14: DIFFERENTIATION

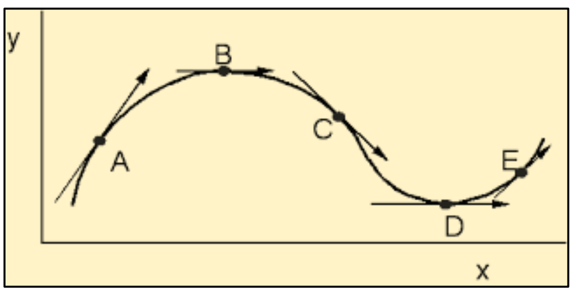
The gradient of a curve

A curve has a variable gradient, unlike that of a straight line which is fixed. So, the gradient of a curve is defined at any given point on the curve.

> The gradient of a curve at a point is defined as the gradient of the tangent to the curve at that point.

To determine the gradient of the curve at a point, we may draw a straight line which 'just touches' the curve at that point. This is the tangent to the curve at that point. Then, we calculate the gradient of the tangent using any appropriate method.

In the curve shown below, clearly, the gradient at A is different from the gradient at B or C or in fact at any other point.



The method of finding the gradient of a curve by drawing the tangent and estimating the gradient of the tangent is rather tedious. Furthermore, the result depends on the degree of accuracy in reading the scales on the graph, that is, the result is subjected to human error and errors due to the accuracy of the instrument in measuring.

In this chapter, we introduce a technique which provides a shorter, simpler and an exact method for finding the gradient of a curve at any point. This technique is called differentiation and once we know the equation of the curve, we can apply the technique.

The gradient function

If y is a function of x, then the derivative of y with respect to x, written as $\frac{dy}{dx}$, is a new function that is called the gradient function.

To obtain the gradient function from a polynomial function of the form $y = ax^n$, we simply multiply the exponent (power) by the coefficient of x, then subtract one from the exponent.

Differential of polynomials of the form $y = ax^n$

We use the following rules to differentiate polynomials.

$y = ax^n$	$y = x^n$
$\frac{dy}{dx} = nax^{n-1}$	$\frac{dy}{dx} = nx^{n-1}$

As the examples below show, if the polynomial is not in the form $y = x^n$, we may need to perform some algebraic manipulations so that the law can be applied. The rules of indices are often helpful here.

Example 1

Differentiate each of the following functions.

(i) $y = x^4 \qquad \frac{dy}{dx} = 4x^{4-1} = 4x^3$

(ii) $y = 3x^2 \qquad \frac{dy}{dx} = 3\left(x^{2-1}\right) = 6x$

(iii) $y = \frac{1}{4}x^6 \qquad \frac{dy}{dx} = \frac{1}{4}\left(6x^{6-1}\right) = \frac{3}{2}x^5$

(iv) $y = \frac{1}{x^3} = x^{-3} \qquad \frac{dy}{dx} = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$

(v) $y = \frac{4}{x^2} = 4x^{-2} \qquad \frac{dy}{dx} = 4(-2x^{-2-1}) = -8x^{-3}$

$$= -\frac{8}{x^3}$$

(vi) $y = \sqrt{x} = x^{\frac{1}{2}} \qquad \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2\sqrt{x}}$

(vii) $y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}} \qquad \frac{dy}{dx} = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

$$= \frac{1}{2}\sqrt{x^3}$$

(viii) $y = \frac{1}{6x^3} = \frac{1}{6}x^{-3} \qquad \frac{dy}{dx} = \frac{1}{6}(-3x^{-3-1}) = -\frac{3}{6}x^{-4}$

$$= -\frac{1}{2x^4}$$

Differential of $y = k$ and $y = kx$

If $y = k$ then $\frac{dy}{dx} = 0$. If $y = kx$, then $\frac{dy}{dx} = k$

We can verify these results by recalling that the gradient of any horizontal line, $y = k$ is zero, and the gradient of the line $y = kx$ is k.

We may also derive these results by applying the rule for differentiation.

$y = 4 \equiv 4x^0$, $\frac{dy}{dx} = 4\left(0x^{0-1}\right) = 0$, and

$y = 4x$, $\frac{dy}{dx} = 4\left(1x^{1-1}\right) = 4$

Example 2

Find the gradient of the curve $y = x^3$ at the point $x = -\frac{1}{4}$.

