21. QUADRATIC FUNCTIONS

DEFINING QUADRATIC FUNCTIONS

In our study of linear functions, we may recall that a linear function has the general form $y = mx + c$, where x and y are variables and m and c are constants. A graph of a linear function is always a straight line with gradient m and whose intercept on the y-axis is c.

Another property of the linear function is that the power of the unknown is one. If we were to draw the graph of y versus x for the linear function, $y = mx + c$, the straight line will cut the x- axis at one point only. This is a characteristic feature of a linear function.

Some examples of linear equations are shown in the table below.

Linear Functions
$y = 4x$
$y = 2x + 7$
$y = \dfrac{3x}{5}$

The Quadratic Function

We will now study a function in which the power of the unknown is no more than two. This function is called a quadratic function. Its special features and characteristics will be explored fully in this section. We must first learn to recognise quadratic expressions.

A quadratic expression is one in which the highest power of the unknown is two. A quadratic expression in x, has a general form, $ax^2 + bx + c$, where a, b and c are real numbers, $a \neq 0$.

Quadratic expressions in x, for example, may take the following forms.

$3x^2 + 2x + 5$ where $a = 3, b = 2$ and $c = 5$
$5x^2 - 2x$ where $a = 3, b = -2$ and $c = 0$
$-x^2 + 4$ where $a = -1, b = 0$ and $c = 4$
$-x^2$ where $a = -2, b = 0$ and $c = 0$

Note it is the term, ax^2, that identifies the quadratic. The numerical value of b and/or c may be zero. A quadratic equation is said to be of degree 2. If the

degree of the expression or equation is greater than 2, that is, 3 or more, they are called polynomials. Polynomials of different degrees have different names.

$2x^3 - 3x^2 + 4x - 6 \rightarrow$ polynomial, degree 3
$-5x^4 + 2x^3 - 3x^2 + 6x - 8 \rightarrow$ polynomial of degree 4
$4x + 1 \rightarrow$ linear expression, degree 1
$2x^2 + 4x + 1 \rightarrow$ quadratic expression, degree 2
$\dfrac{1}{3}x^3 + 2x^2 + 4x + 1 \rightarrow$ polynomial, degree 3
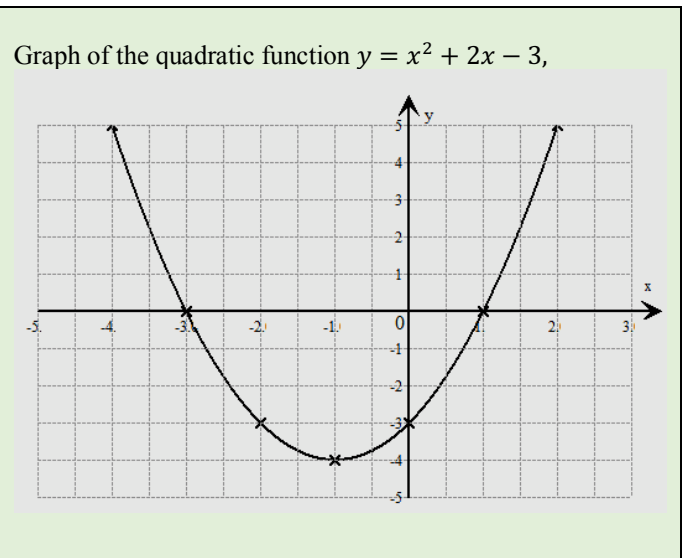
The graph of a quadratic function

To draw the graph of the quadratic function, say, $y = x^2 + 2x - 3$, between $-4 \leq x \leq 2$, we first create a table of values as shown:

x	-4	-3	-2	-1	0	1	2
y	5	0	-3	-4	-3	0	5

Sample calculations

When $x = -4, y = (-4)^2 + 2(-4) - 3$
$\qquad\qquad = 16 - 8 - 3 = 5$
When $x = -3, y = (-3)^2 + 2(-3) - 3$
$\qquad\qquad = 9 - 6 - 3 = 0$
When $x = 2, y = (2)^2 + 2(2) - 3$
$\qquad\qquad = 4 + 4 - 3 = 5$

The points are then plotted on graph paper with carefully labelled axes and a smooth curve is drawn to connect all plotted points as shown below.

