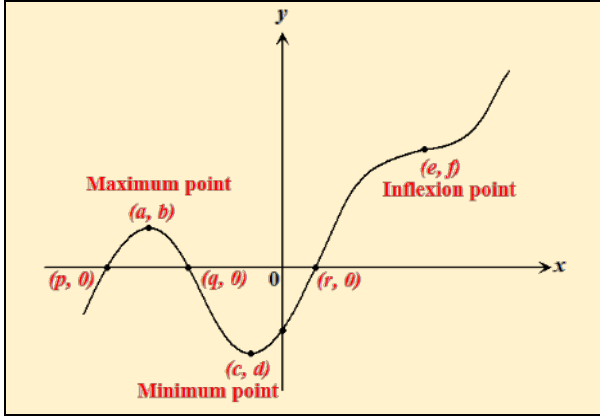


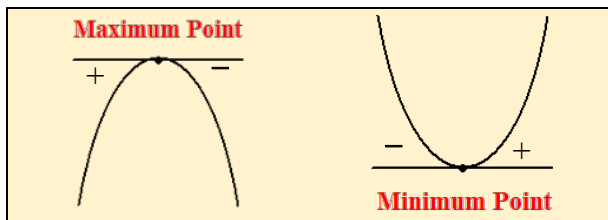
## 15: APPLICATIONS OF DIFFERENTIATION

### Stationary Points

Stationary points are points on a graph where the gradient is zero. A stationary point can be any one of a maximum, minimum or a point of inflexion. These are illustrated below.



Let us examine more closely the maximum and minimum points on a curve. We notice that a tangent to the curve, drawn at a maximum point, is a horizontal line and hence its gradient is zero. This is also true for the tangent drawn at a minimum point and at a point of inflexion (at a higher level of mathematics, we would discover inflexion points where the tangent is not horizontal). Just before and just after a maximum or a minimum point, the gradient of a curve changes to the opposite sign. In the case of a point of inflexion, though, the gradient does not change sign.



If we know the equation of a curve, we can determine the stationary points. This is because at no other point on the graph is the gradient zero, except at a stationary point. The converse is also true. That is, if the gradient at any point is zero, then the point is a stationary point.

To obtain the coordinates of the stationary point, we need to find out where on the curve the gradient is

zero. Therefore, we first obtain the gradient function,  $\frac{dy}{dx}$ , then, equate it to zero. Next, we solve for  $x$ .

The values of  $x$  obtained will be the  $x$  coordinates of the stationary points.

We can substitute these values of  $x$  into the equation of the curve to determine the corresponding  $y$  coordinates of the stationary points.

### Example 1

Find the stationary points on the graph of  $y = 4x^3 - 3x^2 - 6x$ .

### Solution

$$y = 4x^3 - 3x^2 - 6x$$

$$\frac{dy}{dx} = 12x^2 - 6x - 6$$

$$\text{Letting } \frac{dy}{dx} = 0$$

$$12x^2 - 6x - 6 = 0$$

$$2x^2 - x - 1 = 0$$

$$(2x + 1)(x - 1) = 0$$

$$x = -\frac{1}{2}, x = 1$$

$$\text{At } x = -\frac{1}{2}, y = 4\left(-\frac{1}{2}\right)^3 - 3\left(-\frac{1}{2}\right)^2 - 6\left(-\frac{1}{2}\right) = 1\frac{3}{4}$$

$$\text{At } x = 1, y = 4(1)^3 - 3(1)^2 - 6(1) = -5$$

Hence, there are two stationary points on the curve with coordinates,  $(-\frac{1}{2}, 1\frac{3}{4})$  and  $(1, -5)$ .

### The nature of the stationary points

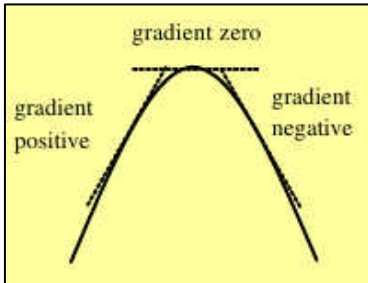
To determine whether a point is a maximum or a minimum point or inflexion point, we must examine what happens to the gradient of the curve in the vicinity of these points. In order to do so, we need not sketch the curve. We can use differential calculus to investigate what happens. There are two methods for carrying out this step and both methods will be described.