Solution

$y = x^3$

The gradient function, $\frac{dy}{dx} = 3x^{3-1} = 3x^2$

$\therefore$ The gradient of the curve at $x = -\frac{1}{4}$

$= 3\left(-\frac{1}{4}\right)^2 = \frac{3}{16}$

Differential of a sum or difference

When there is more than one term in the expression, each term is differentiated separately.

Example 3

Find $\frac{dy}{dx}$ for $y = 5x^2 - 2x^3$

Solution

$y = 5x^2 - 2x^3$

$\frac{dy}{dx} = 5\left(2x^{2-1}\right) - 2\left(3x^{3-1}\right) = 10x - 6x^2$

Example 4

$y = 6x^3 - \frac{4}{x^2}$, obtain an expression for $\frac{dy}{dx}$

Solution

First, ensure that each term is expressed in a form that is conformable to differentiation.

$y = 6x^3 - 4x^{-2}$

$\frac{dy}{dx} = 6\left(3x^{3-1}\right) - 4\left(-2x^{-2-1}\right) = 18x^2 + \frac{8}{x^3}$

Example 5

Obtain an expression for $\frac{dy}{dx}$ if $y = x^2(3 - x)$.

Solution

We multiply before differentiating

$y = x^2(3 - x) = 3x^2 - x^3$

$\frac{dy}{dx} = 3\left(2x^{2-1}\right) - 3x^{3-1} = 6x - 3x^2$

Example 6

Obtain an expression for $\frac{dy}{dx}$ if $y = \frac{x^4 - 3x + 2}{x^2}$.

Solution

We divide before differentiating

$y = \frac{x^4 - 3x + 2}{x^2} = x^2 - \frac{3}{x} + \frac{2}{x^2} = x^2 - 3x^{-1} + 2x^{-2}$

$\frac{dy}{dx} = 2x - 3\left(-1x^{-1-1}\right) + 2\left(-2x^{-2-1}\right) = 2x + \frac{3}{x^2} - \frac{4}{x^3}$

Example 7

Obtain an expression for $\frac{dy}{dx}$ if $y = (3x - 1)^2$.

Solution

We may choose to expand before differentiating

$y = (3x-1)^2 = (3x-1)(3x-1) = 9x^2 - 6x + 1$

$\frac{dy}{dx} = 9\left(2x^{2-1}\right) - 6 = 18x - 6$

The Chain Rule

Sometimes the technique of expansion is not practical and may even fail to simplify an expression. For example, examine the following expressions:

$y = (2x-1)^{46}$ $y = \sqrt[3]{x+4} = (x+4)^{\frac{1}{3}}$ and

$y = \frac{1}{\sqrt{3x-1}} = (3x-1)^{-\frac{1}{2}}$

If these expressions are expanded the numbers of terms would be large. More so, those with negative and/or fractional indices would have an infinite number of terms. Therefore, we need to find a more efficient way of differentiating this type of expression. The chain rule is used to differentiate such composite functions

It is sometimes called 'function of a function' and the method is often referred to as the 'method of substitution'.

If we have a composite function such that

$y = f(t)$ and $t = g(x)$

then by the Chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Example 8

Differentiate $y = (2x-1)^{10}$.

Solution

Applying the chain rule, we have:

$y = (2x-1)^{10}$

Let $t = 2x-1$, and $y = t^{10}$

$\frac{dt}{dx} = 2 \qquad \frac{dy}{dt} = 10t^9$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 10t^9 \times 2 = 20t^9 = 20(2x-1)^9$

Example 9

Differentiate $y = \frac{1}{\sqrt{3x-2}}$.

Solution

$y = \frac{1}{\sqrt{3x-2}} = (3x-2)^{-\frac{1}{2}}$

Let $t = 3x-2$ and $y = t^{-\frac{1}{2}}$

$\frac{dt}{dx} = 3 \qquad \frac{dy}{dt} = -\frac{1}{2}t^{-\frac{1}{2}-1}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -\frac{1}{2}t^{-\frac{1}{2}-1} \times 3$

$= \frac{-3}{2\sqrt{t^3}} = \frac{-3}{2\sqrt{(3x-2)^3}}$

The Product Rule

The product rule is useful when differentiating a product of two functions and in which a simple multiplication such is not possible. The rule states:

If y is of the form, $y = uv$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$.

