14. RELATIONS AND FUNCTIONS

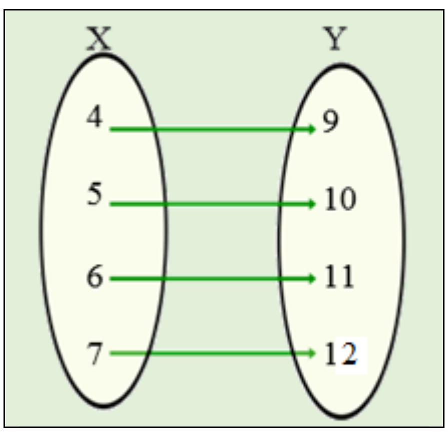
RELATIONS

In mathematics, we study relations between two sets of numbers, where members of one set are related to the other set by a rule. Relations are also described as mappings. When we map a set of numbers onto another set of numbers, we often express the rule for the mapping using mathematical relationships instead of words.

Representing relations – Arrow diagrams and Ordered pairs

An Arrow Diagram is often used to represent a relation. The members of each set are listed inside an enclosed shape and arrows are drawn to connect related members. For a relation to exist between the sets, there must be a rule connecting pairs of elements and this rule must hold for all mappings from set X to set Y.

In the example, shown below, we define a relation between the set {4, 5, 6, 7} and the set {9, 10, 11, 12} as 'add 5'. Notice that 4 is mapped onto 9, 5 onto 10, 6 onto 11 and 7 onto 12. This is shown by drawing arrows to connect members of the set X to the members Y.



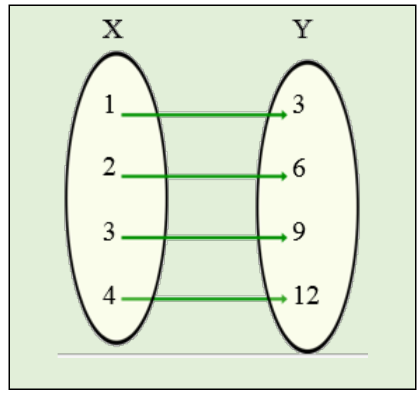
We refer to the members of the set X as the input and members of the set Y as the output. The direction of the arrows is always from the input to the output.

The mapping can also be represented as the set of ordered pairs:

(4, 9) (5, 10) (6, 11) (7, 12).

The ordered pair preserves the directional property of the relation. It is consistent with the order of points plotted on a Cartesian Plane represented by (x, y).

In the Arrow Diagram that follows, we define a relation between the set {1, 2, 3, 4} and the set {3, 6, 9, 12} as 'multiply by 3'. Notice that 1 is mapped onto 3, 2 onto 6, 3 onto 9 and 4 onto 12. The arrows are drawn from the set X to the set Y.

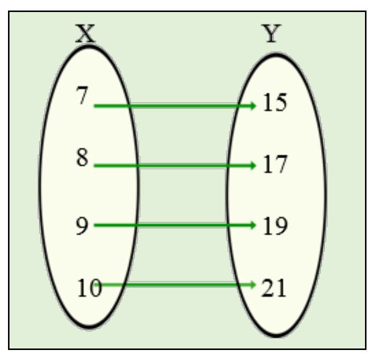


The mapping can also be represented as the set of ordered pairs:

(1, 3) (2, 6) (3, 9) (4, 12).

In the mapping below, the members are related to the rule. '× 2 and then +1'. We may say that x is mapped onto $2x + 1$. This can be represented as the set of ordered pairs.

(7, 15) (8, 17) (9, 19) (10, 21)



A relation exists between two sets of numbers if we can find a rule that maps members of the first set (***domain***) onto members of the second set (***codomain***). The rule must hold for all possible pairs that are connected. So that when we select a value of x, also called an ***input***, and apply the rule of the relation, we obtain the y value, also called the ***output***. We can also refer to the y value as the ***image*** of the x value.