#### Method 1: Use of the first derivative

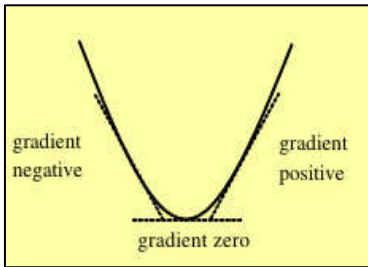
We can use the first derivative or the gradient function to determine how the gradient changes in the vicinity of the stationary point. At any stationary

point, the gradient is zero. By drawing tangents 'just before' and 'just after' the stationary point we observe that the changes in gradient differ in sign depending on the type of stationary point.

**Figure A: Maximum Point**



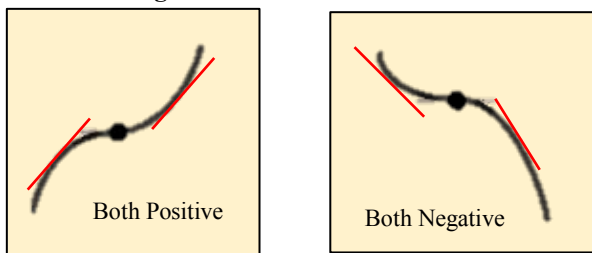
**Figure B; Minimum Point**



A change from positive to zero to negative indicates that the point is a maximum, this is illustrated in Figure A. On the other hand, a change from negative to zero to positive indicates that the point is a minimum as illustrated by the Figure B.

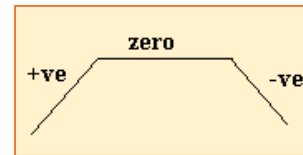
When there is no change in the sign, it indicates an inflexion point. As shown in Figure C, these points, the signs of the gradient can either be both positive or both negative, as shown below. This is because they change from positive, to zero at the point, and then to positive again OR from negative, to zero at the point, and then to negative again.

**Figure C: Points of inflexion**

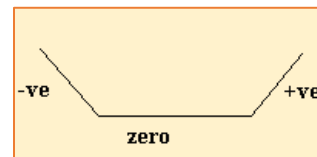


In using the first differential to determine the nature of a stationary point, we use the procedure outlined below:

1. Choose a value of  $x$ , 'just smaller' and one that is 'just larger' than the  $x$  value of the stationary point. For example, if there is a stationary point at  $x = 2$ , we may choose  $x = 1.9$  and  $x = 2.1$ .
2. Calculate  $\frac{dy}{dx}$  for each of these two values of  $x$ .
3. By observing the sign change of  $\frac{dy}{dx}$ , we can deduce the nature of the stationary point.
4. If the gradient changes from positive to negative before and after the stationary point, the point is a maximum.



5. If the gradient changes from negative to positive before and after the stationary point, the point is a minimum.



6. If there is no change in the sign of the gradient, the stationary point is a point of inflexion.

**Example 2**

A curve whose equation is  $y = 4x^3 - 3x^2 - 6x$  has stationary points at  $(-0.5, 1.75)$  and  $(1, -5)$ . Determine the nature of these points.

**Solution**

We apply the procedure outlined above, noting that For the stationary point,  $(-0.5, 1.75)$ , the  $x$ -coordinate is  $-0.5$ .

1. We can choose  $x = -0.6$  as the value 'just

smaller' than  $-0.5$ , and we can choose  $x = -0.4$  as the value 'just larger' than  $-0.5$ .