Graph of the quadratic function $y = x^2 + 2x - 3$,

Features of a quadratic graph

1. The graph of a quadratic function has a characteristic shape called a parabola.

2. This is a curve with a single maximum or a minimum point.

3. The sign of the constant, a, in the quadratic function, indicates whether the parabola has a maximum or a minimum point. For $a > 0$, the parabola has a minimum point and for $a < 0$, the parabola has a maximum point.

4. Its degree of concavity depends on the actual values of the constants a, b and c. The quadratic curve has a single axis of symmetry (a vertical) which passes through the maximum or the minimum point.

5. The quadratic function is not a one to one function. If we draw a horizontal line on the graph, it cuts at two points, except at the maximum or the minimum point.

6. The x-coordinates of the point of intersection of the curve and the x-axis are called the roots or solutions of the quadratic equation

$$ax^2 + b + c = 0.$$

Interpreting the quadratic graph

The diagram below shows the graph of the function $y = x^2 - 2x - 3$ for the domain $-2 \leq x \leq 4$.
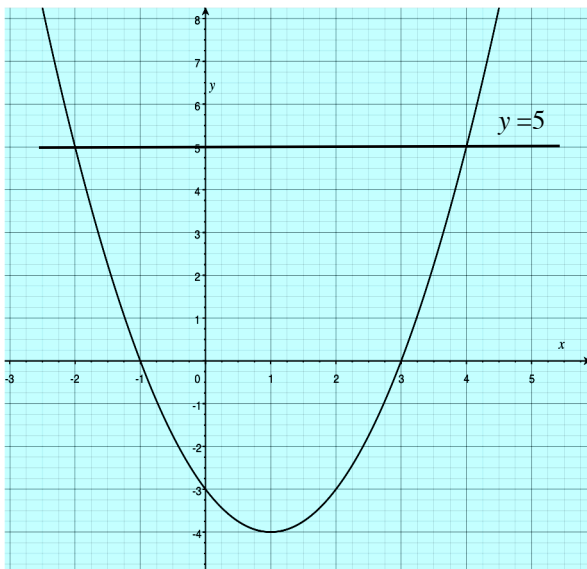
We wish to use the graph to determine:

(i) The values of x for which $x^2 - 2x - 3 = 0$.

(ii) The coordinates of the minimum point on the graph.

(iii) The values of x for which $y \leq 0$.

(iv) The values of x for which $y \leq 5$.

(v) The equation of the axis of symmetry.

(i) To determine the values of x for which

$x^2 - 2x - 3 = 0$, we consider the solution of the equations
$$y = x^2 - 2x - 3 \text{ and } y = 0$$
The values will be the x-coordinate of the point of intersection of the curve and the line $y = 0$ (which is the x-axis)
These values are $x = -1$ and $x = 3$.

(ii) The coordinates of the minimum point on the graph.

The point $(1, -4)$ is the minimum point as shown in the graph.

(iii) The values of x for which $y \leq 0$.

We wish to state the interval on the domain for which the function is negative.
The values of x in the interval $-1 \leq x \leq 3$ correspond to negative values of the function ($y \leq 0$).

(iv) The values of x for which $y \leq 5$.

To answer this question, we draw the line $y = 5$ on the graph.
We note that the line cuts the graph at $x = -2$ and $x = 4$
Hence, the values of x in the interval $\{x: -2 \leq x \leq 4\}$ correspond to values of the function below the horizontal line and hence is the set of solutions for $y \leq 5$).

(v) The axis of symmetry is the line $x = 1$.

Alternatively, that the equation of the axis of symmetry can be obtained by substitution in the formula $x = -\dfrac{b}{2a}$, where a and b are coefficients of the quadratic $ax^2 + b + c$.