Defining a relation

Based on our discussion so far, there are three conditions that must be present when a relation exist, these are:

- Two sets are involved.
- There must be a clear rule describing the relationship.
- There is a directional property, that is, the relation is defined from one set called the **domain** on to another set called the **codomain**.

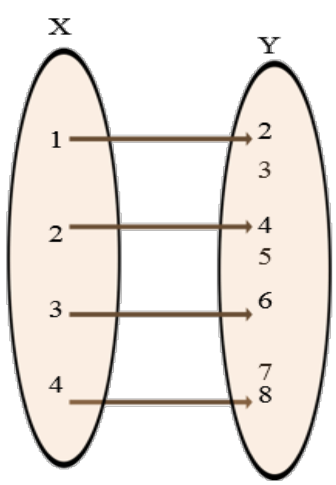
Codomain and Range

Now assume we define a relation from the set, X= {1, 2, 3, 4} to the set, Y = {2,3,4,5,6,7,8}.

We will define the relation as $x \to 2x$. The arrow diagram is shown below. Notice that some members of the set Y are not outputs. These are the odd numbers 3, 5 and 7.

The subset of Y consisting of the even numbers {2, 4, 6, 8} is called the ***range***.

The entire set, Y, is called the ***codomain***.

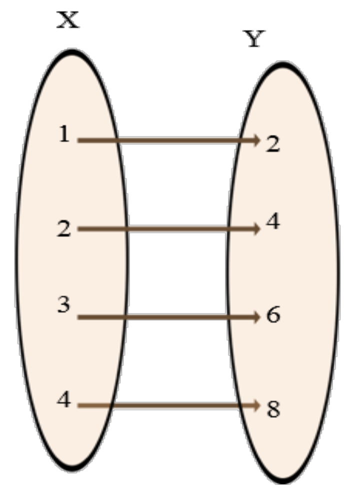


Range is a subset of the codomain

The range is defined as those members of the codomain that are 'used', that is they are connected to some member of the domain by the rule that defines the relation. They are also the output values or the images of the input values.

If all the members of the codomain are connected to members of the domain, then the range is equal to the codomain. If this is not so, then the range is a subset of the codomain.

By defining Y as a set of even numbers {2, 4, 6, 8}, we can have the situation where the range is equal to the codomain.



Range is equal to the codomain

Example 1

A relation is represented by the ordered pairs shown below:

(1, 5) (2, 7) (3, 9) (4, ?) (?, 23)

i. State the rule for the relation.
ii. What is the image of 4?
iii. What is the input for an output of 23?

Solution

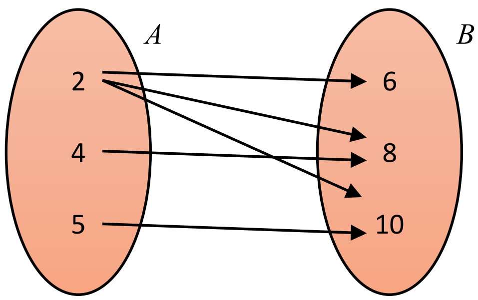
i. By inspection, the rule for the relation is $2x + 3$.

ii. The image of 4 is calculated by substituting $x = 4$ in $2x+3$
$2(4) + 3 = 11$. The image of 4 is 11.

iii. When the output, $y = 23$.
$2x + 3 = 23$
$2x = 23 – 3 = 20$ a
$x = 20 \div 2 = 10$.
The input is 10.

Types of Relations

Let the set $A = \{2, 4, 5\}$ and the set $B = \{6, 8, 10\}$. We define a relation from the set A to the set B as "is a factor of". This relation is illustrated below.



