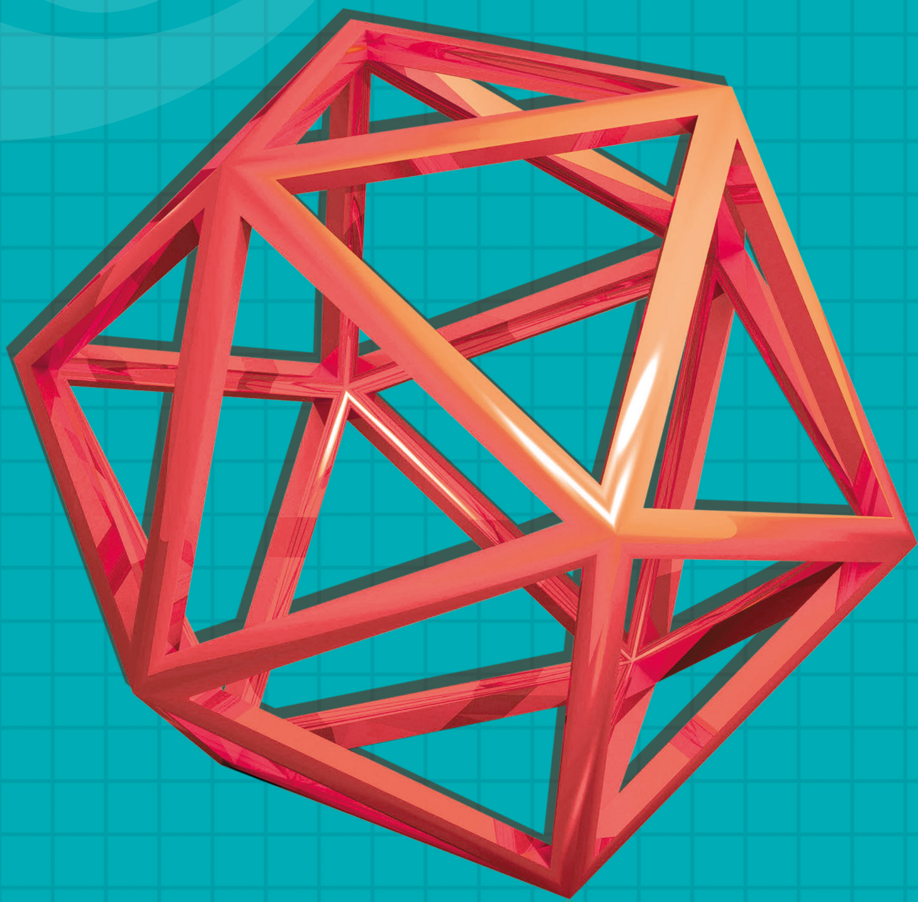
DIPCHAND BAHALL

PURE MATHEMATICS Unit 1
FOR CAPE® EXAMINATIONS



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Contents

MODULE 2 TRIGONOMETRY AND PLANE GEOMETRY

MODULE 3 CALCULUS I

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Introduction

These two volumes provide students with an understanding of pure mathematics at the CAPE® level taken from both a theoretical and an application aspect and encourage the learning of mathematics. They provide the medium through which a student can find problems applied to different disciplines. The concepts are developed step by step; they start from the basics (for those who did not do additional mathematics) and move to the more advanced content areas, thereby satisfying the needs of the syllabus. Examination questions all seem to have answers that are considered 'nice' whole numbers or small fractions that are easy to work with; not all real-world problems have such answers and these books have avoided that to some extent. Expect any kind of numbers for your answers; there are no strange or weird numbers.

The objectives are outlined at the beginning of each chapter, followed by the keywords and terms that a student should be familiar with for a better understanding of the subject. Every student should have a section of their work book for the language of the subject. I have met many students who do not understand terms such as 'root' and 'factor'. A dictionary developed in class from topic to topic may assist the students in understanding the terms involved. Each objective is fulfilled throughout the chapters with examples clearly explained. Mathematical modelling is a concept that is developed throughout, with each chapter containing the relevant modelling questions.

The exercises at the end of each section are graded in difficulty and have adequate problems so that a student can move on once they feel comfortable with the concepts. Additionally, review exercises give the student a feel for solving problems that are varied in content. There are three multiple choice papers at the end of each Unit, and at the end of each module there are tests based on that module. For additional practice, the student can go to the relevant past papers and solve the problems given. After going through the questions in each chapter, a student should be able to do past paper questions from different examining boards for further practice.

A checklist at the end of each chapter enables the student to note easily what is understood and to what extent. A student can identify areas that need work with proper use of this checklist. Furthermore, each chapter is summarised as far as possible as a diagram. Students can use this to revise the content that was covered in the chapter.

The text provides all the material that is needed for the CAPE® syllabus so that teachers will not have to search for additional material. Both new and experienced teachers will benefit from the text since it goes through the syllabus chapter by chapter and objective to objective. All objectives in the syllabus are dealt with in detail and both students and teachers can work through the text, comfortably knowing that the content of the syllabus will be covered.

Mathematical Modelling

A mathematical model is a mathematical description of the behaviour of some real-life system or some aspect of a real-life system. The mathematical model may be used to find an optimal solution to a problem, answer a number of questions, date fossils by analysing the decay of radioactive substance or model the population change of a community. The mathematical model makes use of identifying variables, setting up equations or inequalities and stating any assumptions and limitations within the model.

In considering the motion of a ball when thrown at an angle to the horizontal, we may ignore the weight of the ball, treat the ball as a particle (we ignore the size of the ball) and assume that there is no resistance to motion. We simplify the problem and create a mathematical model to find distances travelled, maximum height reached by the ball or any aspect of the motion that interests us.

Construction of a mathematical model for a system

Step 1

Identify all variables that are responsible for changing the system.

Step 2

State any assumptions or hypotheses about the system.

Step 3

Formulate the problem in mathematical terms: equations, inequalities etc.

Step 4

Solve the mathematical problem.

