find the equation using $\frac{y-y_1}{x-x_1}=m$ and

$$(x_1, y_1)=(2,-2)$$
$$\frac{y-(-2)}{x-2}=1$$
$$y+2=x-2$$
$$y=x-4$$

Intercepts on the x-axis and the y-axis

The intercepts on the $x-$ axis and the $y-$ axis are the points where a straight line intersects or cuts the axis. To determine the coordinates of these points, we must note the following.

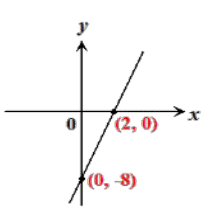
A straight line cuts the x-axis at $y = 0$.
Any intercept on the x-axis has coordinates $(a, 0)$
A straight line cuts the y-axis at $x = 0$.
Any intercept on the y-axis has coordinates $(0, b)$

Example 7

Determine the coordinates of the x and y-intercepts of the line $y = 4x - 8$.

Solution

When $x = 0$, $y = -8$, When $y = 0$, $x = 2$



The points, (2, 0) and (0, −8) are called the x and y-intercepts respectively, of the line shown

The midpoint of a straight line

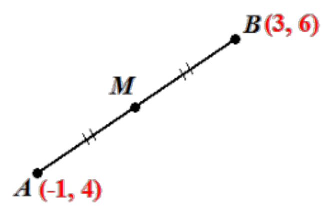
Once we know the coordinates of two points on a straight line we can find the mid-point of the line.

If A is (x_1, y_1) and B is (x_2, y_2), then the midpoint, M, of the straight line joining A to B, is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Example 8

State the coordinates of the midpoint of the line joining $A(-1, 4)$ and $B(3, 6)$.

Solution



$$M=\left(\frac{-1+3}{2}, \frac{4+6}{2}\right)=(1, 5)$$

We can use this same formula to find the endpoint of a line segment, given the midpoint and the other endpoint. For example, if we were given the coordinates of A and M, we can find the coordinates of B.

Example 9

The mid-point of the line joining P $(-2, 5)$ and R (a, b) is M $(2, 1)$. Calculate the value of a and of b.

Solution

Since we know M (2, 1) is the mid-point, we use the mid-point formula, substituting the given values so that:

$$\frac{-2+a}{2}=2, \quad \text{and} \quad \frac{5+b}{2}=1$$
$$-2+a=4 \quad \text{and} \quad 5+b=2$$
$$a=6 \quad \text{and} \quad b=-3 \quad \text{Hence, } R \text{ is } (6,-3).$$

Point of intersection of two straight lines

To determine the point of intersection of two straight lines, given their equations, it is not necessary to represent the lines graphically. We use algebraic methods and solve the equations simultaneously.

To obtain the point (s) of intersection of any two lines, we solve their equations simultaneously.

Example 10

Find the point of intersection of the lines whose equations are $y=2x-3$ and $2y-x=0$.

Solution

Let $y=2x-3 \quad \ldots(1)$

$2y-x=0 \quad \ldots(2)$

Substitute (1) into (2)

$2(2x-3)-x=0$

$4x-6-x=0$

$\therefore x=2$

When $x=2,\ y=1$.

$\therefore$ The point of intersection is $(2, 1)$.

Alternative Method

We may rewrite the equations as

$y-2x=-3 \quad \ldots(1)$

$2y-x=0 \quad \ldots(2)$

$2y-4x=-6 \quad$ Eq(1)$\times 2$

$2y-x=0 \quad$ Eq (2)

Subtracting Eq (2) from Eq (1)

$-3x=-6$

$x=2$

Substituting $x=2$ in Eq (1)

$y-2(2)=-3$

$y=-3+4$

$y=1$

$\therefore$ The point of intersection is $(2, 1)$.