5. EQUATIONS AND INEQUALITIES

SOLUTION OF EQUATIONS

Now, we draw reference to the additive and multiplicative inverse, as it is used quite often in the solution of algebraic equations.

Additive Inverse

We know that

$$3 + -3 = 0$$
$$-8 + 8 = 0$$

In the first example, we refer to -3 as the additive inverse of 3 (and vice versa), and in the second example, 8 as the additive inverse of -8.

We refer to zero as the **Additive Identity Element**.

Multiplicative Inverse

We know that

$$3 \times \frac{1}{3} = 1$$
$$\frac{1}{8} \times 8 = 1$$

In the first example, $\frac{1}{3}$ is the multiplicative inverse of 3 (and vice versa) and in the second example, 8 is the multiplicative inverse of $\frac{1}{8}$.

We refer to one as the **Multiplicative Identity Element**.

Equations involving one inverse

Some simple types of these types of equations are shown in the four examples below. They can be solved by one step.

In solving the following equations, we use the additive inverse to isolate the variable.

$x+3=5$ -3 from both sides $x=5-3$ $x=2$	$x-4=9$ $+4$ to both sides $x=9+4$ $x=13$

In solving the following equations, we use the multiplicative inverse to isolate the variable.

$3x=6$ Divide both sides by 3 $x=\frac{6}{3}$ $x=2$	$\frac{x}{4}=5$ Multiply both sides by 4 $x=5\times 4$ $x=20$

Equations involving two inverses

The following examples require two steps to isolate the unknown. In these examples, both additive and multiplicative inverses are used.

$3x+1=7$ $3x=7-1$ $3x=6$ $x=\frac{6}{3}$ $x=2$	$\frac{x}{2}-5=7$ $\frac{x}{2}=7+5$ $\frac{x}{2}=12$ $x=12\times 2$ $x=24$

Equations with unknown on both sides

When equations have unknowns on both sides, we use the inverse properties to collect all the unknowns on one side and the constants on the other side.

$4x+2=x+14$ $4x-x=14-2$ $3x=12$ $x=\frac{12}{3}$ $x=4$	$7x+2=5x+6$ $7x-5x=6-2$ $2x=4$ $x=\frac{4}{2}$ $x=2$

Equations involving brackets

When brackets are involved, expansion is usually necessary, before simplification can take place.

$2(1-3x)=x+4$ $2-6x=x+4$ $-6x-x=4-2$ $-7x=2$ $x=-\frac{2}{7}$	$-3(2x+1)=2(x-4)$ $-6x-3=2x-8$ $-3+8=2x+6x$ $5=8x$ $x=\frac{5}{8}$

WORD PROBLEMS

We now have the necessary tools required to solve word problems involving linear equations. We often encounter situations in the real world where we have to find an unknown quantity. In these situations, we must formulate our own equation and then solve it.

Example 1

A number is doubled and then increased by 3. If the resulting number is 73, find the original number.

Solution

Let the original number be represented by x The number is doubled: $x \times 2 = 2x$ Increasing $2x$ by 3 gives $2x+3$. Hence, we can equate to obtain $2x+3 = 73$	Solving the equation $2x+3=73$ $2x=73-3$ $2x=70$ $x=35$

Example 2

A father divides a collection of 36 pearls among his three daughters, Amy, Beth and Carolyn. The youngest, Carolyn, gets 1 pearl less than the second youngest daughter, Beth, who gets 1 less than the eldest daughter Amy. How many pearls did each daughter get?

Solution

Let the number of pearls received by Amy be represented by x.
The number of pearls received by Beth will be $=x-1$
The number received of pearls received by Carolyn will be
$=(x-1)-1=x-2$
Since the total number of pearls is 36, we can now write the equation

$$x+(x-1)+(x-2)=36$$
$$3x-3=36$$
$$3x=36+3$$
$$=39$$
$$x=13$$

Therefore, Amy received, x which is 13.
Beth received x – 1 which is 13 – 1 = 12
Carolyn received x – 2 which is 13 – 2 = 11

ALGEBRAIC FRACTIONS

Algebraic fractions, like arithmetical fractions, have a numerator and a denominator. However, algebraic fractions have symbols, rather than numerals in the numerator or the denominator or in both the numerator and the denominator.
We would have encountered algebraic fractions when we were attempting to divide two algebraic terms. For example,
$8c^5d^3 \div 2c^2d$, was written in fraction form, $\frac{8c^5d^3}{2c^2d}$ to make it easier to perform the operation of division.