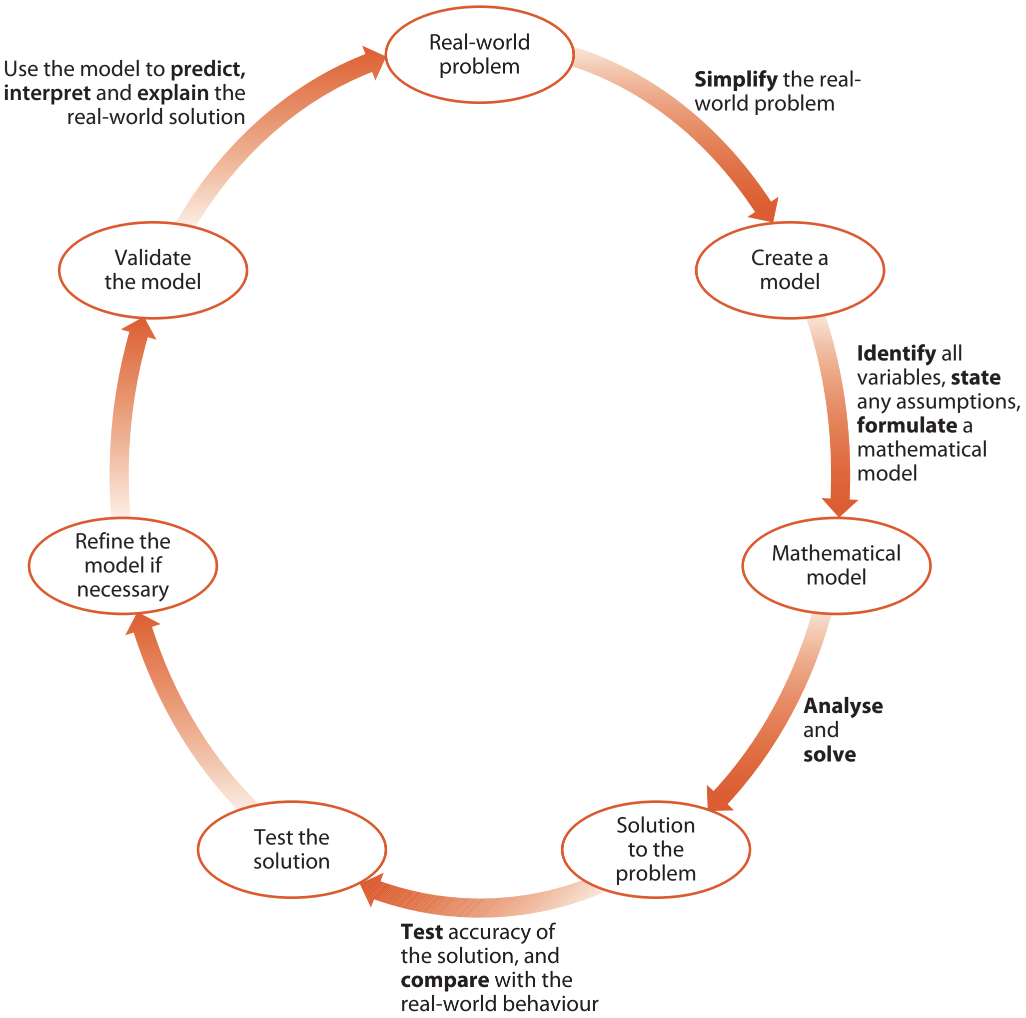
Step 5

Test the accuracy of the solution against the real-world behaviour.

Step 6

Refine the model if necessary.

Mathematical modelling is not restricted to any particular topic. The real-world situation being analysed can be based on topics inclusive of algebra, trigonometry, calculus and statistics or a combination of these. Throughout the text there are modelling questions for the relevant chapters.


Real-world problem
Simplify the real-world problem
Create a model
Identify all variables, **state** any assumptions, **formulate** a mathematical model
Mathematical model
Analyse and **solve**
Solution to the problem
Test accuracy of the solution, and **compare** with the real-world behaviour
Test the solution
Refine the model if necessary
Validate the model
Use the model to **predict, interpret** and **explain** the real-world solution

1

Basic Algebra and Functions

CHAPTER 1

Reasoning and logic

At the end of this section you should be able to:

- Identify simple and compound propositions
- Identify connectives (conjunction, disjunction, negation, conditional, biconditional)
- Draw truth tables and identify the truth value of compound statements
- State the converse of a conditional statement
- State the contrapositive of a conditional statement
- State the inverse of a conditional statement
- Identify the logical equivalence of statements

KEYWORDS/TERMS

proposition • statement • simple statement • compound statement • connectives • conjunction • disjunction • negation • conditional • biconditional • converse • contrapositive • inverse • logical • equivalence • truth table • tautology • contradiction • contingency

Mathematics has been called the 'universal language'. All equations and calculations are interpreted in the same manner in any country of the world. 'Reasoning and logic' is an area that demands your understanding of this 'language'. It is a branch of mathematics that tests one's ability to manipulate words and letters. Indeed, a certain amount of 'common sense' is required.

Consider this statement.

Today is Monday.

DEFINITION

A **proposition** is a statement that makes a declaration that is either true or false, but not both.

Is this a true statement? Is it false? To make any inference, one has to 'investigate' the 'facts'. If this statement is read on Monday, then obviously it is true. If it is read on any other day, it becomes false. At no time can it be true and false.

The following pages are aimed to ensure a comprehensive understanding of this sometimes confusing and misleading topic. We show how the use of simple grammar can alter the meaning, interpretation and validity of a statement. It will help you to appreciate the completeness and lack of ambiguity found in mathematical logic and reasoning.

EXAMPLE 1

Are these propositions?

(a) Port-of-Spain is the capital of Trinidad and Tobago.

(b) Rain is falling. **(c)** $4 + 5 = 12$ **(d)** $17 + 15 = 32$

SOLUTION

All of the statements are propositions. They make a declaration that can be true or false. **(a)** and **(d)** are both true, whereas proposition **(c)** is false. **(b)** will be true if rain is falling and false if it is not falling.

EXAMPLE 2

Are these propositions? Give a reason for each of your answers.

(a) Where are you?

(b) $2x + 3 = 12$

(c) $x + y + z = 4$

(d) What is an even number?

(e) $\sqrt{2}$ is a rational number.

SOLUTION

(a) 'Where are you?' is not a proposition since it does not declare anything.

(b) $2x + 3 = 12$ is not a proposition. It contains a variable x, so we cannot tell whether it is true or false.

(c) $x + y + z = 4$ is not a proposition. It contains a variables x, y and z, so we cannot tell whether it is true or false.

(d) This is not a proposition since we cannot determine whether it is true or false.

(e) This is a proposition as we can declare it true or false. Since $\sqrt{2}$ is an irrational number, the proposition is false.

Note

The truth or falsity of a propositional statement is called its **truth value**.

Notation

Letters are used to denote propositional variables. If the proposition is true, its value is denoted by T or 1. If the proposition is false, its truth value is denoted by F or 0.

For example, we can denote the statement 'π is irrational' by the letter p. Its truth value is T since this proposition is true.

Simple statement

A **simple statement** is a statement that cannot be decomposed into separate statements. 'The grass is green' is a simple statement. This statement cannot be separated into separate statements.

Negation

DEFINITION

Let q be a proposition.
The **negation** of q is denoted by $\sim q$ (or $-q$ or $\bar{q}$) and is the statement 'it is not the case that q'.
The truth value of $\sim q$ (read as 'not q') is the opposite of the truth value of q.