In this relation,

- an element of A (2) is associated with more than one element of B (6, 8, 10).
- more than one element of A (2 and 5) is associated with one element of B (10)

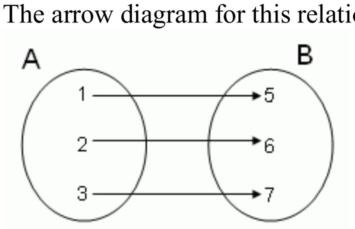
A relation is **one to many** if an element of the domain is mapped onto more than one element of the codomain.

A relation is **many to one** if, two or more elements of the domain have the same image in the codomain.

If a relation has both **one to many** and **many to one** associations, then the relation is **many to many**

The relation "is a factor of" has both of the above types of relationships. This is an example of a **many to many** relation.

We now define a relation from a set A= {1, 2, 3} to a set B = {5, 6, 7} such that "B is four more than A". The arrow diagram for this relation is shown below.



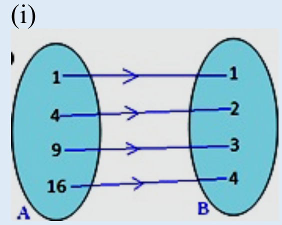
Note that it is not possible to have one member of A associated with more than one member of B. Also, two members of A cannot be associated with one member of B. Such a relation is said to be **one to one**.

A relation is **one to one** if no two elements of the domain have the same image in the codomain.

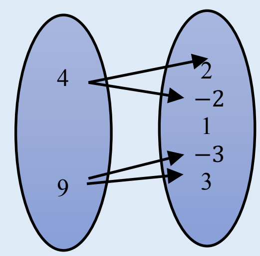
Example 2

For each of the examples below, state the type of relation.

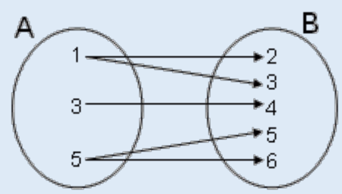
(i)



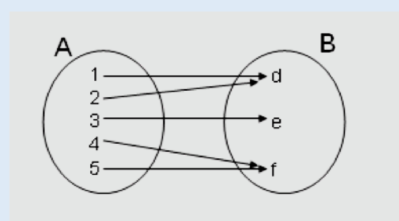
(ii)



(iii)



(iv)



Solution

i. Each member of B is related to only one member of A, we conclude that this relation is **one-one**.
ii. There is at least one example of one-many relationships, we conclude that this relation is **one-many**.
iii. There is at least one example of a one-many relationship, and at least one example of a many-one relationship, we conclude that this relation is **many-many**.
iv. Since there is at least one example of a many-one relationship, we conclude that this relation is **many-one**.

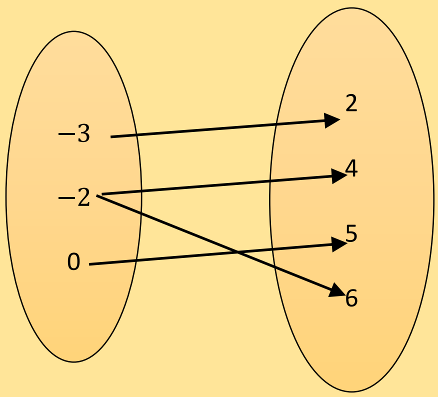
Example 3

A relation, R is defined by the set of ordered pairs; $(-3,2),(-2,4),(0,5),(-2,6)$

(i) List the members of the domain.
(ii) List the members of the codomain.
(iii) What type of relation is R?

Solution

(i) Members of the domain are $\{-3,-2,0\}$
(ii) Members of the codomain are $\{2,4,5,6\}$
(iii) We can draw an arrow diagram to represent the relation.



The arrow diagram shows one example of a one-to-many relation. We conclude that this relation is one-to-many.