2. Calculating  $\frac{dy}{dx}$  for each of these two values, we have

$$\left(\frac{dy}{dx}\right)_{x=-0.6} = 12(-0.6)^2 - 6(-0.6) - 6 = 1.92 > 0$$

$$\left(\frac{dy}{dx}\right)_{x=-0.4} = 12(-0.4)^2 - 6(-0.4) - 6 = -1.68 < 0$$

3. The gradient changes from positive to negative. Hence,  $(-0.5, 1.75)$  is a maximum point.

Repeating these steps for the point  $(1, -5)$ , we can choose  $x = 0.9$  as the value 'just smaller' than  $1$ , and choose  $x = 1.1$  as the value 'just larger' than  $1$ .

$$\left(\frac{dy}{dx}\right)_{x=0.9} = 12(0.9)^2 - 6(0.9) - 6 = -1.68 < 0$$

$$\left(\frac{dy}{dx}\right)_{x=1.1} = 12(1.1)^2 - 6(1.1) - 6 = 1.92 > 0$$

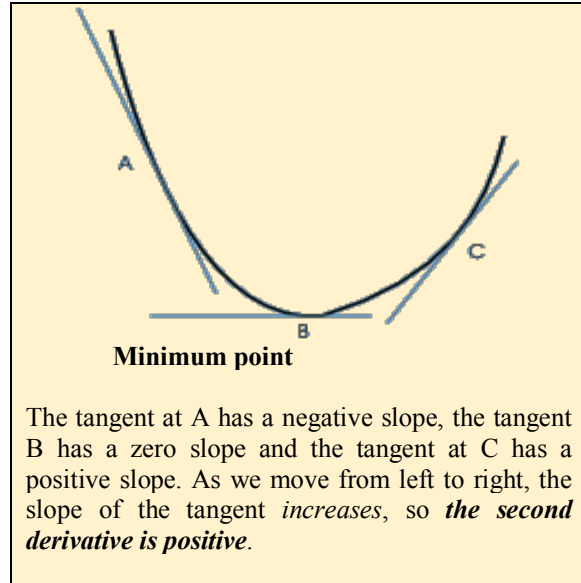
The gradient changes from negative to positive. Hence,  $(1, -5)$  is a minimum point.

*(Remember, if the gradient ever changes from negative, to zero, at the point, then to negative again OR from positive, to zero at the point, then to positive again, then the point would have been an inflexion point.)*

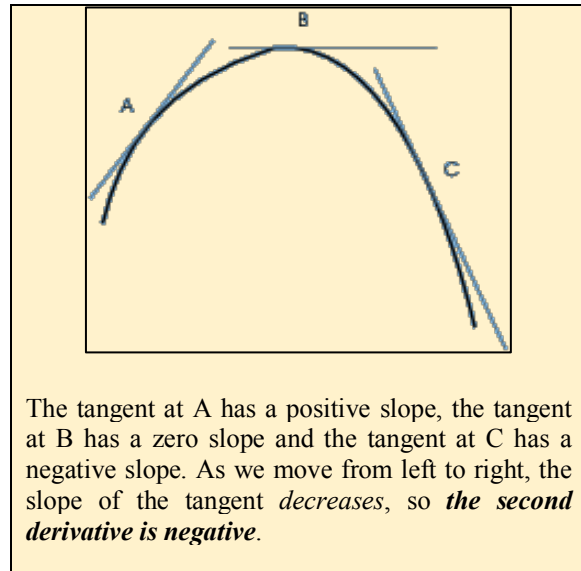
### Method 2: Use of the second derivative

To determine the nature of the stationary points, we can obtain the second derivative and examine its sign at the turning point. The second differential is also a function and obtained by differentiating the first derivative. It represents the derivative of the gradient function and it specifies the rate of change of the gradient function.

If the second derivative is positive then the gradient *increases* as you move along the curve with the gradient increasing from negative just before the point to zero at the point and to positive just after the point. This pattern indicates that the point is a minimum point.



If the second derivative is negative then the gradient decreases as we move along the curve with the gradient decreasing from positive just before the point to zero at the point and to negative just after the point. This pattern indicates that the point is a maximum point.



While this method is quite efficient, it can become quite a task to find the second derivative when the first derivative is an extremely complex function. In such a case we use Method 1 to determine the nature of the stationary points.

### Alternative Notation

So far, we have used  $\frac{dy}{dx}$  to refer to the first differential. We can also refer to the first differential as  $f'(x)$ .

The second differential is written as  $\frac{d^2y}{dx^2}$  or as  $f''(x)$ .