EXAMPLE 3 Find the negation of the proposition 'The rain is falling.'

SOLUTION The negation is 'It is not the case that the rain is falling.' We can write this as 'The rain is not falling.'

EXAMPLE 4 Find the negation of the proposition '6 is an even number'.

SOLUTION The negation is '6 is not an even number' or '6 is odd'.

We can let p be the proposition '6 is an even number' and write $\sim p$ is the proposition '6 is not an even number'.

Truth tables

A convenient way to identify the truth value of propositions is to set up a table identifying the truth value of each statement. This table is called a truth table. A truth table for the negation is as follows:

p	$\sim p$
T	F
F	T

Further we can substitute 0s and 1s to rewrite the truth table as follows.

p	$\sim p$
1	0
0	1

In a truth table, the first row identifies the statements we are interested in (in this case p and $\sim p$). The following rows give the truth value of the statement and its negation. For example, in the second row we see that when p is true, $\sim p$ is false. The third row shows the reverse scenario.

EXAMPLE 5 Draw a truth table for the negation of 'the answer is wrong'.

SOLUTION Let p be the statement 'the answer is wrong'. $\sim p$ is the statement 'the answer is not wrong' and the truth table is:

p	$\sim p$
T	F
F	T

Try these 1.1 Write the negation of the following statements.

(a) Today is Sunday. (b) The music is loud.

(c) Lorraine teaches Physics.

Compound statements

A **compound statement** is formed from two or more simple statements. The simple statements are called parts or components of the compound statements.

The truth value of a compound statement is determined by the truth value of the simple statements that make up the compound statement as well as the way in which they are connected. The compound statement '$2 + 4 = 6$ and $5^2 = 25$' is a compound statement formed from the simpler statements '$2 + 4 = 6$' and '$5^2 = 25$'. The truth of an 'and' compound statement is related to the truth of the simple statement as follows: The compound statement is true if both simple statements are true and is false if any one of the simple statements is false.

EXAMPLE 6 Are the following compound statements? If so, identify the components of the statement.

(a) The book has 200 pages and the book is yellow.

(b) Trishan ate the cake.

(c) $\sqrt{2}$ is irrational and 5 is odd.

SOLUTION

(a) The book has 200 pages and the book is yellow is a compound statement which has components 'The book has 200 pages' and 'the book is yellow'.

(b) 'Trishan ate the cake' is not a compound statement since it cannot be broken down into simpler parts.

(c) '$\sqrt{2}$ is irrational and 5 is odd' is a compound statement which has components '$\sqrt{2}$ is irrational' and '5 is odd'.

Connectives

Connectives are used for making compound propositions. The main connectives are negation, conjunction, disjunction, implication and biconditional.

Conjunction

DEFINITION

The proposition 'p and q' is called the conjunction of p and q and is denoted by $p \wedge q$. The statement is true when both p and q are true and is false otherwise.

Any two statements can be joined by the word 'and' to form a compound statement. This is the coordinating **conjunction** of the original statements.

The proposition 'p and q' is called the conjunction of p and q and is denoted by $p \wedge q$.

The truth value of the compound statement p and q is true if both p and q are true. It is false if either p or q is false.

Let us draw a truth table for $p \wedge q$.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

EXAMPLE 7

Draw a truth table for the following compound statement: 'Bridgetown is the capital of Barbados and Castries is the capital of St Lucia.'

SOLUTION

Let p be the statement 'Bridgetown is the capital of Barbados' and q be the statement 'Castries is the capital of St Lucia'.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

If Bridgetown is the capital of Barbados and Castries is the capital of St. Lucia, then the statement is true.

If Bridgetown is the capital of Barbados and Castries is not the capital of St Lucia, then the statement is false.

Similarly, if Bridgetown is not the capital of Barbados and Castries is the capital of St Lucia, then the compound statement is false.

Lastly, if both statements are false, then the compound statement is false.

EXAMPLE 8

Write the following in symbolic form.

(a) The exam is difficult and rain is falling.

(b) The coconut is green and the water is cold.

SOLUTION

(a) Let p be 'the exam is difficult' and q be 'rain is falling'.
We write $p \wedge q$ for 'The exam is difficult and rain is falling.'

(b) Let c be 'the coconut is green' and w be 'the water is cold'.
'The coconut is green and the water is cold' can be written as $c \wedge w$.

Disjunction ('or')

DEFINITION

The proposition 'p or q' is called the **disjunction** of p and q and is denoted by $p \vee q$. This statement is true when p is true, q is true or both p and q are true. It is false only when p and q are both false.

The compound proposition '$2 + 4 = 6$ or $3 + 5 = 10$' is true when either of the statements are true or when both are true. Since '$2 + 4 = 6$' is true and '$3 + 5 = 10$' is false, then the compound proposition is true.

Let p and q be two propositions.
'p or q' is false when both p and q are false and is true otherwise.

When the word 'or' is used in the English language, it is used in a similar way. When both components of a compound statement are true, the disjunction is true; and when any one of the statements is true, the disjunction is also true. It is inclusive in this case.

There are times when 'or' can be used in an exclusive sense. When the exclusive 'or' is used, the proposition 'p or q' is true when either p is true or q is true. The proposition is false when both p and q are true as well as when both p and q are false.

Assume that the disjunctions below are inclusive, unless stated otherwise.

DEFINITION

Let p and q be two propositions. The proposition $p \oplus q$ is the exclusive 'or' of p and q. This proposition is true when one and only one of p and q is true. It is false otherwise.

Truth table for disjunction ($p \vee q$)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p \vee q$ is false when both p and q are false and $p \vee q$ is true for all other cases.

EXAMPLE 9 Draw a truth table to represent the truth values of the proposition: 'Faheim studied applied mathematics in sixth form or he lived in Couva.'

SOLUTION Let p be the proposition 'Faheim studied applied mathematics in sixth form' and q be the proposition 'Faheim lived in Couva.'

The proposition $p \vee q$ is false when both p and q are false and is true otherwise. If Faheim studied applied mathematics in sixth form, then the compound statement is true. If Faheim lived in Couva, the compound statement is also true. If he studied mathematics in sixth form and lived in Couva, the compound statement is again true. The statement $p \vee q$ is false only if Faheim did not study applied mathematics in sixth form and if he did not live in Couva.

This is the truth table.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

EXAMPLE 10 Write the following in symbolic form.

(a) Laura scored 6 runs or the sun is shining.

(b) Calculus is easy or the printer is not working.

SOLUTION

(a) Let x be 'Laura scored 6' and y be 'the sun is shining'. The statement 'Laura scored 6 runs or the sun is shining', can be written as $x \vee y$. The two statements are connected by the disjunction.

(b) Let p be 'calculus is easy' and q be 'the printer is not working'. The whole statement can now be written as $p \vee q$.

