6: QUADRATIC INEQUALITIES

Inequalities

In mathematics, an open sentence which involves any one of the signs, $>, \geq, <$ or $\leq$ is called an . The meanings associated with these inequalities are:

$>$ greater than
$\geq$ greater than or equal to
$<$ less than
$\leq$ less than or equal to

Equations, on the other hand, are mathematical statements that show equality between two expressions. These are all examples of **equations**.

$2x = 5, 3x - 4 = 7$ and $x^2 + 6x + 5 = 0$

Inequalities are mathematical statements in which one expression is less than or greater than another we These are all examples of **inequalities**.

$2x > 5, \ 3x - 4 < 7, \ x^2 + 6x + 5 \geq 0$ and $x^2 - 9 \leq 0$ are

When the highest power of the variable is one, we have a linear. When the highest power of the variable is two, we have a **quadratic inequality**. These are examples of quadratic inequalities.

$x^2 + 5x + 6 \leq 0, \quad 3x^2 - 4x + 5 > 0 \quad$ and $x^2 - 9 \geq 0$

Solution of linear inequalities in one unknown

In our study of linear inequalities, we learned that the solution of an inequation differs from the solution of an equation. When we solve the equation
$x + 5 = 8$ we obtain $x = 3$

The solution is unique in that there is a single value that satisfies the equation. However, when we solve the inequation
$x + 5 < 8$ we obtain $x < 3$

The solution gives a range of values rather than a single value. If $x \in \Re$ then we can represent the solution on a number line as shown below.

The solution consists of all the values on the number line to the left of 3. Since $x = 3$ is **NOT** a solution, there is an empty circle at 3.
The graph below shows the solution of the inequation $x \geq 2$. Note that the circle at 2 is filled or shaded, indicating that 2 is part of the solution.

When the unknown is a real number, we usually have an infinite number of solutions that may lie within a given range. When the unknown in an is an integer, a natural number or a whole number, the solution is restricted and it is best described by listing the members.

Solution of Quadratic Inequalities

If we solve the quadratic equation:
$x^2 - 4x + 3 = 0$, the solutions or roots obtained will be $x = 1$ or $x = 3$. This quadratic equation has two unique or distinct roots.

If we solve the quadratic inequality $x^2 - 4x + 3 \leq 0$, the number of solutions is infinite, but, they will lie within a given range.

In solving a quadratic inequality, we seek the range of values of x that satisfy the inequality. We may obtain this range of values by sketching the curve. We note the points where the curve cuts the x-axis (sometimes called the critical values) and then deduce the solution set by reading off the values from the sketch. The following steps are used to arrive at the solution.

Example 1

Solve for x: $x^2 - 6x + 8 \geq 0$.

Solution
1. Determine the critical values
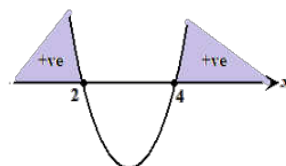   Let $x^2 - 6x + 8 = 0$, and obtain the roots.
   When $x^2 - 6x + 8 = 0$
 $(x-2)(x-4) = 0$
   The graph of $x^2 - 6x + 8 = 0$ cuts the x-axis at and at $x = 4$.

2. Sketch the graph of the quadratic function.
   We note the coefficient of $x^2 > 0$. So, the quadratic graph has a minimum point. This is sufficient to make a basic sketch of the quadratic curve. We wish only to know where the curve cuts the horizontal axis. This is shown below.

3. Determine the range of values of x that satisfy the inequality:
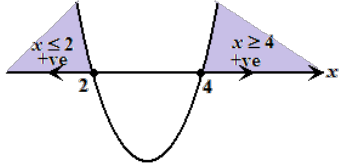
From the above graph, we can see that when $x \geq 4$, $y = x^2 - 6x + 8 \geq 0$. That is, the y coordinates are all positive and hence satisfies the inequality.

Also, when $x \leq 2$, $y = x^2 - 6x + 8 \geq 0$. That is, the y coordinates are also all positive and hence satisfies the inequality.

4. Write the solution

For the inequality, $x^2 - 6x + 8 \geq 0$, the solution is the union of two disjoint sets. We combine both sets and write the solution as
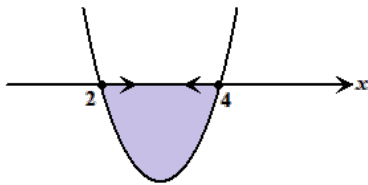
$\{x : x \geq 4\} \cup \{x : x \leq 2\}$

Example 2

Solve for x: $x^2 - 6x + 8 \leq 0$.