The following are examples of algebraic fractions.

$$\frac{a}{3},\quad \frac{5}{b},\quad \frac{3cd}{f},\quad \frac{p^2}{5},\quad \frac{(3m+1)}{4},\quad \frac{1}{(n-6)}$$

When we performed the operations of addition and subtraction of arithmetic fractions we ensured that all the fractions had the same denominator. This entailed finding the LCM of all the numbers in the denominators. When adding and subtracting algebraic fractions, we do likewise, and so we must be able to find the LCM of algebraic terms.

LCM of algebraic terms

We can apply the same procedure used in arithmetic to obtain the LCM of algebraic terms. It is easy to check for divisibility in algebraic terms. For example,

$3p$ is a multiple of p because, $\frac{3p}{p}=3$.

p^3 is a multiple of p because, $\frac{p^3}{p}=p^2$.

Example 3

Find the LCM of
(a) x and $2x$ (b) x^2 and x^5 (c) a and b

Solution

(a) Since $2x$ is a multiple of x, the LCM is $2x$.

(b) Since x^5 is a multiple of x^2, the LCM is x^5

(c) There are no common factors of a and b, hence, the LCM is ab

Adding and subtracting algebraic fractions

When the denominators are the same, we simply add (or subtract) the numerators.

Example 4

Add

(a) $\frac{x}{3}+\frac{y}{3}$ (b) $\frac{5}{a}+\frac{3}{a}$ (c) $\frac{b}{c+2}+\frac{2b}{c+2}$

Solution

$\frac{x}{3}+\frac{y}{3}$ $=\frac{x+y}{3}$	$\frac{5}{a}-\frac{3}{a}$ $=\frac{5-3}{a}$ $=\frac{2}{a}$	$\frac{b}{c+2}+\frac{2b}{c+2}$ $=\frac{b+2b}{c+2}$ $=\frac{3b}{c+2}$

When the denominators are different, we compute the LCM and use this as the common denominator

Example 5

Add

(a) $\frac{4}{pq}+\frac{5}{qr}$ (b) $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}$

Solution

(a) The LCM of pq and qr is pqr. $\frac{4}{pq}+\frac{5}{qr}$ $=\frac{4r}{pqr}+\frac{5p}{pqr}$ $=\frac{4r+5p}{pqr}$	(b) The LCM of x, x^2, and x^3 is x^3 $\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^3}$ $=\frac{x^2+2x+3}{x^3}$

Example 6

Simplify

(a) $\frac{m}{m+2}-\frac{n}{(m+2)^2}$

(b) $\frac{6}{y+1}-\frac{3}{y-1}$

(a) The LCM is $(m+2)^2$ because $(m+2)$ is a multiple of $(m+2)^2$, $\frac{m}{m+2}-\frac{n}{(m+2)^2}$ $=\frac{m(m+2)+n}{(m+2)^2}$ $=\frac{m^2+2m+n}{(m+2)^2}$	(b) There are no common factors in both denominators. Hence, the LCM is the product of the denominators. $\frac{6}{y+1}-\frac{3}{y-1}$ $=\frac{6(y-1)-3(y+1)}{(y+1)(y-1)}$ $=\frac{6y-6-3y-3}{(y+1)(y-1)}$ $=\frac{3y-9}{(y+1)(y-1)}$

Multiplication and division of algebraic fractions

The procedure for multiplication and division of algebraic fractions is the same as that for arithmetic fractions. It is always best to reduce fractions to its lowest terms in stating the answer.

Example 8

(a) Multiply $\frac{2p}{a}\times\frac{5q}{b}$ (b) Divide $\frac{t}{2k}\div\frac{5m}{n}$

Solution

(a)
$$\frac{2p}{a}\times\frac{5q}{b}=\frac{2p\times 5q}{a\times b}=\frac{10pq}{ab}$$

(b)
$$\frac{t}{2k}\div\frac{5m}{n}=\frac{t}{2k}\times\frac{n}{5m}=\frac{tn}{10km}$$

Example 9

Multiply $\frac{a^2}{2c^2}\times\frac{5c^4}{ab}$

Solution

$$\frac{a^2}{2c^2}\times\frac{5c^4}{ab}=\frac{5\times a\times a\times c\times c\times c\times c}{2\times c\times c\times a\times b}=\frac{5\times a\times c\times c}{2\times b}=\frac{5ac^2}{2b}$$

Solving equations involving fractions

If an equation has fractions we use algebraic techniques to express the equations in a non-fractional form and solve accordingly.

Solving equations of the form $\frac{a}{b} = \frac{c}{d}$

When we are solving an equation in the above form, we can simplify the equation by using the principle of **cross multiplication**.

We know from arithmetic that

If $\frac{3}{5} = \frac{6}{10}$, then $3 \times 10 = 6 \times 5$

Cross multiplication is an efficient technique to eliminate fractions. Alternatively, we can multiply both sides of an equation by the LCM of the fractions. The following examples illustrate both methods.