SOLVING QUADRATIC EQUATIONS

Quadratic equations take the form $ax^2 + bx + c = 0$ where a, b and c are real numbers, $a \neq 0$.

When we solved a linear equation in x, we will have found the value of x that satisfied the equation. To solve the linear equation, we simply use the laws of basic algebra to isolate the unknown, for example, if

$$2x - 1 = 7$$
$$2x = 7 + 1$$
$$2x = 8$$
$$x = 4.$$

If we substitute $x = 4$ in the equation, we will get $2(4) - 1 = 8 - 1 = 7$.
In the same way, when we solve a quadratic equation, we obtain the value(s) of the unknown that satisfies the equation.

A quadratic equation will have at most two solutions. Some though may have only one solution **and** yet still, some quadratic equations may not have any solutions at all. A solution of a quadratic equation (or for that matter any equation) is also called a **root** of the equation.

In solving a quadratic equation, we may use graphical or algebraic methods. Both techniques will be illustrated in the sections below.

Solving quadratic equations by graphical methods

Graphical methods have the advantage of visualising the function but can produce results which at times can only be fairly accurately read. When solutions lie between, say two integers, they usually have to be estimated. However, the graphical approach can be quite tedious and algebraic methods are generally more efficient.

In using this method, we draw a graph of a quadratic function by creating a table of values. The solutions or roots are obtained by reading-off the x-coordinates of the point of intersection of the curve and the horizontal axis (when the equation $= 0$)
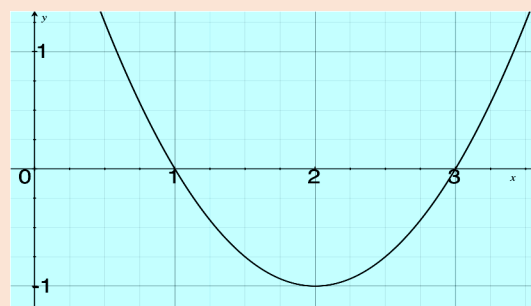
Recall that the quadratic can have a maximum of two roots – this occurs when the graph cuts the x-axis at two distinct points.

If the x-axis is a tangent to the curve, then the two roots are equal to each other and so there is just one solution.
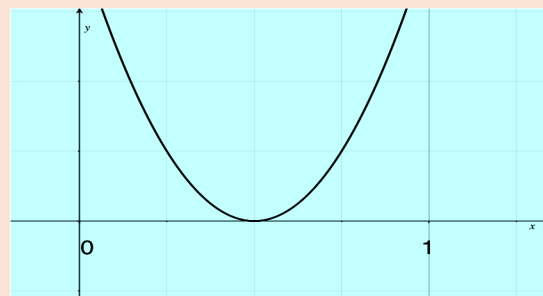If the curve does not cut or touch the x-axis, there are no solutions.
These cases are illustrated below.

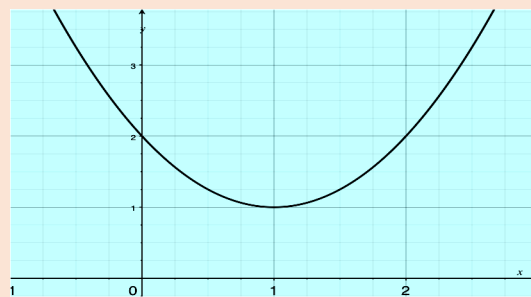Graph of $x^2 - 4x + 3 = 0$

The roots are $x = 1$ and $x = 3$

Graph of $4x^2 - 4x + 1 = 0$

There is only one root, $x = \frac{1}{2}$

Graph of $x^2 + 2x + 8 = 0$

There are no solutions – the graph does not cut or touch the x-axis.

Solving quadratic equations by factorisation

Factorising is a short and simple technique to solve a quadratic equation. The method of factorisation works only when the roots are integers or simple fractions.