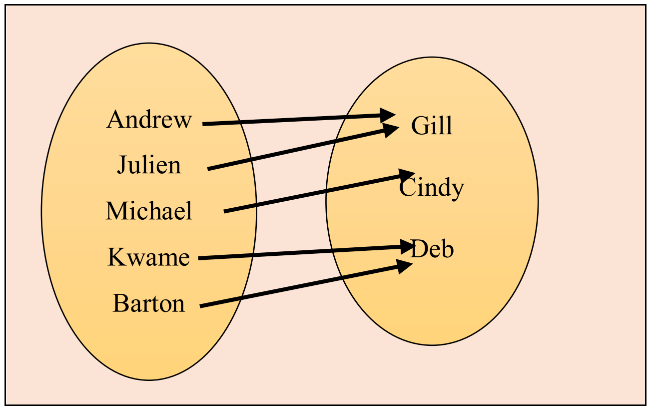
Relations and Functions

A ***function*** is a relation that has **exactly one output** for each input **in the domain**.

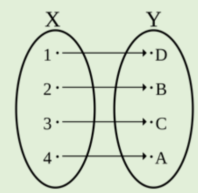
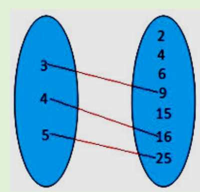
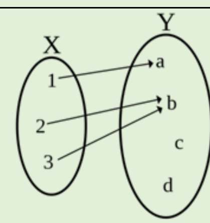
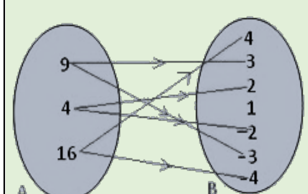
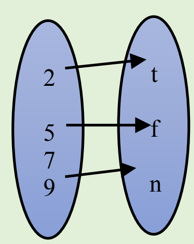
The following are characteristic features of a function defined from a set X to a set Y:

- Every member of X is mapped onto one and only one member of Y
- An input cannot have more than one output.
- Two or more members of the set X can be mapped onto the same member of the set Y

In the real world, we may think of a function as a mapping onto the set of sons (X) to a corresponding set of biological mothers (Y). Each son will be associated with one and only one mother, and two or more sons can be associated with the same mother but one son cannot be associated with two or more mothers. This is shown in the arrow diagram below.
Functions form a subset of relations that are one-one or many-one.



Examples and non-examples of a function

	This is a **one to one function.** The range is the same as the codomain.
	This is a one to one function. The range is a subset of the codomain.
	This is a many to one function. The range is a subset of the codomain.
	This mapping is **not** a function. Each member of the domain is associated with two distinct members of the co-domain. It is a one-many relation.
	This mapping is **not** a function. At least one member of the domain is not associated with any member of the co-domain.

Notation

We can describe a function using mathematical notation, written as $f(x)$, read 'f of x'. The letter f stands for the function itself and x for the input number. Three notations are illustrated below.

$f(x) = 2x + 1$ or
$f: x \to 2x + 1$ or
$y = 2x + 1$

Using any of the above examples, we can calculate the output for a given input. When $x = 1$,

$f(1) = 2(1) + 1 = 3$
$f: 1 \to 2(1) + 1 = 3$
$y = 2(1) + 1 = 3$

If we have more than one function, we can use another letter, say g or h so that we can write, for example:

$$g(x) = 2x + 1$$

Representing relations and functions graphically

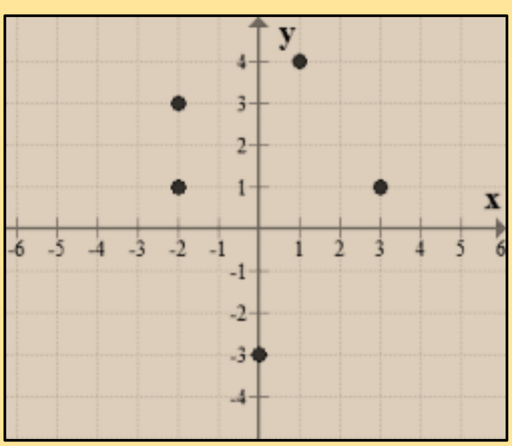
Functions of any type as well as relations can be represented graphically. We merely plot the ordered pairs using the Cartesian plane.