### Example 3

A curve whose equation is  $y = 4x^3 - 3x^2 - 6x$  has stationary points at  $(-0.5, 1.75)$  and  $(1, -5)$ . Determine the nature of these points.

### Solution

We solved this problem by using method 1 already. Now we will use the second differential to determine the nature of the stationary points.

$$y = 4x^3 - 3x^2 - 6x$$

$$\frac{dy}{dx} = 12x^2 - 6x - 6$$

$$\frac{d^2y}{dx^2} = 24x - 6$$

At  $(-0.5, 1.75)$ ,

$$\frac{d^2y}{dx^2} = 24(-\frac{1}{2}) - 6$$

$$= -18 < 0$$

$\Rightarrow$  Maximum

At  $(1, -5)$ ,

$$\frac{d^2y}{dx^2} = 24(1) - 6$$

$$= 18 > 0$$

$\Rightarrow$  Minimum

### Increasing and decreasing functions

A sketch of a curve gives us not just the coordinates of the stationary but we can also observe intervals for which the curve is increasing or decreasing. We say that a function is increasing in an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing in an interval if the function values decrease as the input values increase over that interval.

For an increasing function, the rate at which  $y$  changes with respect to  $x$  is positive, or

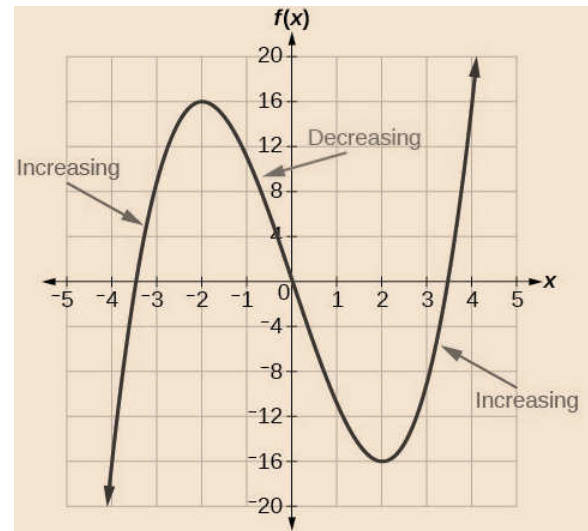
$$\frac{dy}{dx} > 0$$

For a decreasing function, the rate at which  $y$  changes with respect to  $x$  is negative.

$$\frac{dy}{dx} < 0$$

Figure 4 shows the function  $f(x) = x^3 - 12x$ . The intervals on which the function are increasing and decreasing intervals are indicated.

Figure 4: Increasing and decreasing functions



The function  $f(x) = x^3 - 12x$  is increasing in the interval  $(-\infty, -2) \cup (2, \infty)$  and is decreasing in the interval  $(-2, 2)$ .

In summary, for any function,  $f(x)$

If  $\frac{dy}{dx} > 0$  in an interval, then the function is increasing in the interval.

If  $\frac{dy}{dx} < 0$  in an interval, then the function is decreasing in the interval.

### Maxima and minima

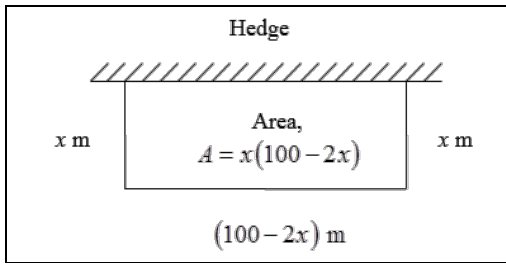
Differential calculus can be applied to many situations in real life where it is necessary to maximise or minimize functions.

### Example 5

A farmer wishes to fence off a rectangular sheep pen and uses all of 100 m of fencing. An adjoining hedge is to be used as one side of the pen. Determine the maximum area of the pen.

**Solution**

We create a diagram based on the data, for a clearer understanding of the information.