Example 1

Solve by factorisation:

(i) $x^2 - 4x + 3 = 0$

(ii) $3x^2 - 2x - 5 = 0$

Solution

(i)
$x^2 - 4x + 3 = 0$
$(x - 1)(x - 3) = 0$
It follows that either
$(x - 1) = 0$ or $(x - 3) = 0$

Solving for x gives $x = 1$ or $x = 3$

(ii)
$3x^2 - 2x - 5 = 0$
$(3x - 5)(x + 1) = 0$
It follows that either
$(3x - 5) = 0$ or $(x + 1) = 0$

Solving for x, gives $x = \frac{5}{3}$ or $x = -1$

Solving quadratic equations using the quadratic formula

The method of factorisation works only when the solutions are integers or simple fractions. We must be careful to bear in mind that although a quadratic equation may not be factorisable, it may still have solutions.
One of the methods of solving a quadratic equation is to express it in the form $ax^2 + bx + c = 0$ and use the quadratic equation formula.

If $ax^2 + bx + c = 0$, where a, b and c are constants:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 2

Solve for x in $x^2 + 4x + 2 = 0$, giving the answer correct to 2 decimal places.

Solution

$x^2 + 4x + 2 = 0$ is of the form $ax^2 + bx + c = 0$, where $a = 1$, $b = 4$ and $c = 2$.
Appling the quadratic formula:

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$= \frac{-4 \pm \sqrt{8}}{2}$$

$$= \frac{-4 \pm 2.828}{2}$$

$$= \frac{-4 + 2.828}{2} \text{ or } \frac{-4 - 2.828}{2}$$

$$= -3.41\underline{2} \text{ or } -0.58\underline{6}$$

$$= -3.41 \text{ or } -0.59 \text{ (correct to 2 decimal places)}$$

Example 3

Solve for x in $x^2 + 2x + 6 = 0$.

Solution

$x^2 + 2x + 6 = 0$ is of the form $ax^2 + bx + c = 0$, where $a = 1$, $b = 2$ and $c = 6$.
Appling the quadratic formula:

$$x = \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4 - 24}}{2}$$

$$= \frac{-2 \pm \sqrt{-20}}{2}$$

We cannot (at this level) find the square root of a negative number, therefore, $x^2 + 2x + 6 = 0$ has no solutions or roots.

Solving quadratic equations by completing the square

Consider the quadratic equation:

$$x^2 + 6x + 9 = 10$$

By factorization, $(x + 3)(x + 3) = 10$
or

$$(x + 3)^2 = 10$$

By taking the square root, we can isolate x,

$$x + 3 = \pm\sqrt{10}$$
$$x = -3 \pm \sqrt{10}$$

Therefore, $x = -3 + \sqrt{10}$ or $x = -3 - \sqrt{10}$, when expressed in exact form.

The above quadratic was relatively easy to solve because the left-hand side of the equation was in the form of a **perfect square**. This made it possible to take the square root of both sides and hence isolate the unknown, x.

An algebraic expression is a perfect square if it can be expressed as a product of two identical linear factors. Hence, like in arithmetic, a perfect square has an exact square root.

If, however, we had to solve the quadratic equation
$$x^2 + 6x - 2 = 0$$
This may seem challenging as the left-hand side is not a perfect square. Let us transpose the constant so that we have:

$$x^2 + 6x = 2$$

Now, we know that $x^2 + 6x + 9$ is a perfect square. So it is convenient to add 9 to both sides of the equation to maintain the equality and so obtain:

$$x^2 + 6x + 9 = 2 + 9$$

Our equation can now be expressed as
$$(x + 3)^2 = 11$$

In this form, it is easy to isolate x by taking the square root. To continue,

$$x + 3 = \pm\sqrt{11}$$
$$x = -3 \pm \sqrt{11}$$

Therefore, $x = -3 + \sqrt{11}$ or $x = -3 - \sqrt{11}$, when expressed in exact form.

In using this method, we need to have an efficient technique to convert any expression of the form

$x^2 + ax$ to a perfect square.

This technique is called completing the square. Let us examine some perfect squares and try to deduce a pattern.