Try these 1.2

(a) Write the following statements in symbolic form.

(i) The measuring cylinder is full or the beaker is empty.

(ii) Breakfast is at the house and the coffee is hot.

(b) Write the negation of the following statements.

(i) The air-condition unit is broken.

(ii) December 25th is Christmas.

De Morgan's laws

For any two statements p and q:

(a) $\sim(p \wedge q)$ is the same as $\sim p \vee \sim q$.

(b) $\sim(p \vee q)$ is the same as $\sim p \wedge \sim q$.

EXAMPLE 11 Let p and q and q be the statements:

p: Nikki is a singer

q: Nikki lives in Trinidad.

Write the following using symbols.

(a) Nikki is a singer and she lives in Trinidad.

(b) Nikki is neither a singer nor does she live in Trinidad.

(c) Either Nikki is a singer or she lives in Trinidad.

SOLUTION

(a) This compound statement is joined by the conjunction 'and'. Hence, we can write the statement as $p \wedge q$.

(b) Nikki is not a singer is the negation of Nikki is a singer and this is represented by $\sim p$.

'Nikki does not live in Trinidad' is the negation of 'Nikki lives in Trinidad' and this is represented by $\sim q$.

The statement is joined by the conjunction 'and'.

We write: $\sim p \wedge \sim q$.

From De Morgan's law we can also write this as: $\sim(p \vee q)$.

(c) The compound statement is joined by 'or'. Hence, the disjunction is used and we write: $p \vee q$.

EXERCISE 1A

1 Determine whether each of the following sentences is a proposition.

(a) Today is Saturday.

(b) How far do you live from your school?

(c) All CAPE students must take Caribbean Studies.

(d) Drive to the supermarket.

(e) Curry chicken is delicious.

2 Write each of the following statements in symbolic form.

(a) This is May, and CAPE examinations must begin.

(b) I will take Spanish or Additional Mathematics.

(c) The music is loud, but Ryan is speaking.

(d) I will not drive to Montego Bay. However, I will go by bus or plane.

3 Is the following statement a disjunction, conjunction or negation?

The cup is broken and the coffee is cold.

4 Let p be 'Alvin is tall' and q be 'Sintra is short'. Write the following in words.

(a) $p \vee \sim q$

(b) $\sim(p \wedge q)$

(c) $p \wedge \sim q$

(d) $\sim(p \vee q)$

5 Write the negation of each of the following.

(a) 4 is a complete square.

(b) The iPod is white.

(c) Robin does not like to work overtime.

(d) $\sqrt{7}$ is irrational.

6 Let s be 'statistics is difficult' and p be 'probability is easy'. Write the negation of each of the following in words.

(a) $s \vee p$

(b) $s \wedge \sim p$

(c) $\sim s \wedge p$

7 Let p be 'Jamaica is beautiful' and q be 'Watson likes jerk chicken'. Write the negation of each of the following in words.

(a) $p \vee q$

(b) $p \wedge \sim q$

(c) $\sim p \wedge \sim q$

8 Let x be 'Jassodra lives in England' and y be ' Jassodra takes yoga classes'. Write each statement using symbols.

(a) Jassodra lives in England and takes yoga classes.

(b) Jassodra neither lives in England nor takes yoga classes.

(c) It is not the case that Jassodra lives in England and takes yoga classes.

9 Let p be 'I will drive to school' and q be 'I will arrive late'.

Form the disjunction of p and q and discuss its truth values.

10 Let r be 'I passed 8 CSEC subjects' and c be 'I will be in sixth form'.

Form the conjunction of r and c and discuss its truth values.

Conditional statements

DEFINITION

Let p and q be propositions. The conditional statement or implication $p \to q$ is the proposition 'p implies q' or 'if p, then q'.

Many mathematical statements are implications or conditional statements. These are statements of the form 'p implies q' where p and q are propositions. In the statement '$p \to q$', p is called the hypothesis (antecedent or premise) and q is called the conclusion (or consequent).

The conditional statement $p \to q$ is false only when p is true and q is false. $p \to q$ is true otherwise.

The conditional statement is false only when the antecedent is true and the consequent is false otherwise the statement is true.

Truth table for $p \to q$

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

EXAMPLE 12 Draw the truth table for the following statement.

If it rains, then the grass is green.

SOLUTION

It rains	The grass is green	If it rains, then the grass is green.
T	F	F
T	T	T
F	F	T
F	T	T

If the statement 'it rains' is true and 'the grass is not green', then the statement becomes false. (The statement implies that once it rains the grass should be green.)

If both statements are true, then the implication is true as well.

Similarly, if it does not rain and the grass is not green, then the implication is again true.

Finally, if it does not rain, it is possible that the grass could still be green, and the implied statement is still true.

Interpretation of $p \to q$

There are many ways of interpreting $p \to q$. Some of these are:

'p implies q'

'if p, then q'

'q is necessary for p'

'a sufficient condition for q is p'

'p only if q'

'q when p'

'q unless not p'

EXAMPLE 13 Let p be the statement 'Ian learns calculus' and q be the statement 'Ian will pass Pure Mathematics'. Express the statement $p \to q$ in words.

SOLUTION We can write $p \to q$ in any of the following ways.

- 'If Ian learns calculus, then he will pass Pure Mathematics.'
- 'Ian will pass Pure Mathematics, when he learns calculus.'
- 'Ian will pass pure Mathematics unless he does not learn calculus.'

EXAMPLE 14 Give the truth value of each of the following.

(a) If Kingston is in Jamaica, then $2 + 2 = 5$. **(b)** If $5 > 6$, then $5^2 = 25$.

(c) If $\sqrt{2}$ is irrational, then London is in England.

SOLUTION These are all conditional statements of the form $p \to q$, where p is the antecedent and q is the consequent.

Note

If you start with a false assumption, you can prove anything to be true.

(a) Since 'Kingston is in Jamaica' is true and '$2 + 2 = 5$' is false, the statement 'If Kingston is in Jamaica, then $2 + 2 = 5$' is false.

(b) Since '$5 > 6$' is false, the statement '$5 > 6$, then $5^2 = 25$' is true.

(c) Since '$\sqrt{2}$ is irrational' is true and 'London is in England' is true, the statement 'If $\sqrt{2}$ is irrational, then London is in England' is true.

The contrapositive

The contrapositive of the implication $p \to q$ is the implication $(\sim q) \to (\sim p)$.

EXAMPLE 15 What is the contrapositive of the conditional statement 'Ryan passes Mathematics whenever he studies'?

SOLUTION We can write this statement as 'If Ryan studies, then he will pass Mathematics'. The contrapositive of this conditional statement is 'Ryan does not pass Mathematics, then he did not study'.