Let the width of the pen be  $x$  m  
 Since there are three sides to this fence, and the perimeter is 100 m

$$\begin{aligned} \therefore \text{The length of the pen will be:} \\ &= 100 - (x + x) \\ &= 100 - 2x \end{aligned}$$

The area of the pen,  $A = x(100 - 2x) = 100x - 2x^2$

$A$  is now a function of  $x$

For a stationary value of  $A$ , we must equate

$$\begin{aligned} \frac{dA}{dx} &= 0 \\ \frac{dA}{dx} &= 100 - 2(2x) = 100 - 4x \end{aligned}$$

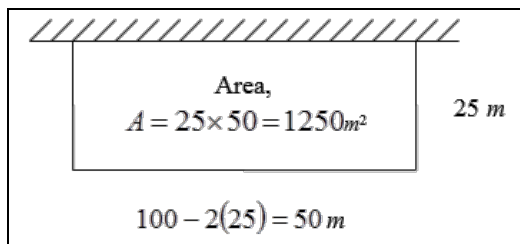
$$\text{Let } 100 - 4x = 0 \quad \therefore x = 25$$

In our example, we know that a stationary value of  $A$  occurs at  $x = 25$ .

We need to find the second derivative to determine or confirm its nature.

$$\frac{d^2A}{dx^2} = -4 \quad (\text{which is negative})$$

$\therefore A$  is a maximum value when  $x = 25$ .



The maximum area of the pen is:  
 $25 \times 50 = 1250 \text{ m}^2$ .

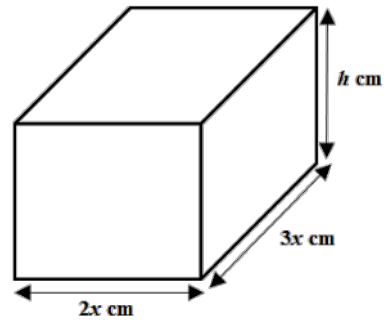
**Example 6**

A rectangular block has base dimensions  $2x$  cm by  $3x$  cm. Given that its volume is  $1800 \text{ cm}^3$ , prove that the total surface area,  $A \text{ cm}^2$ , is

$$A = 12x^2 + \frac{3000}{x}$$

Calculate the value of  $x$  for which  $A$  has a stationary value. Find this value of  $A$  and determine whether  $A$  is a maximum or a minimum.

**Solution**



Let the height of the block be  $h$  cm. Hence,

$$\begin{aligned} 2x \times 3x \times h &= 1800 \\ 6x^2h &= 1800 \\ h &= \frac{1800}{6x^2} = \frac{300}{x^2} \end{aligned}$$

The total surface area,  $A \text{ cm}^2 =$  Area of the front and back faces + Area of left and right faces + Area of the top and base.

$$\begin{aligned} A &= 2(2x \times h) + 2(3x \times h) + 2(2x \times 3x) \\ A &= 4hx + 6hx + 12x^2 \\ A &= 10hx + 12x^2 \end{aligned}$$

Substitute  $h = \frac{300}{x^2}$ , we get

$$A = 10\left(\frac{300}{x^2}\right)x + 12x^2 = \frac{3000}{x} + 12x^2$$

At a stationary value of  $A$ ,  $\frac{dA}{dx} = 0$

$$\begin{aligned} A &= 3000x^{-1} + 12x^2 \\ \frac{dA}{dx} &= 3000(-1x^{-2}) + 24x = -\frac{3000}{x^2} + 24x \end{aligned}$$

$$\frac{dA}{dx} = \frac{-3000}{x^2} + 24x = 0$$

$$\frac{3000}{x^2} = 24x$$

$$24x^3 = 3000$$

$$x^3 = 125$$

$$x = \sqrt[3]{125} = 5$$

When  $x = 5$ , the value of

$$A = 12(5)^2 + \frac{3000}{5} = 12(25) + 600 = 900 \text{ cm}^2$$

$$\frac{dA}{dx} = -3000x^{-2} + 24x$$

$$\frac{d^2A}{dx^2} = 6000x^{-3} + 24$$

$$= 24 + \frac{6000}{x^3}$$

When  $x = 5$ ,

$$\frac{d^2A}{dx^2} = 24 + \frac{6000}{(5)^3} > 0$$

$\therefore A$  is a minimum when  $x = 5$ .