We know that
$$x^2 + 6x + 9 = (x + 3)^2$$
$$x^2 + 6x = (x + 3)^2 - 9$$

$$x^2 + 8x + 16 = (x + 4)^2$$
$$x^2 + 8x = (x + 4)^2 - 16$$

$$x^2 + 10x + 25 = (x + 5)^2$$
$$x^2 + 10x = (x + 5)^2 - 25$$

$$x^2 + 12x + 36 = (x + 6)^2$$
$$x^2 + 12x = (x + 6)^2 + 36$$

In general,

$$x^2 + ax = \left(x + \frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$$

We will now use the above rule to solve quadratic equations by the method of completing the square.

Example 4

Solve $x^2 - 4x + 1 = 0$ by the method of completing the square.

Solution

Let $x^2 - 4x = (x - 2)^2 - 4$
We can now rewrite our quadratic equation as:
$(x - 2)^2 - 4 + 1 = 0$

$(x - 2)^2 - 3 = 0$

$(x - 2)^2 = 3$

$x - 2 = \sqrt{3}$

$x = 2 \pm \sqrt{3}$

$x = 2 + \sqrt{3}$ or $2 - \sqrt{3}$
$x = 2.73$ or 0.27 (to 2 d.p)

Example 5

Solve $4x^2 - 3x - 1 = 0$ by the method of completing the square.

Solution

When the coefficient of x^2 is **NOT** one. We factorise to ensure that the coefficient of x^2 is one and then use the same rule within the brackets.

$$4(x^2 - \frac{3}{4}x) - 1 = 0$$

$$4[(x - \frac{3}{8})^2 - \frac{9}{64}] - 1 = 0$$

$$4(x - \frac{3}{8})^2 - \frac{9}{16} - 1 = 0$$

$$4(x - \frac{3}{8})^2 - \frac{25}{16} = 0$$

$$4(x - \frac{3}{8})^2 = \frac{25}{16}$$

$$(x - \frac{3}{8})^2 = \frac{25}{64}$$

$$(x - \frac{3}{8}) = \sqrt{\frac{25}{64}}$$

$$x = \frac{3}{8} \pm \frac{5}{8}$$

$$= 1 \text{ or } -1/4$$

Example 6

Solve $\quad 1 - 4x - 2x^2 = 0 \quad$ by the method of completing the square.

Solution

$$1 - 4x - 2x^2 = 0$$

$$1 - 2(2x + x^2) = 0$$

$$1 - 2[x^2 + 2x] = 0$$

$$1 - 2[(x + 1)^2 - 1] = 0$$

$$1 - 2(x + 1)^2 + 2 = 0$$

$$3 - 2(x + 1)^2 = 0$$

$$-2(x + 1)^2 = -3$$

$$(x + 1)^2 = 1.5$$

$$x + 1 = \sqrt{1.5}$$

$$x = -1 \pm 1.22$$

$$x = -2.22 \quad \text{or} \quad 0.22$$
(correct to 2 decimal places)

Determining the maximum and minimum of a quadratic function

Any quadratic expression can be written in the form $a(x + h)^2 + k$ or $k - a(x + h)^2$. These forms are very useful in determining the minimum or maximum value of the function.

When $a > 0$, the expression $a(x + h)^2 \geq 0$, for all values of x
Since $f(x) = a(x + h)^2 + k$
$\Rightarrow \quad f(x) = 0$ or a positive umber $+ k$.
$\Rightarrow f(x) \geq k$ for all values of x,

Therefore, the minimum value of $f(x)$ is k, when $a(x + h)^2 = 0$ and it will occur when $x = -h$,

When $a < 0$, the expression $a(x + h)^2 \leq 0$, for all values of x
Since $f(x) = a(x + h)^2 + k$
$\Rightarrow \quad f(x) = $ Negative Number $+ k$.
$\Rightarrow f(x) \leq k$ for all values of x,

Therefore, the maximum value of $f(x)$ is k, when $a(x + h)^2 = 0$ and it will occur when $x = -h$.

