5: ROOTS OF A QUADRATIC EQUATION

The general form of a quadratic equation

We have grown accustomed to recognising a quadratic equation in the form $ax^2 + bx + c = 0$. In this section, we will be introduced to a new format for such a quadratic equation. This format would express the quadratic in the form of its roots. It is a convenient form to know and it allows us the flexibility to switch from this form to the standard form.

Roots of a quadratic equation ($\propto$ *and* β)

A quadratic equation in x is of the general form $ax^2 + bx + c = 0$, where a, b and c are constants.

If we divide each term by a, then the quadratic equation can be expressed in an equivalent form with the coefficient of x^2 is equal to one as shown below.

$$ax^2 + bx + c = 0$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (1)$$

Now consider $\propto$ and β as the roots of the quadratic.
We can now rewrite the quadratic in the form:

$$(x-\alpha)(x-\beta) = 0.$$

By expanding we get,

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0. \qquad (2)$$

Equation (2) is an equivalent form of equation (1). In fact, any quadratic equation, in x, can always be expressed in the form of its roots.
We can replace ($\alpha + \beta$) by the 'sum of the roots' and $\alpha\beta$ by the 'product of the roots', to obtain the following form for a quadratic equation.

$$x^2 - (\text{sum of roots})x + \text{product of roots} = 0$$

Sum and product of the roots of a quadratic equation

Equations (1) and (2) above are two equivalent forms of a quadratic equation.
Equating both forms we get:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (\alpha+\beta)x + \alpha\beta$$

When we equate coefficients, the following is obtained:

$$\alpha + \beta = -\frac{b}{a} \qquad \text{and} \qquad \alpha\beta = \frac{c}{a}.$$

We can now make a general statement about the roots of a quadratic.

> For the quadratic equation
> $$ax^2 + bx + c = 0,$$
> the sum of the roots $= -\frac{b}{a}$ and
>
> the product of the roots $= \frac{c}{a}$.

Example 1

If α and β are the roots of the quadratic equation $x^2 - 3x + 2 = 0$, determine
(i) the sum of the roots and
(ii) the product of the roots.

Solution

In the quadratic equation $x^2 - 3x + 2 = 0$
$a = 1$, $b = -3$ and $c = 2$.

(i) The sum of the roots, $\alpha + \beta = -\frac{b}{a}$

$$\alpha + \beta = \frac{-(-3)}{1} = 3$$

(ii) The product of the roots, $\alpha\beta = \frac{c}{a}$

$$\alpha\beta = \frac{2}{1} = 2.$$

Example 2

The quadratic equation $x^2 - 4x + 3 = 0$ has roots α and β.

a) Obtain the equation whose roots are $\alpha + 1$ and $\beta + 1$.

b) Obtain the equation whose roots are α^2 and β^2.

Solution

If the equation $x^2 - 4x + 3 = 0$ has roots α and β, then $a = 1, b = -4$ and $c = 3$. Hence,
$(\alpha + \beta) = 4$ and $\alpha\beta = 3$
To obtain an equation whose roots are $\alpha + 1$ and

$\beta+1$, we can substitute these roots in the following equation:

$$x^2-(\text{sum of roots})x+\text{product of roots}=0$$

$$x^2-[(\alpha+1)+(\beta+1)]x+[(\alpha+1)(\beta+1)]=0$$

$$x^2-[(\alpha+\beta)+2)]x+[\alpha\beta+(\alpha+\beta)+1]=0$$

$$x^2-(4+2)x+(3+4+1)=0$$

$$x^2-6x+8=0$$

This is the required equation.

Part b) To obtain an equation whose roots are α^2 and β^2, we substitute these roots in:

$$x^2-(\text{sum of roots})x+\text{product of roots}=0$$

$$x^2-(\alpha^2+\beta^2)x+(\alpha^2\times\beta^2)=0$$

$$x^2-(\alpha^2+\beta^2)x+(\alpha^2\beta^2)=0$$

[Recall: $(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$]

$$x^2-\left((\alpha+\beta)^2-2\alpha\beta\right)x+(\alpha\beta)^2=0$$

$$x^2-\left\{(4)^2-2(3)\right\}x+(3)^2=0$$

$$x^2-10x+9=0$$

This is the required equation.

Example 3

Given that $x^2+(k-5)x-k=0$ has real roots which differ by 4, determine i. the value of each root ii. the value of k.

Solution

If we let α be the smaller real root, then the other will be $(\alpha+4)$.

Then the sum of the roots is : $\alpha+(\alpha+4)=2\alpha+4$

The product of the roots is $\alpha(\alpha+4)$.

From the given equation $x^2+(k-5)x-k=0$,

The sum of the roots is: $-(k-5)$

The product of the roots is: $-k$

Equating coefficients, we have:

Sum of roots	Product of roots
$2\alpha+4=-(k-5)$ $2\alpha+4=-k+5$ $k=1-2\alpha$ (1)	$\alpha(\alpha+4)=-k$ $k=-\alpha(\alpha+4)$ (2)

Equating equations (1) and (2) to eliminate k, we have:

$$-\alpha^2-4\alpha=1-2\alpha$$

$$\alpha^2+2\alpha+1=0$$

$$(\alpha+1)(\alpha+1)=0$$

$$\alpha=-1$$

The value of k: $k=1-2\alpha=1-2(-1)=3$

$\therefore$ Roots are -1 and $-1+4$

The roots are -1 and 3.

Alternative Method

If we let α be the smaller real root, then the other will be $(\alpha+4)$.

Hence the quadratic equation may be expressed as

$$(x-a)(x-(\alpha+4))=0$$

$$(x-a)(x-\alpha-4)=0$$

$$x^2-\alpha x-ax+\alpha^2-4x+4\alpha=0$$

$$x^2+(-2\alpha-4)x+(\alpha^2+4\alpha)=0$$

Equating coefficient of x, we obtain

$$-2\alpha-4=k-5$$

$$-2\alpha-4=k-5$$

$$k=1-2\alpha$$

$$\therefore\alpha=\frac{1-k}{2}$$

Equating constant terms, we obtain

$$\alpha^2+4\alpha=-k$$

$$\therefore\left(\frac{1-k}{2}\right)^2+4\left(\frac{1-k}{2}\right)=-k$$

$$\frac{1-2k+k^2}{4}+2-2k+k=0$$

$$1-2k+k^2+8-8k+4k=0$$

$$k^2-6k+9=0$$

$$(k-3)^2=0$$

$$k=3$$

When, $k=3$, $\alpha=\frac{1-3}{2}=-1$

$\therefore$ Roots are -1 and $-1+4$

The roots are -1 and 3.