Example 4

Draw the graph of the relation represented by the set of ordered pairs

$(-2, 1), (-2, 3), (0, -3), (1, 4), (3, 1)$

Solution



The graph of a relation provides a visual method of determining whether it is a function or not. The graph of the relation shown in example 4 above shows that the image of -2 is both 1 and 3. This relation cannot be a function because it has a one-many mapping.

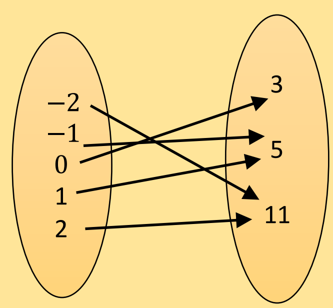
Example 5

A function is defined as
$y = 2x^2 + 3$, where $x = \{-1, 0, 1, 2, 3\}$.
(i) Calculate the output values for this function.
(ii) Draw an arrow diagram for the function.
(iii) Sketch the graph of $y = 2x^2 + 3$

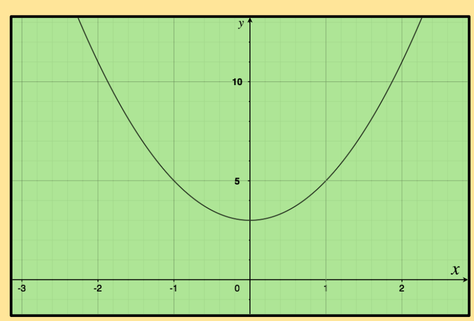
Solution

(i) Substitute the input values to find the output values.
$y = 2(-2)^2 + 3 = 8 + 3 = 11$
$y = 2(-1)^2 + 3 = 2 + 3 = 5$
$y = 2(0)^2 + 3 = 0 + 3 = 3$
$y = 2(1)^2 + 3 = 2 + 3 = 5$
$y = 2(2)^2 + 3 = 8 + 3 = 11$

(ii) The arrow diagram is shown below.



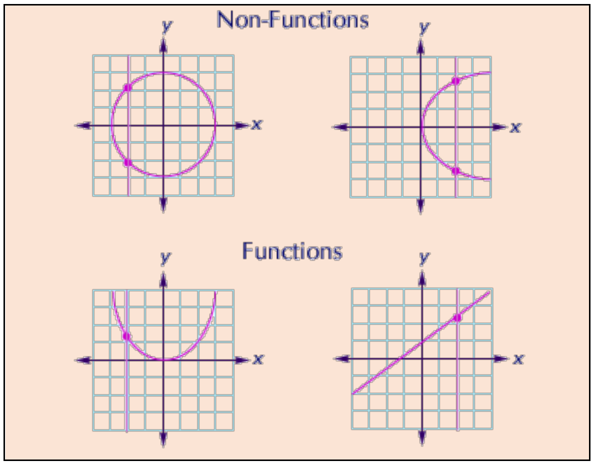
(iii) The graph is shown below.



In example 5, there are no one-many mappings, that is, no value on the domain (or x-axis) is mapped onto more than one image point. This indicates that the graph represents a function.

Vertical Line Test for a function

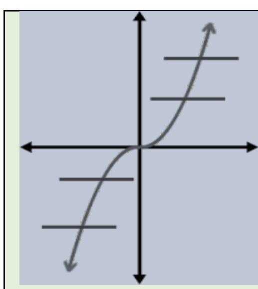
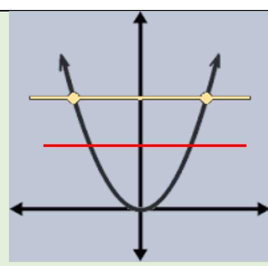
In a function, an input cannot have more than one output. A member of the domain must be assigned to a unique member of the range. A vertical line drawn through any input must be associated with only one output, intersecting the graph only once. If a vertical line cuts the graph more than once then the graph does not represent a function.