Solution

To solve, $x^2 - 6x + 8 \leq 0$ we again follow the same four steps. But we are now interested in finding the range of values for x when the function, y is negative.

The graph shows that the y coordinates are negative below the x-axis and so the solution lies in the region where $x \geq 2$ and $x \leq 4$, as shown below.

This solution set for the $x^2 - 6x + 8 \leq 0$ lies in a continuous interval and may be combined as

$\{x : 2 \leq x \leq 4\}$

Example 3

Solve for x, $6x - 8 - x^2 > 0$.

1. Determine the critical values
   Let $6x - 8 - x^2 = 0$
   We may wish to multiply by -1 to get the roots.
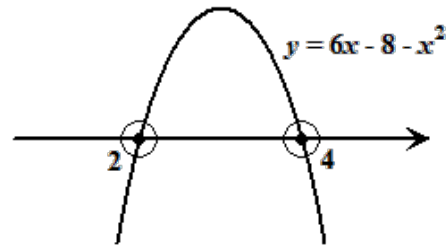
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$

$\therefore$ Curve $y = 6x - 8 - x^2$ cuts the x-axis at $x = 2$ and $x = 4$.

2. Sketch the graph of the quadratic function
   The coefficient of x^2 is negative. Hence, the quadratic curve has a maximum point.
   If the graph of $y = 6x - 8 - x^2$ is sketched, it would look like:

3. Determine the range of values of x that satisfy the inequality
   The empty circles at $x = 2$ and at $x = 4$ indicate that these values are not to be included in the solution.
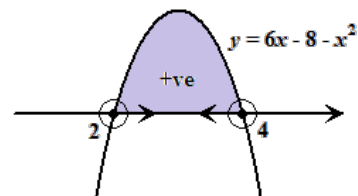   Notice that y is positive when $x > 2$ and also when $x < 4$.

4. Write the solution
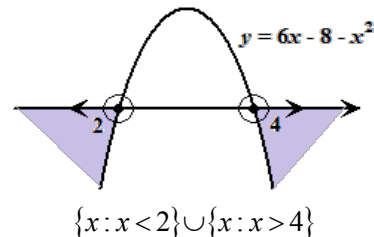   The solution is a continuous interval.
   $\therefore 6x - 8 - x^2 > 0$ for $x > 2$ **and** $x < 4$.
   This result can be combined to read
   $$\{x : 2 < x < 4\}.$$

If we required the solution of $6x - 8 - x^2 < 0$, then the solution will now lie in the range $x > 4$ **or** $x < 2$, as shown below.

$\{x : x < 2\} \cup \{x : x > 4\}$

Solution of inequalities of the type
$$\frac{ax+b}{cx+d} > 0; \geq 0; < 0; \leq 0$$

Example 4

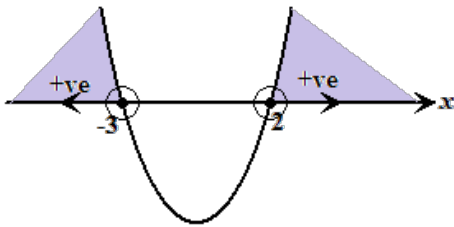Solve for x in $\dfrac{x-2}{x+3} > 0$.

Solution

Multiply by $(x+3)^2$ which is always positive (except for $x = -3$) so that the sign of the inequality is maintained and it is now in a more manageable form

$$\therefore (x-2)(x+3) > 0$$

The roots of $(x-2)(x+3) = 0$ are 2 and -3.

The coefficient of $x^2 > 0 \Rightarrow$ the curve has a minimum point.

The graph of $y = (x-2)(x+3)$ looks like:

Hence, the inequality, $\dfrac{x-2}{x+3} > 0$ for $x > 2$ or $x < -3$.

The solution is the union of two disjoint sets. We combine both sets and write the solution as:
$$\{x : x > 2\} \cup \{x : x < -3\}.$$

If we had to solve for $\dfrac{x-2}{x+3} < 0$, then the solution would have been the continuous interval
$$\{x : -3 < x < 2\}$$

We can use another method that does not involve the sketch of the curve but still requires the critical values.
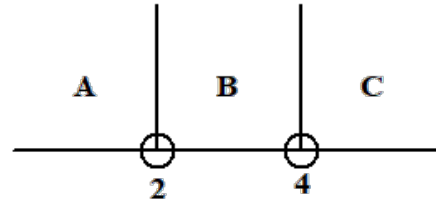
Example 5

Solve for x in $x^2 - 6x + 8 < 0$.