### Small Changes

If two variables are related then we can use calculus to find the approximate change in one variable if there is a **small** change in the other. It is important to emphasise that the method used is valid only for **small changes** and also that the result is only **approximate**.

If two variables are related, then a change in  $x$  will produce a change in  $y$ . So too, a change in  $y$  will also produce a change in  $x$ .

A change could either be an increase (indicated by a positive sign) or a decrease (indicated by a negative sign).

### Notation

The change in any variable is defined as the difference between the final value and the initial value of the variable. We use the Greek symbol,  $\Delta$ , (*Capital Delta*) to denote a change in the value of a variable. If, say,  $x$  changes from an initial value of  $x_1$ , to a final value,  $x_2$ , then, we write

$$\Delta x = x_2 - x_1$$

We also use the Greek symbol  $\delta$ , (*common delta*), to mean a small change in the value of a variable. So, if  $x$  changes from a value of  $x_1$ , to a new slightly different value,  $x_2$ , then we say

$$\delta x = x_2 - x_1$$

### Small changes and gradient

We recall that the gradient of a curve at a point is the gradient of the tangent at the point or  $\left(\frac{dy}{dx}\right)_{x=a}$ . We

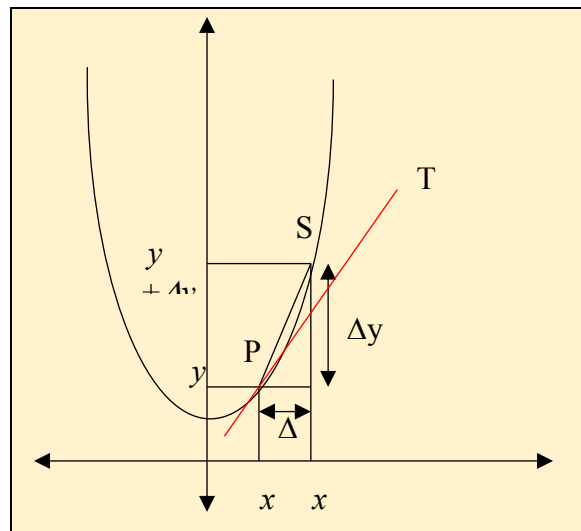
also recall that gradient is the change of  $y$  with respect to  $x$  or  $\frac{\Delta y}{\Delta x}$  or  $\frac{\delta y}{\delta x}$ .

We will now illustrate how we can use calculus to calculate the change in one variable when we know the small change in another related variable.

Consider the points  $P(x, y)$  and  $S(x + \Delta x, y + \Delta y)$  on the curve shown below.

The gradient of a chord,  $PS = \frac{\Delta y}{\Delta x}$ .

Note that the point  $S$  is some distance away from  $P$ . But since we are interested in very small changes, let us move  $S$  closer and closer to  $P$ , such that it almost touches  $P$ . When this happens, the gradient of the chord,  $PS$  is approximately equal to the gradient of the tangent  $PT$ .



Hence,  $\frac{\Delta x}{\Delta y} \approx \frac{dy}{dx}$  at the point  $P$ . This relationship is used to calculate  $\Delta y$  when given  $\Delta x$  or to calculate  $\Delta x$  when given  $\Delta y$ .

For infinitely small changes

$$\frac{\Delta y}{\Delta x} \approx \frac{dy}{dx} \quad \text{or} \quad \Delta y \approx \frac{dy}{dx} \Delta x$$

If the variables are different from  $x$  and from  $y$ , the principle still holds and we may adjust them accordingly.

If  $V$  is a function of  $r$ , then we can calculate the change in  $V$  ( $\Delta V$ ), given the change in  $r$  ( $\Delta r$ ) using

$$\Delta V \approx \frac{dV}{dr} \Delta r.$$

If  $A$  is a function of  $r$ , then we can calculate the change in  $A$  ( $\Delta A$ ), given the change in  $r$  ( $\Delta r$ )

$$\Delta A \approx \frac{dA}{dr} \Delta r.$$

### Example 7

The variables  $x$  and  $y$  are related by  $y = \frac{1}{x}$ . If  $x$  changes from 8 to 7.9, find (i) the approximate change in  $y$ , stating whether it is an increase or a decrease (ii) the new value of  $y$  (iii) the percentage change in  $y$ .