Horizontal Line Test for a One to One Function

In a similar fashion, we can develop a test for a one-one function. Since each member of the domain is associated with one and only one member of the codomain, horizontal lines drawn through image points should cut the graph only once.

If a horizontal line intersects a graph at most once, then the graph represents a one-to-one function.

	
This graph represents a one to one function since the horizontal lines cut the curve once. Each input is associated with one output.	This graph does **NOT** represent a one to one function since the horizontal lines cut the curve more than once. Two input values are associated with the same output value.

The inverse of a function

A function maps a set of points from a set X onto a set Y using a given rule. For example, if the rule is "multiply x by 2", a function, $f: x \longrightarrow 2x$ can be represented by the set of ordered pairs:

$$(1, 2), (2, 4), (3, 6), (4, 8), (5, 10)$$

The domain for $f(x)$ is $\{1, 2, 3, 4, 5\}$
The range for $f(x)$ is {2, 4, 6, 8, 10}

If we were to interchange the domain and the range, we will have the following set of ordered pairs:

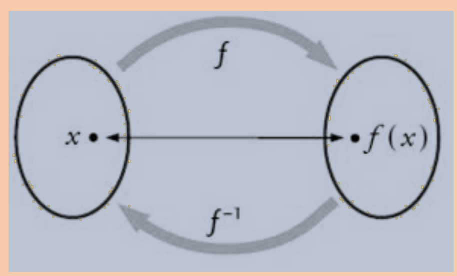
$$(2, 1), (4, 2), (6, 3), (8, 4), (10, 5)$$

The domain for the new function is {2, 4, 6, 8, 10}
The range for the new function is $\{1, 2, 3, 4, 5\}$

The new function which maps members of the range onto members of the domain is called the inverse of $f(x)$, written as $f^{-1}(x)$**.** The rule for this function is "divide by 2", expressed as $f^{-1}(x) = \frac{1}{2}x$**.**

The inverse of f, denoted $f^{-1}(x)$ is the rule that maps members of the range back onto members of the domain.

If f maps x onto $f(x)$ such that
$f: x \rightarrow f(x)$, then
the inverse of f maps $f(x)$ back onto f such that
$f^{-1}(x): f(x) \rightarrow x$
This is illustrated in the diagram below.



Conditions for an inverse to exist

For a function to have an inverse, there must be a one to one correspondence between the members of X and Y. We already know that all functions are **one-one** but the function must also be **onto**, that is - all the y values must be associated with an X value, in other words, there must be no unused output values.

To determine the inverse of a function

Consider the function, $f(x) = 3x + 2$, x is a real number. We can think of the function as a sequence of operations, illustrated below.

Start with x, multiply by 3 then add 2 to the result. This is represented symbolically as

$$x \longrightarrow 3x \longrightarrow +2$$

To obtain the rule for the inverse, we reverse the process. In doing so, what was done last must be done first. We always start with x.

Start with x, subtract 2 then divide the result by 3. This is represented symbolically as

$$x \longrightarrow x - 2 \longrightarrow \frac{x-2}{3}$$

The above method is useful in finding the inverse of simple functions. For more complex ones, the following steps are followed.

Step 1: Rewrite the function letting $y = f(x)$

So $f(x) = 3x + 2$ is written as $y = 3x + 2$

Step 2: Make x the subject

$$\frac{y-2}{3} = x$$

Step 3: Replace y by x to get f^{-1}

$$f^{-1}(x) = \frac{x-2}{3}$$

Example 6

Find the inverse of the function $f(x) = \frac{3x-1}{x+2}$

Solution

Let $y = \frac{3x-1}{x+2}$

Next we make x the subject.