The contrapositive has the same truth value as $p \to q$.

Truth table for the contrapositive

p	q	$\sim p$	$\sim q$	$(\sim q) \to (\sim p)$
T	F	F	T	F
T	T	F	F	T
F	F	T	T	T
F	T	T	F	T

Converse

Let p and q be propositions. The proposition $q \to p$ is called the converse of $p \to q$.

EXAMPLE 16 Write the converse of the following statement: 'If 6 is an integer, then 12 is also an integer.'

SOLUTION The converse of this statement is: 'If 12 is an integer, then 6 is also an integer.'

Inverse

The proposition $\sim p \to \sim q$ is the inverse of $p \to q$.

EXAMPLE 17 Let p be 'I will go to the party' and let q be 'the DJ is good'. Write the following in symbols and words.

(a) If q then p

(b) The converse of $q \to p$

(c) The inverse of $q \to p$

(d) The contrapositive of $q \to p$

SOLUTION

(a) $q \to p$

'If the DJ is good, then I will go to the party.'

(b) The converse of $q \to p$ is $p \to q$.

'If I will go to the party, then the DJ is good.'

(c) The inverse of $q \to p$ is $\sim q \to \sim p$.

'If the DJ is not good, then I will not go to the party.'

(d) The contrapositive of $q \to p$ is $\sim p \to \sim q$.

'If I do not go to the party, the DJ is not good.'

Equivalent propositions

When two propositions have the same truth value, they are called equivalent.

If p and q are equivalent we write $p \Leftrightarrow q$.

We say that a conditional statement and its contrapositive are equivalent. The truth value of the converse and the inverse are the same, and therefore they are logically equivalent.

EXAMPLE 18 Draw a truth table for each of the following.

(a) $\sim p \vee q$

(b) $\sim(p \wedge \sim q)$

(c) Are these two propositions logically equivalent?

SOLUTION

(a)

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(b)

p	q	$\sim q$	$p \wedge \sim q$	$\sim(p \wedge \sim q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

(c) For two propositions to be logically equivalent, their truth values must be the same. From the two tables we have the truth values as follows.

$\sim p \vee q$	$\sim(p \wedge \sim q)$
T	T
F	F
T	T
T	T

Since the truth values are the same, $\sim p \vee q$ is logically equivalent to $\sim(p \wedge \sim q)$.

EXAMPLE 19 Use truth tables to show that the statements $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

SOLUTION Let us draw a truth table for $p \vee (q \wedge r)$ and one for $(p \vee q) \wedge (p \vee r)$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

p	q	r	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	F	F	F

The last column of the two tables have the same truth values. Hence, the two statements are logically equivalent.

Biconditional statements

The statement 'If the calculator is working, then I will solve the problem, and if I solve the problem, then the calculator is working' uses the conditional statement twice. A statement which uses the conditional statement twice is said to be **biconditional**.

Note

$p \leftrightarrow q$ is true when p and q have the same truth values and is false otherwise.

DEFINITION

Let p and q be propositions. The proposition 'p if and only if q' denoted by $p \leftrightarrow q$ or $(p \Leftrightarrow q)$ is called the biconditional statement. The biconditional statement can also be written as $p \rightarrow q \wedge q \rightarrow p$.

Let us draw a truth table for $p \leftrightarrow q$.

$p \leftrightarrow q$ states that $p \rightarrow q$ and $q \rightarrow p$, that is $p \rightarrow q \wedge q \rightarrow p$.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

When $p \rightarrow q$ and $q \rightarrow p$ is true, $p \leftrightarrow q$ is also true otherwise $p \leftrightarrow q$ is false.

Remember

Statements A and B are logically equivalent if they have identical truth tables. To show that two statements are logically equivalent we can draw the truth tables for each and look at the last column. Once the last columns of both are the same, they are logically equivalent.

EXAMPLE 20 Draw a truth table for the following statement.

You ride to school if and only if you have a bicycle.

SOLUTION Let p be the statement 'you ride to school' and q be the statement 'you have a bicycle'.

This is the truth table.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

EXAMPLE 21 Is the statement $(20 + 8 = 28) \leftrightarrow (7 \times 8 = 56)$ true or false?

SOLUTION This is a biconditional statement and the biconditional statement is true when both statements have the same truth value. Since $(20 + 8 = 28)$ is true and $(7 \times 8 = 56)$ is also true, then the given biconditional statement is true.

EXAMPLE 22 Is the statement 'All dogs can talk if and only if all donkeys can fly' true or false?

SOLUTION This statement is of the form 'p if and only if q' where p is 'all dogs can talk' is a false statement and q is 'all donkeys can fly' is also a false statement. Since the two components of this biconditional statement has the same truth value, then the given biconditional statement is true.

Try these 1.3

(a) Is the statement 'Paris is in France if and only if Rome is in Italy' true or false?

(b) Is the statement $(8 - 4 = 12) \leftrightarrow (30 + 24 = 64)$ true or false?

(c) What is the truth value of this statement?

Port of Spain is the capital of Trinidad and Tobago if and only if $\frac{4}{6} = \frac{1}{2}$.

EXAMPLE 23 Let p be 'you go to school everyday' and q be 'you get an award'. Write the following using symbols.

(a) You get an award only if you go to school everyday.

(b) Going to school everyday is a sufficient condition to get an award.

(c) To get an award, it is necessary that you go to school everyday.

(d) Going to school everyday is a necessary and sufficient condition to get an award.

(e) You do not get an award unless you go to school everyday.

SOLUTION

(a) $q \to p$ (b) $p \to q$

(c) $q \to p$ (d) $p \leftrightarrow q$

(e) $\sim p \to \sim q$

Methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments.

Tautology and contradiction

DEFINITION

A **tautology** is a compound proposition that is always true. A compound proposition that is always false is called a **contradiction**.
A **contingency** is a compound proposition that is neither a tautology nor a contradiction.

EXAMPLE 24 Show that $p \vee \sim p$ is a tautology.

SOLUTION We can draw a truth table for $p \vee \sim p$ and show that all the possibilities will be true.

p	$\sim p$	$p \vee \sim p$
T	F	T
T	F	T
F	T	T
F	T	T

From the last column, $p \vee \sim p$ is always true and hence it is a tautology.

EXAMPLE 25 Show that $p \wedge \sim p$ is a contradiction.

SOLUTION

p	$\sim p$	$p \wedge \sim p$
T	F	F
T	F	F
F	T	F
F	T	F

From the truth table, $p \wedge \sim p$ is always false. By definition, a contradiction is a compound proposition that is always false. Hence, $p \wedge \sim p$ is a contradiction.

Algebra of propositions

The following laws of the algebra of propositions allow us to simplify compound propositions.