If a quadratic expression is written in the form $a(x + h)^2 + k$, the maximum or the minimum value is always k and it will occur at $x = -h$.

Example 9

Find the coordinates of the minimum point of the function

$f(x) = x^2 + 4x - 5$.

Solution

We need to express $x^2 + 4x - 5$ in the form $(x + h)^2 + k$.
Let $x^2 - 4x = (x - 2)^2 - 4$
$x^2 + 4x - 5 = [(x + 2)^2 - 4] - 5$
$\qquad\qquad = (x + 2)^2 - 9$
Now, $(x + 2)^2 \geq 0$ for all values of x.
The minimum value occurs when $(x + 2)^2 = 0$, that is, $x = -2$
$f(x)_{min} = 0 - 9 = -9$
Hence, the minimum value is -9 when $x = -2$.
The coordinates of the minimum point are (-2, -9).

Example 10

Find the maximum value of the function $f(x) = 3 + 8x - 2x^2$ and the value of x at which it occurs.

Solution

$f(x) = 3 + 8x - 2x^2$

$f(x) = 3 - 2(x^2 - 4x)$

Now, $x^2 - 4x = (x - 2)^2 - 4$, by the process of completing the square

$f(x) = 3 - 2(x^2 - 4x)$

$\quad = 3 - 2[(x - 2)^2 - 4]$

$\quad = 3 - 2(x - 2)^2 + 8$

$\quad = -2(x - 2)^2 + 11$

Now, $-2(x - 2)^2 \leq 0$ for all values of x.
The maximum value occurs when $-2(x - 2)^2 = 0$, that is, $x = 2$

$\therefore f(x)_{max} = 11 - (0) = 11$

Hence, the maximum value is 11 when $x = 2$.
The coordinates of the minimum point are (2, 11).

Express a quadratic function in the form $a(x + h)^2 + k$ – by equating coefficients

Instead of completing the square, we can obtain the values of a, h and k using an alternative method. The method of equating coefficients is illustrated in the examples below.

Example 11

Express $f(x) = x^2 + 4x + 1$ in the form $(x + h)^2 + k$

Solution

$x^2 + 4x + 1 = (x + h)^2 + k$

$\quad = x^2 + 2hx + h^2 + k$

Equating coefficients of x, $2h = 4, h = 2$
Equating the constant term: $h^2 + k = 1$,
$\Rightarrow 2^2 + k = 1, k = 1 - 4 = -3$
Hence, $x^2 + 4x + 1 = (x + 2)^2 - 3$

Example 12

Express $y = 3x^2 + 4x - 5$ in the form $a(x + h)^2 + k$ Hence, find the coordinates of the minimum point.

Solution

$3x^2 + 4x - 5 = a(x + h)^2 + k$

$\quad = a(x^2 + 2hx + h^2) + k$

$\quad = ax^2 + 2ahx + ah^2 + k$

Equating coefficients of x^2: $a = 3$ Equating coefficients of x: $2ah = 4$ $2(3) \times h = 4$ $h = \dfrac{4}{6} = \dfrac{2}{3}$	Equating the constants: $ah^2 + k = -5$ $a = 3$ and $h = \dfrac{2}{2}$ $3 \times \dfrac{4}{9} + k = -5$ $k = -5 - \dfrac{4}{3} = -6\dfrac{1}{3}$

$$y = 3x^2 + 4x - 5 = 3\left(x + \dfrac{2}{3}\right)^2 - 6\dfrac{1}{3}$$

Example 13

Express $f(x) = 10 - 4x - x^2$ in the form $c - (b + x)^2$. Hence, state the maximum value of the expression and the value of x for which the maximum occurs.