It is necessary in cases where the functions u and v cannot be combined with simple multiplication.

The Quotient Rule

We use the quotient rule when there is a quotient that cannot be simplified using a simple division. The rule states:

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Example 10

Differentiate $y = (2x+1)\sqrt{3x+2}$.

Solution

Let $u = 2x+1$ and $v = \sqrt{3x+2}$

$\frac{du}{dx} = 2$

To determine $\frac{dv}{dx}$, we use the chain rule

$v = \sqrt{3x+2}$

Let $t = 3x+2$, so $v = t^{\frac{1}{2}}$

$\frac{dv}{dx} = \frac{1}{2}(3x+2)^{-\frac{1}{2}}(3) = \frac{3}{2\sqrt{3x+2}}$

$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} = \sqrt{3x+2} \times 2 + (2x+1) \times \frac{3}{2\sqrt{3x+2}}$

$= 2\sqrt{3x+2} + \frac{6x+3}{2\sqrt{3x+2}} = \frac{18x+11}{2\sqrt{3x+2}}$

Example 11

Differentiate $y = \frac{4x+1}{2x-4}$.

Solution

Let $u = 4x+1$ and $v = 2x-4$.

$$\frac{du}{dx} = 4 \qquad \frac{dv}{dx} = 2$$

Applying the quotient law,

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$

$$\frac{dy}{dx} = \frac{(2x-4)4-(4x+1)2}{(2x-4)^2} = -\frac{18}{(2x-4)^2}$$

Example 12

The normal to the curve $y = 2x^3 - 5x^2$ at the point $(2, -4)$ crosses the x-axis at A.
Find the coordinates of A.

Solution

$y = 2x^3 - 5x^2$

The gradient function of the curve,

$$\frac{dy}{dx} = 2\left(3x^{3-1}\right) - 5\left(2x^{2-1}\right) = 6x^2 - 10x$$

The gradient of the tangent at $(2, -4)$ is

$$= 6(2)^2 - 10(2) = 4$$

Hence, the gradient of the normal is $-\frac{1}{4}$ (the product of the gradients of perpendicular lines $= -1$).

To find the equation of the normal, we use

$$\frac{y - y_1}{x - x_1} = m$$

$$\frac{y-(-4)}{x-2} = -\frac{1}{4}$$

$$4y + 16 = -x + 2$$

$$4y = -x - 14$$

When $y = 0$, $x = -14$.

Therefore, $A(-14, 0)$.

Differentiation as a limit

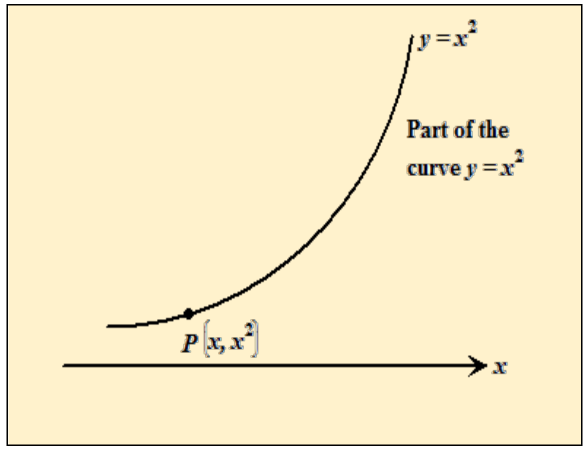
Earlier, we used differentiation to find the gradient of a curve at a point. We did so by obtaining the gradient function, $\frac{dy}{dx}$ which is itself a function that gives us the gradient at any point on the curve. For polynomial functions of the form $y = ax^n$, we simply used the procedure, $\frac{dy}{dx} = nax^{n-1}$.

We will now discover how these procedures were derived through another method.