1 *Idempotent laws*

(a) $p \vee p \equiv p$

(b) $p \wedge p \equiv p$

The idempotent laws states that 'p or p is identical to p' and 'p and p is identical to p'.

2 *Associative Laws*

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

3 *Commutative laws*

(a) $p \vee q \equiv q \vee p$

(b) $p \wedge q \equiv q \wedge p$

4 *Distributive laws*

(a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

5 *Identity laws*

(a) $p \vee F \equiv p$ or $p \vee 0 \equiv p$

(b) $p \wedge T \equiv p$ or $p \wedge 1 \equiv p$

(c) $p \vee T \equiv T$ or $p \vee 1 \equiv 1$

(d) $p \wedge F \equiv F$ or $p \wedge 0 \equiv 0$

(c) and (d) are also known as the domination laws.

6 *Complement laws*

(a) $\sim\sim p \equiv p$

(b) $\sim T \equiv F, \sim F \equiv T$

(c) $p \vee \sim p \equiv T$

(d) $p \wedge \sim p \equiv F$

These are also known as the double negation laws.

7 *De Morgan's laws*

(a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

(b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

8 *Absorption laws*

(a) $p \wedge (p \vee q) = p$

(b) $p \vee (p \wedge q) = p$

Some of the laws will remind you of the rules for sets with intersection and union. In fact if you remember your rules for sets and how they are used, you will be able to use the laws for the algebra of propositions. Even the symbols are similar. Let us look at some of them.

Sets	Propositional logic
$\cap$ intersection (and)	$\wedge$ logical and
$\cup$ union (or)	$\vee$ logical or
$A \cap B \equiv B \cap A$	$p \wedge q \equiv q \wedge p$
$A \cup B \equiv B \cup A$	$p \vee q \equiv q \vee p$
$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Let us use these results to simplify some propositions.

EXAMPLE 26 Simplify the expression $(p \vee q) \wedge (p \vee r)$.

SOLUTION

$$\begin{aligned}
(p \vee q) \wedge (p \vee r) &= (p \wedge p) \vee (p \wedge r) \vee (q \wedge p) \vee (q \wedge r) && \text{(Distributive law)}\\
&= p \vee (p \wedge r) \vee (q \wedge p) \vee (q \wedge r) && \text{(Idempotent law)}\\
&= p \vee (q \wedge p) \vee (q \wedge r) && \text{(Absorption law)}\\
&= ((p \vee q) \wedge (p \vee p)) \vee (q \wedge r) && \text{(Distributive law)}\\
&= ((p \vee q) \wedge p) \vee (q \wedge r) && \text{(Idempotent law)}\\
&= (p \wedge (p \vee q)) \vee (q \wedge r) && \text{(Commutative law)}\\
&= p \vee (q \wedge r) && \text{(Absorption law)}
\end{aligned}$$

EXAMPLE 27 Simplify $y \wedge (x \vee y')$.

SOLUTION

$$\begin{aligned} y \wedge (x \vee y') &= (y \wedge x) \vee (y \wedge y') && \text{(Distributive law)} \\ &= (y \wedge x) \vee 0 && \text{(Complement law, } y \wedge y' = 0\text{)} \\ &= (y \vee 0) \wedge (x \wedge 0) && \text{(Distributive law)} \\ &= y \wedge x && \text{(Identity property, } y \vee 0 = y, x \wedge 0 = x\text{)} \end{aligned}$$

EXAMPLE 28 Simplify the expression $p \vee (\sim p \wedge q)$.

SOLUTION

$$\begin{aligned} p \vee (\sim p \wedge q) &= (p \wedge 1) \vee (\sim p \wedge q) && \text{(Identity law)} \\ &= (p \wedge (1 \vee q)) \vee (\sim p \wedge q) && \text{(Identity law, } 1 \equiv 1 \vee q\text{)} \\ &= (p \wedge 1) \vee (p \wedge q) \vee (\sim p \wedge q) && \text{(Distributive law)} \\ &= p \vee (p \wedge q) \vee (\sim p \wedge q) && \text{(Identity law)} \\ &= p \vee (q \wedge (p \vee \sim p)) && \text{(Distributive law)} \\ &= p \vee (q \wedge 1) && \text{(Complement law)} \\ &= p \vee q && \text{(Identity law)} \end{aligned}$$

EXERCISE 1B

1. Write the contrapositive of this statement: 'If the music is good, then I will dance.'

2. Let r be 'I will go running'. Let s be 'the sun is shining'. Write the following in words.

   (a) The conjunction of r and s

   (b) The negation of s

   (c) The disjunction of r and s

3. Determine the contrapositive of each of the following statements.

   (a) If the board is clean, then the teacher will write.

   (b) Only if Chris studies, will he pass calculus.

4. Construct a truth table for $p \rightarrow \sim(q \vee p)$.

5. Show that $((p \vee (\sim q)) \wedge (\sim p \wedge q))$ is a contradiction.

6. Simplify the statement $(\sim(x \vee y)) \vee ((\sim x) \wedge y)$.

7. What are the contrapositive, the converse, and the inverse of this conditional statement?

   Trinidad and Tobago cricket team wins whenever the sun is shining.

8. Construct a truth table for the proposition $(p \vee \sim q) \rightarrow (p \wedge q)$.

9 Construct a truth table for each of the following.

(a) $\sim(\sim p \vee \sim q)$

(b) $\sim(\sim p \wedge \sim q)$

(c) $(p \vee q) \wedge \sim(p \wedge q)$

(d) $p \wedge (q \vee r)$

10 Write the following sentence as a logical expression.
You can access the staff room only if you are a member of staff or you are not a first year student.

11 Write the following as a logical expression.
You cannot watch the movie if you are under 13 years-old unless you are accompanied by an adult.

12 Let p and q be the following propositions.

p: I bought peanut butter this week.

q: I made peanut punch on Saturday.

Express each of the following propositions in words.

(a) $\sim p$

(b) $p \vee q$

(c) $p \wedge q$

(d) $\sim p \vee (p \wedge q)$

13 Let p and q be the following propositions.

p: You got over 85% in Mathematics.

q: You get an A in Mathematics.

Write the following propositions using logical connectives.

(a) You did not get over 85% in Mathematics.

(b) You got over 85% in Mathematics but you did not get an A in Mathematics.

(c) Getting over 85% in Mathematics is sufficient for getting an A in Mathematics.

(d) Whenever you get an A in Mathematics, you got over 85% in Mathematics.