Solution

Let $x^2 - 6x + 8 = 0$
$$(x-2)(x-4) = 0$$

Hence, $x = 2$ and $x = 4$ are critical values. We divide the number line into three regions separated by the critical values.

The regions are A, B and C are as shown. To determine which region(s) contain the solution, select any point from any region and substitute in the inequality. If the inequality is satisfied then the region is included in the solution.

Region A

Select the point $x = 1$, substitute in the inequality:
$$(1)^2 - 6(1) + 8 < 0$$
$$3 < 0 \text{ which is false.}$$

Hence, the solution cannot lie in region A.

Note: If 0 is not a critical value, then it is useful to choose it to avoid unnecessary longer computation.
In this example, $x = 0$ lies in region A. Substituting $x = 0$, we have
$$(0)^2 - 6(0) + 8 < 0$$
$$8 < 0, \text{ which is false}$$
This verifies the above result, that is, the solution cannot lie in region A.

Region B

Select the point $x = 3$. We substitute $x = 3$ to obtain:
$$(3)^2 - 6(3) + 8 < 0$$
$$-1 < 0 \text{ which is true}$$
Hence, the required region is B and where $x > 2$ and $x < 4$, stated as $\{x : 2 < x < 4\}$.

Region C

Although it is not necessary, we could have tested $x = 5$ (in C) and obtain
$$(5)^2 - 6(5) + 8 < 0$$
$$3 < 0 \text{ which is false}$$
Hence, the solution cannot lie in region C.

In using the above method to obtain the solution of a quadratic inequality, we observe a certain pattern and can easily predict the solution without having to test all three regions.

The pattern holds for quadratics with maximum and minimum values. This pattern is illustrated in the following diagram.
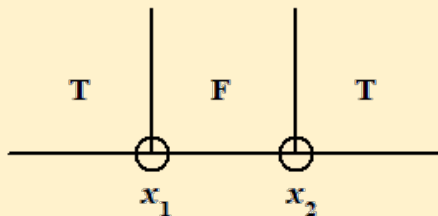
If x_1 and x_2 are the critical values and we test any value of x in the inequality, we will obtain the following pattern.
EITHER – Pattern 1
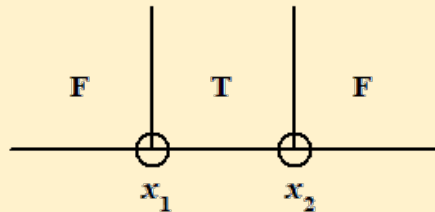
OR – Pattern 2

The regions alternate in truth or falsity as indicated in the diagrams above. So we may choose to test only one region and depending on the result we can deduce what is happening in the other regions.

Example 6
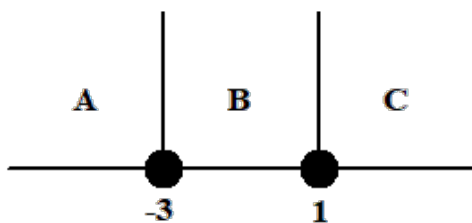
Solve for x in $x^2 + 2x - 3 \geq 0$.

Solution
Let $x^2 + 2x - 3 = 0$
$(x+3)(x-1) = 0$
The critical values are $x = 1$ and $x = -3$.
We divide the number line into three regions separated by the critical values.

Test $x = 0$ in B.
$(0)^2 + 2(0) - 3 \geq 0$
$-3 \geq 0$ False

∴ The region at the sides of B, that is, A and C are TRUE (second pattern in the diagram) and would be the correct regions that satisfy the inequality.

Hence, the solution is

$x \leq -3$ or $x \geq 1$ or $\{x : x \leq -3\} \cup \{x : x \geq 1\}$.

Had we tested a value in A or in C , say, $x = 2$ in C:
$(2)^2 + 2(2) - 3 \geq 0$
$5 \geq 0$ True

So, the regions would have been A and C.

Remember we can test any value of x in any region. and if true, then the region we chose is correct. We can then predict the outcome of the other regions by filling out the other regions using either pattern 1 or pattern 2.