$y(x + 2) = 3x - 1$

$xy + 2y = 3x - 1$

$xy - 3x = -1 - 2y$

$x(y - 3) = -1 - 2y$

$x(3 - y) = 1 + 2y$ [multiply by -1]

$x = \frac{1+2y}{3-y}$

Replacing y by x on the right side to obtain

$f^{1}(x) = \frac{1+2x}{3-x}$

Composite functions

Sometimes it is necessary to obtain a single function that is equivalent to two functions. We refer to the single function as the composite of the two separate functions.

A **composite function** is a combination of two functions, where we apply the first function and the output is used as the input into the second function.

In combining two functions, we must note that the order (commutative property) is important. We must first understand order as it applies to composite functions before making observations about the commutative property.

Notation for composite functions

Consider the following functions:

$f(x) = 2x + 1$ and $g(x) = \frac{x+3}{2}$

We are interested in obtaining a function that combines both functions in the order:

$f(x)$ followed by $g(x)$ or ***f* followed by *g***.

This means that $f(x)$ is the first function and $g(x)$ is the second function. In writing this, we place the first function, f, to the left of x and the second function g, to the left of $f(x)$. Hence, in summary,

f followed by *g* written as $\boldsymbol{g[f(x)]}$ read as "*g* of *f*(*x*)".

A more shortened form is $\boldsymbol{gf(x)}$, read as "*g* of *f*(*x*)".

Commutative Property

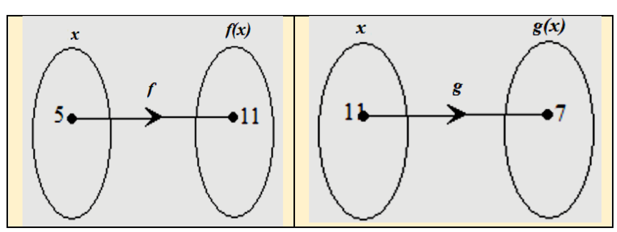
Let us now apply the above principles to evaluate $gf(5)$.

We first find $f(5)$, $f(5) = 2(5) + 1 = 11$.

The output is now the input into $g(x)$, so we need $g(11)$

$g(11) = \frac{11+3}{2} = \frac{14}{2} = 7$.

This process is illustrated in two separate diagrams.



Therefore,
$gf(5) = g[f(5)\} = g[11] = 7$

We now evaluate $fg(5)$

First, calculate $g(5)$, $g(5) = \frac{5+3}{2} = 4$

$fg(5) = f[g(5)\} = f[4] = 2(4) + 1 = 9$

$$fg(5) \neq gf(5)$$

At this point, we may conclude that, when the order is reversed when combining two functions, the result is not the same.

The function *gf* is called a **composite** function, which is equivalent to a combination of $f(x)$ and then $g(x)$. We can say that the function $gf(x)$ means to perform *f* first and then *g* second.

Deriving the Composite Function

Sometimes we are required to derive an algebraic expression for a composite function. That is, we require one expression that will generate any output for a given input. In such a case, we will be given both functions and the order in which the functions are to be carried out. For example, consider the functions:

$f(x) = x + 2$ and $g(x) = \sqrt{x}$

To determine the composite function $gf(x)$, we can carry out the following procedure.

Step 1:
The order is *f* then g, so we write
$$gf(x) = g[f(x)]$$

Step 2:
We replace $f(x)$ by its rule, that is $(x + 2)$
$$g[f(x)] = g[x + 2]$$

Step 3:
We now apply the rule for the function, g, which can be read as 'take the square root' of the input. Since the input is $(x + 2)$, we write:
$$g[x + 2] = \sqrt{x + 2}$$

$$\therefore gf(x) = \sqrt{x+2}$$

We may now use this function to evaluate say, $gf(7)$ in one step.
$gf(7) = \sqrt{7 + 2} = \sqrt{9} = \pm 3$

Example 7

The functions *g* and *f* are defined as:
$f(x) = 2x + 1$ and $g(x) = \dfrac{x+3}{2}$
Derive the composite function $gf(x)$and $fg(x)$.
What conclusion can be made with respect to the commutative property?