14 Construct a truth table for each of the following propositions.

(a) $(p \vee \sim q) \to q$ (b) $(p \vee q) \to (p \wedge q)$

15 Show that $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ are logically equivalent.

16 Simplify $\sim(p \vee (\sim p \wedge q))$.

17 Let x be 'the mango is sweet' and y be 'the mango is yellow'. Write the following in words.

(a) The disjunction of the two statements

(b) The negation of the statement y

(c) The conjunction of the two statements

18 Show by means of a truth table that the statement $p \leftrightarrow \sim q$ is not a contradiction.

19 Construct a truth table for $(p \vee \sim q) \rightarrow (\sim p \wedge q)$.

20 By drawing a truth table prove that the statement $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$ is true.

21 Show by means of a truth table that $(p \wedge q) \rightarrow p$ is a tautology.

22 Are any of the following propositions equivalent?
$(p \vee q) \wedge \sim q, \quad (p \vee q) \wedge \sim(p \wedge q), \quad \sim(p \wedge q) \wedge (p \wedge \sim q)$

23 Propositions p and q are given by the following.

p: Deepak works out everyday.

q: Deepak has muscles.

(a) Express $p \leftrightarrow q$ in words.

(b) Given the statement $p \rightarrow q$, write the following using symbols.

  (a) The contrapositive

  (b) The inverse

  (c) The converse

24 Propositions p and q are given by the following.

p: The plums are green.

q: I will pick the plums.

Write the following in symbolic form.

(a) The plums are green but I pick the plums.

(b) The plums are green and I pick the plums.

(c) Assume that p is true and q is false. Find the truth values of the statements in (a) and (b).

25 Show that $\sim(p \wedge q) \wedge (p \vee \sim q) \equiv \sim(p \wedge q)$.

SUMMARY

Reasoning and logic

A proposition is a statement that makes a declaration that is either true or false, but not both.
A simple statement is a statement that cannot be decomposed into separate statements.
A compound statement is formed from two or more simple statements.

Algebra of propositions

Idempotent laws
(i) $p \vee p \equiv p$ (ii) $p \wedge p \equiv p$

Associative laws
(i) $(p \vee q) \vee r \equiv p \vee (q \vee r)$
(ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Commutative laws
(i) $p \vee q \equiv q \vee p$ (ii) $p \wedge q \equiv q \wedge p$

Distributive laws
(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Identity laws
(i) $p \vee F \equiv p$ (ii) $p \wedge T \equiv p$
(iii) $p \vee T \equiv T$ (iv) $p \wedge F \equiv F$

Complement laws
(i) $\sim\sim p \equiv p$ (ii) $\sim T \equiv F$, $\sim F \equiv T$
(iii) $p \vee \sim p \equiv T$ (iv) $p \wedge \sim p \equiv F$

De Morgan's laws
(i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$
(ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Absorption laws
(i) $p \wedge (p \vee q) \equiv p$
(ii) $p \vee (p \wedge q) \equiv p$

T or 1 represents true.
F or 0 represents false.

$\sim q$, q', $\bar{q}$, **negation** of q: 'It is not the case that q.'

$\wedge$, 'and' **conjunction**: $p \wedge q$ is true only when both p and q are true.

$\vee$, 'or': **disjunction**: $p \vee q$ is false when both p and q are false and it is true otherwise.

$p \rightarrow q$, 'p implies q': $p \rightarrow q$ is false only when p is true and q is false, it is true otherwise.
p is called the antecedent or premise and q the conclusion or consequent.

The **contrapositive** of $p \rightarrow q$ is $(\sim q) \rightarrow (\sim p)$.

The **converse** of $p \rightarrow q$ is $q \rightarrow p$.

The **inverse** of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

Equivalent: Two propositions have the same truth value.

Biconditional: $p \leftrightarrow q$: 'p if and only if q'
$p \leftrightarrow q$ is true when $p \rightarrow q$ and $q \rightarrow p$ are true.
$p \leftrightarrow q$ is false otherwise.

A **tautology** is a compound statement that is always true.

A **contradiction** is a compound statement that is always false.

A **contingency** is a compound statement that is neither a tautology nor a contradiction.

Truth tables
In a truth table
(i) the first row consists of the statements
(ii) the following rows give the truth value of the statements

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Checklist

Can you do these?

- Identify simple and compound propositions?
- Identify connectives (conjunction, disjunction, negation, conditional, biconditional)?
- Draw truth tables and identify the truth value of compound statements?
- State the converse of a conditional statement?
- State the contrapositive of a conditional statement?
- State the inverse of a conditional statement?
- Identify the logical equivalence of statements?

CHAPTER 2

The Real Number System

At the end of this section you should be able to:

- Define a binary operation
- Perform binary operations
- Define commutativity, associativity, identity, distributivity, inverse and closure
- Use the concept of identity, closure, inverse, commutativity, associativity and distributivity and other simple binary operations
- Construct simple proofs: direct proofs, proof by counter example
- Use the axioms of the number system including the non-existence of the multiplicative inverse of zero

KEYWORDS/TERMS

rational numbers • irrational numbers • natural numbers • whole numbers • integers • unary • binary • binary operation • commutativity • associativity • distributivity • closure • identity • inverse • self-inverse • direct proof • proof by counter example

DEFINITION

The rational numbers are numbers which can be written in the form of $\frac{a}{b}$ where a and b are integers and $b \neq 0$. The symbol $\mathbb{Q}$ is used for the set of rational numbers. $\mathbb{Q} = \{x: x = \frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0\}$ By definition, all numbers in fractional form are rational numbers. Rational numbers will include all terminating decimals and all recurring decimals.

The real number system consists of two sets of numbers: the rational numbers and the irrational numbers. The set of rational numbers consists of the natural numbers, the integers and all real numbers that can be written in fractional form. The set of irrational numbers consists of all numbers which cannot be written in fractional form.

Subsets of rational numbers

The set of natural numbers or counting numbers is denoted by $\mathbb{N}$.

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

When 0 is included in the natural numbers, we have another set of numbers called the set of whole numbers, which is denoted by $\mathbb{W}$.

$\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$

The set of integers consists of all the natural numbers, the number zero and the negative of the natural numbers. The symbol used for the set of integers is $\mathbb{Z}$.

$\mathbb{Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$

The set of positive integers is denoted by $\mathbb{Z}^+$ and the set of negative integers is denoted by $\mathbb{Z}^-$.

EXAMPLE 1

Write each of the following numbers as a fraction.