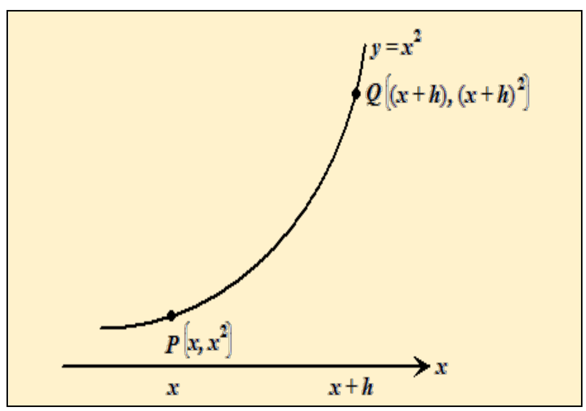
Using the limit of a chord to find the gradient function

Consider the function, $y = x^2$. To obtain the differential from first principles of $y = x^2$, let us first look at its graph, or more precisely, a section of its graph.

For the curve, $y = x^2$ let P be some arbitrary point on the curve with coordinates (x, y). So, the coordinates of P in terms of x will be (x, x^2).



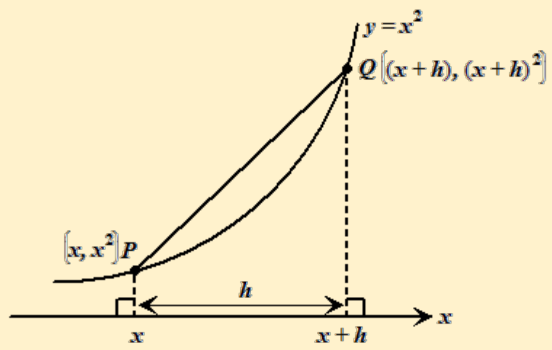
Some distance away, say h, to the right of P, we choose another point, Q, on the curve, with the x coordinate of $(x + h)$. The coordinates of Q will be $[(x+h), (x+h)^2]$. We join P to Q as shown in the diagram below.



Now, we now find the gradient of the chord PQ of the curve. Recall the formula for the gradient of a straight line joining the points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}, \text{ where}$$

$(x_1, y_1) = (x, x^2)$ and $(x_2, y_2) = [(x+h), (x+h)^2]$



$$\text{Gradient of } PQ = \frac{(x+h)^2 - x^2}{(x+h) - x}$$

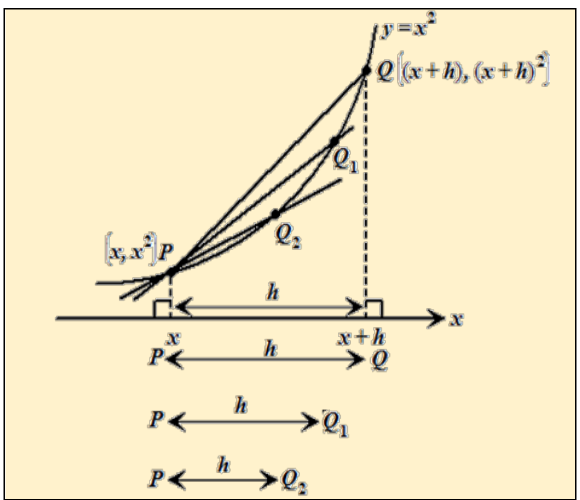
$$= \frac{x^2 + 2hx + h^2 - x^2}{h}$$

$$= \frac{2hx + h^2}{h}$$

$$= 2x + h$$

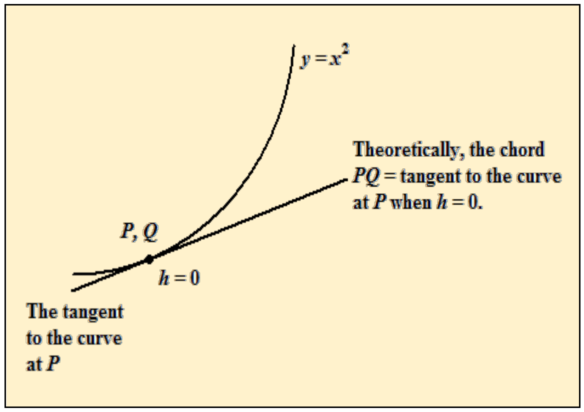
Imagine the chord PQ, hinged at P and being rotated clockwise as shown below. At each new position of Q, along the curve, we will find the chord getting shorter and shorter.

For example, PQ_1 is shorter than PQ, PQ_2 is shorter than PQ_1, PQ_3 is shorter than PQ_2 and so on. In each case, the length of h decreases and Q approaches P.