#### Solution

$$(i) y = \frac{1}{x}$$

$$\Delta x = 7.9 - 8 = -0.1$$

$$\frac{dy}{dx} = -x^{-2} = -\frac{1}{x^2}$$

$$\Delta y \approx \frac{dy}{dx} \Delta x$$

$$\Delta y \approx \frac{-1}{(8)^2} (-0.1) \approx 1.6 \times 10^{-3} \quad [\text{Substitute the original value of } x]$$

value of  $x$ ]

Since the value is positive, there is an increase.

(ii) The new value of  $y$  is the original value of  $y$  added to the corresponding change in  $y$ .

$$\text{When } x = 8, y = \frac{1}{8} = 0.125$$

$$\text{New value of } y \approx 0.125 + 1.6 \times 10^{-3} \approx 0.1266$$

(iii) The percentage change is

$$\approx \frac{1.6 \times 10^{-3}}{0.125} \times 100\% \approx 1.28\%$$

### Example 8

The radius,  $r$  cm, of a circle changes from 5 cm to 5.1 cm. Find the approximate change in the area  $A$  cm<sup>2</sup>, leaving the result in terms of  $\pi$ . State whether this change is an increase or decrease.

#### Solution

We recall the formula for the area of a circle,

$$A = \pi r^2$$

Since  $A$  is a function of  $r$  then,

$$\Delta A \approx \frac{dA}{dr} \times \Delta r, \quad \text{where } \Delta r = \text{a small change in } r$$

and  $\Delta A$  is the corresponding change in  $A$ ,

$$\Delta r = 5.1 - 5 = 0.1$$

$$\frac{dA}{dr} = 2\pi r$$

$$\therefore \Delta A \approx 2\pi r \times 0.1$$

$\therefore \Delta A \approx 2\pi(5) \times 0.1$  (Substitute  $r = 5$ , the original value)  $\Delta A \approx \pi \text{ cm}^2 > 0$ , this indicates there is an increase in  $A$ .

### Example 9

Given,  $y = 3x^2 - 4x + 5$ , Calculate

a.  $\frac{dy}{dx}$  when  $x = -2$

b. the approximate change in  $y$  when  $x$  increases from  $-2$  to  $-1.97$ .

#### Solution

a.  $y = 3x^2 - 4x + 5, \quad \frac{dy}{dx} = 6x - 4$

When  $x = -2$

$$\frac{dy}{dx} = 6(-2) - 4 = -12 - 4 = -16$$

b. If  $y$  is a function of  $x$  then,  $\Delta y \approx \frac{dy}{dx} \times \Delta x$  where

$\Delta x =$  small change in  $x$  and  $\Delta y =$  corresponding change in  $y$

$$\Delta y \approx \{6(-2) - 4\} \times 0.03$$

$$\Delta y \approx -0.48$$

Since  $\Delta y < 0$  this indicates a decrease.

## Differentiation as a Rate of Change

Rate of change refers to the rate at which any variable, say  $x$ , changes with respect to time,  $t$ . The gradient function,  $\frac{dx}{dt}$ , is therefore a rate of change.

The symbol,  $dx$ , refers to the change in the variable that is studied and the symbol  $dt$  refers to the change in time.

The rate of change of the volume,  $V$ , ... is  $\frac{dV}{dt}$ .

The rate of change of the area,  $A$ , ... is  $\frac{dA}{dt}$ .

### Positive and negative rates of change

A positive sign denotes an increasing rate. Sometimes the word increasing or the word decreasing may not be used. In these cases, we are expected to deduce whether the rate is increasing or decreasing by the words of the question.

For example,

A length,  $l$ , of elastic is stretched at the rate of  $0.5 \text{ cms}^{-1}$ ...

Stretched  $\Rightarrow$  increasing length  $\therefore \frac{dl}{dt} = +0.5 \text{ cms}^{-1}$

In this case, the unit of measure indicated that length,  $l$ , is the variable under study.