Solution

$$\begin{aligned} gf(x) &= g[f(x)] \\ &= g[2x+1] \\ &= \frac{(2x+1)+3}{2} \\ &= \frac{2x+4}{2} \end{aligned}$$

$$\begin{aligned} fg(x) &= f[g(x)] \\ &= f\left[\frac{x+3}{2}\right] \\ &= 2\left(\frac{x+3}{2}\right)+1 \\ &= x+3+1 \\ &= x+4 \end{aligned}$$

Conclusion
Since $gf(x) \neq fg(x)$
Reversing the order of the operations affects the result. Hence, the commutative property does not hold.

Example 8

Given
$\boldsymbol{f(x)} = \frac{\mathbf{1}}{\mathbf{2-x}}$, where $x \epsilon R, x \neq 2$.
Evaluate $ff(3)$

Solution

$$f(3) = \frac{1}{2-3} = -1$$
$$\therefore ff(3) = f(-1) = \frac{1}{2-(-1)} = \frac{1}{3}$$

The composition of inverse functions

We saw in the previous section that the commutative property is not obeyed in functions. For example, if f and g are two functions, then $fg \neq gf$. However, there is an exception to this rule when evaluating the composition of a function and its inverse. In this case,

$$ff^{-1}(x) = f^{-1}f(x)$$

We will use an example to illustrate the above rule.

Let us define a function, $f(x) = 3x - 1$

We find f^{-1}using the following steps

Let $y = 3x - 1$

Making x the subject, we have

$x = \frac{y+1}{3}$

Hence $f^{-1}(x)= \frac{x+1}{3}$

$ff^{-1}(x)= f[\frac{x+1}{3}] = 3(\frac{x+1}{3})$ -1 $= x$

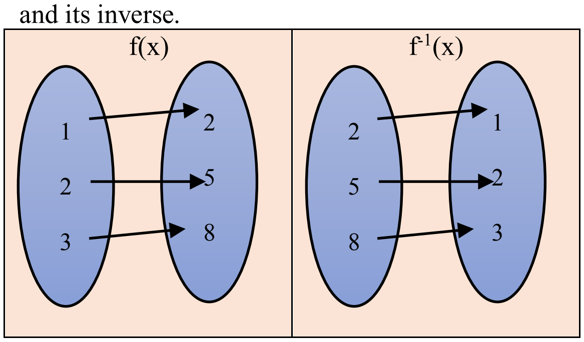
$f^{-1}f(x)= f^{-1}\left(\frac{x+1}{3}\right) = \frac{(3x-1)+1}{3} = x$

Hence, $ff^{-1}(x) = f^{-1}f(x) = x$

We can also illustrate why this is so by considering the images of three points as shown below.

$f(x) = 3(x) - 1$	$f^{-1}(x) = \dfrac{x+1}{3}$
$f(1) = 3(1) - 1 = 2$	$f^{-1}(2) = \dfrac{2+1}{3} = 1$
$f(2) = 3(2) - 1 = 5$	$f^{-1}(5) = \dfrac{5+1}{3} = 2$
$f(3) = 3(3) - 1 = 8$	$f^{-1}(8) = \dfrac{8+1}{3} = 3$

Now we examine the arrow diagrams for the function and its inverse.



The above illustration shows that $f^{-1}f(x) = x$, because

$$1 \xrightarrow{f(x)} 2 \xrightarrow{f^{-1}(x)} 1$$

$$2 \xrightarrow{f(x)} 5 \xrightarrow{f^{-1}(x)} 2$$

$$3 \xrightarrow{f(x)} 8 \xrightarrow{f^{-1}(x)} 3$$

In a similar fashion, we could illustrate that

$$ff^{-1}(x) = x$$

Hence,

$$ff^{-1}(x) = f^{-1}f(x)$$