Cross multiplication	Multiply both sides of the equation by the LCM
$\frac{2x}{3} = \frac{4}{5}$ $2x \times 5 = 4 \times 3$ $10x = 12$ $x = \frac{12}{10}$ $x = 1\frac{1}{5}$	$\frac{2x}{3} = \frac{4}{5}$ $15 \times \frac{2x}{3} = 15 \times \frac{4}{5}$ $5 \times 2x = 3 \times 4$ $10x = 12$ $x = \frac{12}{10} = 1\frac{1}{5}$

Solving equations involving brackets and fractions

In these examples, we may choose to eliminate the fractions using either of the following methods:

Method 1 - Cross multiplication or

Method 2 - Multiplying both sides by the LCM of the fractions

Example 10

Solve for x:

$$\frac{2x+3}{5} - \frac{3x-1}{2} = 2$$

Solution

Method 1 – Cross multiplication

$$\frac{2x+3}{5} - \frac{3x-1}{2} = 2$$
$$\frac{2(2x+3) - 5(3x-1)}{10} = 2$$
$$4x + 6 - 15x + 5 = 2 \times 10$$
$$-11x + 11 = 20$$
$$-11x = 20 - 11$$
$$-11x = 9$$
$$x = -\frac{9}{11}$$

Method 2 – Multiplying both sides by the LCM.

$$\frac{2x+3}{5} - \frac{3x-1}{2} = 2$$
$$\frac{10(2x+3)}{5} - \frac{10(3x-1)}{2} = 10 \times 2$$
$$2(2x+3) - 5(3x-1) = 20$$
$$4x + 6 - 15x + 5 = 20$$
$$-11x + 11 = 20$$
$$-11x = 20 - 11 = 9$$
$$x = -\frac{9}{11}$$

Example 11

Solve for x:

$$\frac{2x+5}{3} - \frac{x}{2} = \frac{5x-1}{4}$$

Solution

Method 1- Cross multiplication

$$\frac{2x+5}{3} - \frac{x}{2} = \frac{5x-1}{4}$$
$$\frac{2(2x+5) - 3x}{6} = \frac{5x-1}{4}$$
$$\frac{4x + 10 - 3x}{6} = \frac{5x-1}{4}$$
$$\frac{x+10}{6} = \frac{5x-1}{4}$$
$$6(5x-1) = 4(x+10)$$
$$30x - 6 = 4x + 40$$
$$30x - 4x = 6 + 40$$
$$26x = 46$$
$$x = \frac{46}{26} = \frac{23}{13}$$

Method 2 – Multiplying both sides by the LCM.

$$\frac{2x+5}{3}-\frac{x}{2}=\frac{5x-1}{4}$$
$$12\times\frac{2x+5}{3}-12\times\frac{x}{2}=12\times\frac{5x-1}{4}$$
$$4(2x+5)-6x=3(5x-1)$$
$$8x+20-6x=15x-3$$
$$2x-15x=-20-3$$
$$-13x=-23$$
$$x=\frac{-23}{-13}$$
$$x=\frac{23}{13}$$

LINEAR INEQUALITIES

If the equal sign of an equation is replaced by any of the four signs shown below, then we have an inequality or an inequality.

$>$	Greater than
$\geq$	Greater than or equal to
$<$	Less than
$\leq$	Less than or equal to

Inequalities differ from equations in that they do not have unique solutions. The variable (or unknown) may have many solutions which are really restricted to a specific range. So, when we solve an inequality, we seek to determine the range of values that the variable can take.

Solution of inequalities

Consider, $2x > 6$
Divide both sides by 2, $x > 3$
In set builder notation, $\{x : x > 3\}$

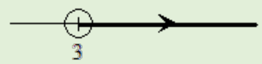
The solution is expressed simply as $x > 3$ or we may use set builder notation and write the solution as, $\{x : x > 3\}$.

To give a more precise description of the solution, we need to define the variable, x. If x is an integer, a natural number or a whole number, the solution is restricted and it is best described by listing the members.

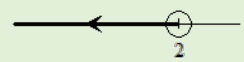
If x represents a whole number (W), then we can describe the solution as members of the set
$x = \{4, 5, 6, \ldots\}$.

If x represents a real number (R), it is impossible to list all the members because there will be fractions and irrational numbers, for example, that belong to the solution set. In such a case, there is an infinite number of solutions and we cannot list all of them. In fact, we cannot even list the first or last one of them. Hence, we express the answer by drawing a graph in one variable (number line) to illustrate the region in which the solution lies.