A fire is spreading at the rate of  $5 \text{ m}^2\text{s}^{-1}$ ...

Spreading  $\Rightarrow$  increasing rate }  $\therefore \frac{dA}{dt} = +5 \text{ m}^2\text{s}^{-1}$  (Let  
Units of  $5 \text{ m}^2\text{s}^{-1} \Rightarrow$  Area }  
 $A$  be area)

A negative sign denotes a decreasing rate.

Water is leaking from a tank at the rate of  $4 \text{ cm}^3\text{s}^{-1}$ ...

Leaking  $\Rightarrow$  decreasing rate  $\therefore \frac{dV}{dt} = -4 \text{ cm}^3\text{s}^{-1}$ .

The units indicated that volume,  $V$ , was the variable.

### Related rates of change

It is common to have a situation involving two changing quantities and we would likely be required to find the rate at which one is changing, given the rate at which the other is changing.

In such a situation, we need to have an equation that connects the two quantities. Then we use the 'chain

rule', to find the corresponding rate of change of one variable, given the change in the other variable.

### Example 10

The radius,  $r$  cm, of a circle is increasing at the rate of  $0.2 \text{ cms}^{-1}$ . Calculate the corresponding rate of change of the area,  $A \text{ cm}^2$ , at the instant when the radius is  $5 \text{ cm}$ . ( $A = \pi r^2$ )

#### Solution

We wish to find the rate of change of the area,  $\frac{dA}{dt}$ , given the rate of change of the radius,  $\frac{dr}{dt}$ .

$$A = \pi r^2 \quad \frac{dr}{dt} = +0.2 \text{ cms}^{-1} \text{ (+ve } \Rightarrow \text{ increase)}$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi(5) \times 0.2 = 2\pi \text{ cm}^2\text{s}^{-1} \text{ (+ve } \Rightarrow \text{ increase)}$$

It is sometimes possible to obtain the rate of change of an area,  $A \text{ cm}^2$ , to be say,  $-6 \text{ cm}^2\text{s}^{-1}$ . The negative sign indicates a decreasing rate. However, if we were asked for the rate of decrease of the area, the answer will be stated as  $6 \text{ cm}^2\text{s}^{-1}$ .

### Example 11

The side of a metal cube,  $x$  cm, increases from  $9.9 \text{ cm}$  at a constant rate of  $0.005 \text{ cms}^{-1}$  as it is heated. Find the rate at which the volume is increasing when  $t = 20$  seconds.

#### Solution

The volume of the cube increases as it is heated. Side increases at the rate of  $0.005 \text{ cms}^{-1}$ .

Required to calculate:  $\frac{dV}{dt}$  at  $t = 20$ .

Let volume of the cube be  $V \text{ cm}^3$ .

$$\therefore V = x^3 \text{ and by differentiation, } \frac{dV}{dx} = 3x^2$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} \quad \frac{dV}{dx} = 3x^2 \times 0.005$$

$$\text{When } t = 20 \quad x = 9.9 + 20(0.005) = 10$$

because after 20 seconds the value of  $x$  is the original value plus the increase after the duration of the 20 s.

$$\text{Hence, } \frac{dV}{dt} = 3(10)^2 \times 0.005 = 1.5 \text{ cm}^3\text{s}^{-1}$$



**Example 12**

A spherical balloon is filled with air at the constant rate of  $200 \text{ cm}^3\text{s}^{-1}$ . Calculate the rate at which the radius is increasing, when the radius = 10 cm.

**Solution**

Let  $V$  = Volume and  $r$  = radius of the balloon.

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = +200 \text{ cm}^3\text{s}^{-1} \text{ (increasing rate)}$$

$$\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$$

$$\frac{dV}{dr} = \frac{4}{3}\pi(3r^2) = 4\pi r^2$$

$$\frac{dr}{dV} = \frac{1}{\frac{dV}{dr}} = \frac{1}{4\pi r^2}$$

When  $r = 10 \text{ cm}$ ,

$$\frac{dr}{dt} = 200 \times \frac{1}{4\pi(10)^2} = \frac{1}{2\pi} \text{ cms}^{-1}$$