Solution

$10 - 4x - x^2 = c - (b^2 + 2bx + x^2)$

$\quad = c - b^2 - 2bx - x^2$

Equating the coefficients of x: $-4 = 2b$, $b = -2$ Equating constants: $10 = c - b^2$ $10 = c - 2^2$ $c = 14$	This can be written in the form $c - (b + x)^2$ as $10 - 4x - x^2 = 14 - (2 + x)^2$ Where $c = 14, b = 2$ Hence, the maximum value of $f(x)$ is 14 at $x = -2$

Minimum/Maximum point from equation of the axis of symmetry

If we know the equation of the axis of symmetry for a quadratic, then we can calculate the coordinates of the maximum or minimum point of a quadratic function. This is because the equation of the axis of symmetry of any quadratic is $x = \frac{-b}{2a}$ and which is really the x-coordinate of the maximum or the minimum point.

Example 14

Find the coordinates of the minimum point on the curve $y = x^2 - 6x + 8$.

Solution

When $y = 0$, $x^2 - 6x + 8 = 0$ $(x - 2)(x - 4) = 0$ $x = 2$ or $x = 4$ The curve cuts the x-axis at the points (2, 0) and (4, 0). The axis of symmetry passes through the minimum point and has the equation: $x = \frac{-b}{2a}$, where $b = -6$ and $a = 1$	$x = \frac{-(-6)}{2(1)} = 3$ Therefore, the x-coordinate of the minimum point is 3. When $x = 3$, $y = (3)^2 - 6(3) + 8$ $y = 9 - 18 + 8$ $y = -1$ Min. point $(3, -1)$

The points of intersection of a line and a curve

When we solved a quadratic equation graphically, we were actually determining the points of intersection of a line and a curve. For example, to solve a quadratic equation, $ax^2, +bx + c = 0$, we actually solved the following pair of equations simultaneously:

$y = ax^2, +bx + c$ and $y = 0$

The equation $y = 0$ is the equation of the x axis. Hence, the solutions occurred where the curve cut the x-axis.

At this point, we will consider the case where the line is not the x-axis, and the curve is not necessarily a quadratic.

Solving a pair of equations in two variables when one is linear and one is not linear

We will first solve a pair of equations where one is quadratic and the other is a linear equation of the form $y = mx + c$. We will use an algebraic approach, similar to the one we used when we solved a pair of linear equations in two unknowns simultaneously.

Example 14

Find the points of intersection of the line with equation $y = 3x + 7$ and the curve with equation $y = x^2 + 5x + 4$.

Solution

To find the points of intersection of a curve and a line, we solve their equations simultaneously.

Let $y = x^2 + 5x + 4 \ldots (1)$
and $y = 3x + 7$ ….(2)
Equating (1) and (2) we get $x^2 + 5x + 4 = 3x + 7$
$$x^2 + 2x - 3 = 0$$
$$(x - 1)(x + 3) = 0$$
Hence, $x = 1$ or $x = -3$.

We substitute these values of x in any of the equations to find y

When $x = 1$, $y = 3(1) + 7 = 10$
When $x = -3$, $y = 3(-3) + 7 = -2$

And so the points of intersection of the line and the curve are
(1, 10) and (−3, −2).

Example 15

Solve for x and y, given
$y + 4x = 27$ and $xy + x = 40$.

Solution

Let
$y + 4x = 27$ …(1)
$xy + x = 40$ …(2)
From (1)
$y = 27 - 4x$
Substitute in (2)
$x(27 - 4x) + x = 40$
$27x - 4x^2 + x = 40$
$4x^2 - 28x + 40 = 0$
$\div 4$
$x^2 - 7x + 10 = 0$
$(x - 2)(x - 5) = 0$
$x = 2$ or 5

When $x = 2$
$y = 27 - 4(2)$
$y = 19$

When $x = 5$
$y = 27 - 4(5)$
$y = 7$

Hence,
$x = 2$ and $y = 19$
OR
$x = 5$ and $y = 7$

The line $y + 4x = 27$ and the curve $xy + x = 40$ intersect at the points (2, 19) and (5, 7).

As a point of interest, if we attempt to solve the equations of two graphs so as to obtain their point or points of intersection and they prove to be unsolvable, then it means that the graphs do not intersect. If it is a line and a curve then a single solution indicates that the line is a tangent at the solution point.