As the chord, PQ approaches tangent to the curve at P, the gradient of $PQ = 2x + h$ (derived above).

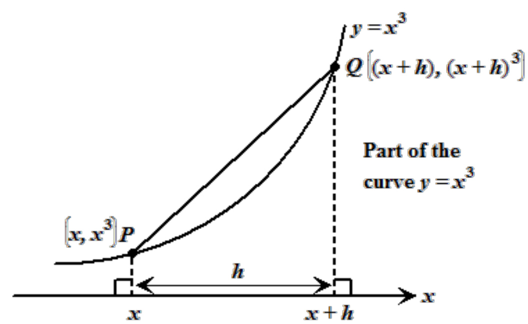
When $h \to 0$, the chord $PQ \to$ the tangent to the curve at P. The gradient of PQ approaches $2x + 0 = 2x$.



The gradient function of the curve, $y = x^2$ is $2x$.

Hence, $\frac{d}{dx}\left(x^2\right) = 2x$.

In a similar fashion, we can repeat the procedure for other simple polynomials such as $y = x^3$ by drawing the curve. We start with a chord PQ and move the point Q until it approaches P. The gradient of the tangent PQ is the gradient of the curve at the point P.



Let P be (x, y) and let the x – coordinate of Q be $x + h$, Hence,

$P = \left(x, x^3\right)$

$Q\left((x+h), (x+h)^3\right)$

The gradient of PQ

$$= \frac{(x+h)^3 - x^3}{(x+h) - x} = \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$$

$$= \frac{3hx^2 + 3h^2x + h^3}{h} = 3x^2 + 3hx + h^2$$

As $h \to 0$, the chord $PQ \to$ the tangent to the curve at P. Let $h = 0$, the gradient of PQ

$= 3x^2 + 3(0)x + (0)^2 = 3x^2$

$\therefore 3x^2$ is the gradient function of the curve $y = x^3$.

Hence, $\frac{d}{dx}\left(x^3\right)=3x^2$ **OR** when $y=x^3$, $\frac{dy}{dx}=3x^2$.

These results enable us to derive a general rule for differentiation.

$\frac{d}{dx}\left(x^2\right)=2x$ $\frac{d}{dx}\left(x^3\right)=3x^2$ $\frac{d}{dx}\left(x^4\right)=4x^3$	In general, $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$.

Differentiation of Trigonometric Functions

So far, we have found the differential of functions such as polynomials and we have used a general rule to obtain their derivatives. To differentiate trigonometric functions we cannot use the same procedure. The process is more complex and involves the study of more advanced calculus.

Only the sine and the cosine functions are to be considered at this level. Their derivatives are:

$\frac{d}{dx}(\sin x)=\cos x$	$\frac{d}{dx}(\cos x)=-\sin x$

We can use these results to determine the derivative of functions such as $\sin ax$ and $\cos ax$.

Example 13

Find $\frac{d}{dx}(\sin 3x)$.

Solution

Let $y=\sin 3x$ and $t=3x$

$\therefore y=\sin t$ and $\frac{dt}{dx}=3$

$\frac{dy}{dt}=\cos t=\cos 3x$

By the Chain Rule:

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\frac{dy}{dx}=cos\,3x\times 3$$

$$\frac{d}{dx}(sin\,3x)=3cos\,3x$$

Example 14

Find $\frac{d}{dx}(\cos 2x)$.

Solution

Let $y=\cos 2x$ and $t=2x$

$\therefore y=\cos t$ and $\frac{dt}{dx}=2$

$\frac{dy}{dt}=-\sin t=-\sin 2x$

t

By the Chain Rule:

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$$

$$\frac{dy}{dx}=-\sin 2x\times 2$$

$$\frac{d}{dx}(cos\,2x)=-2sin\,2x$$

We can now state the following**:**

$$\frac{d}{dx}sin\,ax=a\,cos\,ax$$

$$\frac{d}{dx}cos\,ax=-a\,sin\,ax$$

Example 15

Find the gradient of $y=\sin x$ at $x=\frac{\pi}{4}$.

Solution

$y=\sin x$, $\frac{dy}{dx}=\cos x$

$\therefore$ The gradient of the curve at $x=\frac{\pi}{4}$ is $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}$

$\therefore$ The gradient $=\frac{1}{\sqrt{2}}$