The graph of $x > 3$



The graph of $x < 2$

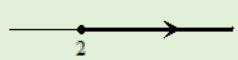


To illustrate $x > 3$, the arrow is drawn to the right of 3 to illustrate that the solution set comprises numbers that are greater than 3. We place a small un-shaded circle around the point that specifies the position of 3, which is the starting point of the solution. The un-shaded circle indicates that 3 is not to be included in the solution set.

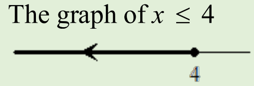
To illustrate $x < 2$, the arrow is drawn to the left of 2 on the number line and this shows the set of solutions are less than 2. The small un-shaded circle at 2 indicates that 2 is not included in the solution set.
The solution is written as, $x < 2$ or in set builder notation, $\{x : x > 2\}$.

When the inequality is of the type $x \geq 2$, we use a shaded circle at 2 to
to indicate that 2 is contained in the solution.

The graph of $x \geq 2$



The graph of $x \leq 4$



The techniques for solving inequalities are the same as those for solving equations. However, we must be careful when using the multiplicative inverse as we shall soon see in the examples below.

Example 12

Solve for x (a) $\frac{x}{3}+1 \geq 7$ (b) $\frac{x}{2}+5>8$

Solution

(a)	(b)
$\frac{x}{3}+1\geq 7$	$\frac{x}{2}+5>8$
$\frac{x}{3}\geq 7-1$	$\frac{x}{2}>8-5$
$\frac{x}{3}\geq 6$	$\frac{x}{2}>3$
$\times 3$	$\times 2$
$x\geq 6\times 3$	$x>3\times 2$
$x\geq 18$ or $\{x: x\geq 18\}$	$x>6$ or $\{x: x>6\}$

When we are not given the number set from which the variable comes, solutions to inequalities are best given using set builder notation.

Example 13

Solve for x
$4x + 1 \leq \boldsymbol{x+13}$

Solution

$$4x+1\leq x+13$$
$$4x-x\leq 13-1$$
$$3x\leq 12$$
$$\div 3$$
$$x\leq 4 \text{ or } \{x: x\leq 4\}$$

Example 14

Solve for x

$$2(3x-1)>2(x+5)$$

Solution

$$2(3x-1)>2(x-5)$$
$$6x-2>2x-10$$
$$4x>-8$$
$$\div 4$$
$$x>-2 \text{ or } \{x: x>-2\}$$

In the above examples, we multiplied or divided across the inequality by a positive number in isolating the unknown, x. We can illustrate this process using numbers as shown below.

$4>1$	$20>6$
$\times 3$	$\div 2$
$3\times 4>3\times 1$	$\frac{20}{2}>\frac{6}{2}$
$12>3$	$10>3$

In each case, we started with a true statement and we ended up with a true statement. Now let us examine what happens when we multiply or divide an inequality by a negative quantity.

$5>2$	$-6<10$
Multiply both sides by -1	Divide both sides by -2
LHS: $-1\times 5=-5$	LHS: $-6\div -2=3$
RHS: $-1\times 2=-2$	RHS: $10\div -2=-5$
But $-5<-2$	But $3>-5$

Hence, if we multiply or divide across an inequality by a negative value, the inequality becomes false. To correct this, we change the direction of the inequality. In the following examples, note how this rule is applied in line 4.

$3-4x<9$	$2+x>3x+8$
$-4x<9-3$	$x-3x>8-2$
$-4x<6$	$-2x>6$
$\times -1$	$\times -1$
$4x>-6$	$2x<-6$
$\div 4$	$\div 2$
$x>-1\frac{1}{2}$	$x<-3$

Rules for solving linear inequalities

1. When multiplying or dividing both sides of an inequality by a positive quantity, the direction of the inequality remains unchanged.

2. When multiplying or dividing both sides of an inequality by a negative quantity we reverse its direction.

3. Attempting to multiply or divide across an inequality by a variable whose sign is unknown, must be avoided at all times, for we are unsure as to whether the sign of the inequality may or may not be preserved.

Sometimes in solving an inequality, it may be convenient to avoid dividing by negative coefficients. This is especially useful in examples such as:

$2-5x \ge 7$	$12+3x < 9+4x$	$4-3(x+2) > 7$
$2-7 \ge 5x$	$12-9 < 4x-3x$	$4-3x-6 > 7$
$-5 \ge 5x$	$3 < x$	$-2-7 > 3x$
$5x \le -5$	$x > 3$	$-9 > 3x$
$x \le -1$		$3x < -9$
		$x < -3$

Notice that in each case we chose to transpose the variable, x, to the right-hand side so that it is positive. Note in the last line of the solution we prefer to rewrite the inequality so that the variable is on the left-hand side.