(a) $1.\dot{6}$

(b) $0.\overline{45}$

SOLUTION

(a) $1.\dot{6} = 1.6666666\ldots$

Let $x = 1.6666666\ldots$	[1]
$10x = 16.666666\ldots$	$[1] \times 10 = [2]$
$10x - x = 9x = 15$	$[2] - [1]$
$x = \frac{15}{9}$	
$x = \frac{5}{3}$	

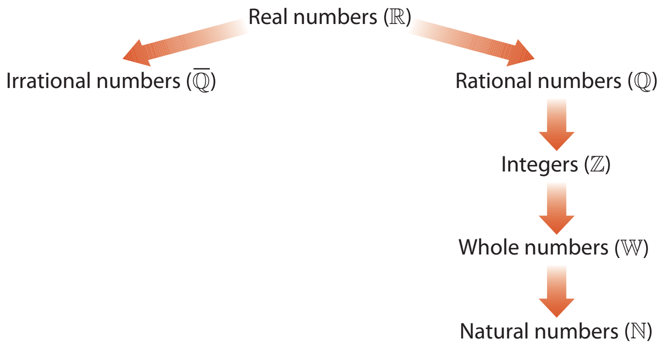
(b) $0.\overline{45} = 0.45454545\ldots$

$x = 0.45454545\ldots$	[1]
$100x = 45.454545\ldots$	$[1] \times 100 = [2]$
$99x = 45$	$[2] - [1]$
$x = \frac{45}{99}$	
$x = \frac{5}{11}$	

The irrational numbers are the numbers which cannot be written as fractions. The symbol $\overline{\mathbb{Q}}$ is used for irrational numbers such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{11}$, π.

Real numbers

The sets $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ are subsets of a larger set of numbers called the set of real numbers, which is denoted by $\mathbb{R}$.



The set of real numbers $\mathbb{R}$ is such that $\mathbb{R} = \mathbb{Q} \cup \overline{\mathbb{Q}}$.

The sets $\mathbb{N}$, $\mathbb{W}$, and $\mathbb{Z}$ are all subsets of $\mathbb{Q}$.

These relationships can be shown like this:

$\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

$\overline{\mathbb{Q}} \subset \mathbb{R}$

Operations

A unary operation is an operation that can be performed on one element of a set or on one number. Finding the square root, square and reciprocal are all examples of unary operations. Some operations such as addition, subtraction, multiplication and division must be applied to two numbers. These operations are examples of binary operations. A binary operation involves two elements or numbers. A formal definition of a binary operation follows.

DEFINITION

A binary operation $*$ on a non-empty set A is a function $*$: $A \times A \rightarrow A$.

Binary operations

Let us look at this definition using the operation of addition and the set of real numbers. If we take any two real numbers and add them together, we get a real number.

For example, $2 + 4 = 6$.

If $a \in \mathbb{R}$ and $b \in \mathbb{R}$, then $a + b \in \mathbb{R}$.

In this case, the binary operation is addition and our set is the set of real numbers.

Closure

The binary operation $*$ is closed with respect to the non-empty set A, if for all $a, b \in A$, $a * b \in A$. This property is called **closure**. We say that for any two elements in the set, if we operate on these two elements and we end up with an element in the set, then the set is closed with respect to that operation.

EXAMPLE 2 Is the set of real numbers closed with respect to addition?

SOLUTION Let $a, b \in \mathbb{R}$ and $+$ be the operation. Since a is a real number and b is a real number, then when we add these two elements, we get $a + b$ which is also a real number.

Note

$\forall$ means 'for all', 'for any', 'for each'.

Therefore, for any pair of real numbers the sum is a real number.

Hence, the set of real numbers is closed with respect to addition.

That is $\forall\ a \in \mathbb{R}, b \in \mathbb{R}, a + b \in \mathbb{R}$.

EXAMPLE 3 Show that the set of real numbers is closed with respect to multiplication.

SOLUTION Let $a, b \in \mathbb{R}$ and $\times$ be the operation. Since a is a real number and b is a real number, then when we multiply these two elements, we get $a \times b$, which is also a real number.

Therefore, for any pair of real numbers the product is a real number.

Hence, the set of real numbers is closed with respect to multiplication.

That is $\forall\ a \in \mathbb{R}, b \in \mathbb{R}, a \times b \in \mathbb{R}$.

EXAMPLE 4 Is the set of integers closed with respect to division?

SOLUTION Let $a, b \in \mathbb{Z}$ and $\div$ be the operation. When we divide two integers, our result is not necessarily an integer. Therefore, the set of integers is not closed with respect to division.

For example $5 \div 4$ is not an integer, even though both 5 and 4 are integers.

Note

When we are deciding about closure, we need two elements in the set and an operation. Also, the language is specific in how we state our conclusion. We state 'the set is closed with respect to the operation'.

Try these 2.1 Which of the following are closed? Justify your answer.

(a) Integers for multiplication

(b) Rational numbers for division

(c) Irrational numbers for addition

Commutativity

Let $a, b \in A$. A binary operation $*$ is said to be commutative on the non-empty set A if and only if $a * b = b * a$. The order in which we perform the operation does not matter and the outcome is the same.

EXAMPLE 5 Show that $+$ is commutative with respect to the set of real numbers.

SOLUTION Let $a, b \in \mathbb{R}$. Now, $a + b$ and $b + a$ give the same value. Therefore, the order in which we add real numbers does not matter. For any $a, b \in \mathbb{R}$, $a + b = b + a$. Hence, $+$ is commutative.

EXAMPLE 6 Show that $\times$ (multiplication) is commutative with respect to the set of real numbers.

SOLUTION Let $a, b \in \mathbb{R}$. Now, $a \times b$ and $b \times a$ give the same value. Therefore, the order in which we multiply real numbers does not matter. For any $a, b \in \mathbb{R}$, $a \times b = b \times a$. Hence, $\times$ is commutative.

EXAMPLE 7 Is subtraction of real numbers a commutative operation?

SOLUTION Since $4 - 9 = -5$ and $9 - 4 = 5$, and $-5 \neq 5$, subtraction is not commutative.

For an operation to be commutative, it must hold for all pairs of elements in the set. In general, for any $a, b \in \mathbb{R}$, $a - b \neq b -$ a. Hence, subtraction is not commutative.

Associativity

Let $a, b, c \in A$. The operation $*$ is associative if and only if $a * (b * c) = (a * b) * c$. The order in which pairs are grouped does not matter when an operation is associative. We get the same outcome however the elements are paired.

To decide whether an operation is associative we need three elements from the set.

EXAMPLE 8 Is the operation $+$ associative with respect to the set of real numbers?

SOLUTION Let $a, b, c \in \mathbb{R}$. When we add three real numbers, we get the same real number no matter what order we add them in. Therefore, $a + (b + c) = a + b + c = (a + b) + c$. The addition of real numbers is an associative operation.

EXAMPLE 9 Is the operation $\times$ associative with respect to the set of real numbers?

SOLUTION Let $a, b, c \in \mathbb{R}$. When we multiply three real numbers, we get the same real number no matter what order we multiply them in. Therefore, $a(bc) = abc = (ab)c$. The multiplication of real numbers is an associative operation.

Note

$a \times b = ab$

EXAMPLE 10 Is the operation subtraction associative with respect to the set of integers?

SOLUTION Now $4 - (5 - 8) = 4 - (-3) = 7$ and $(4 - 5) - 8 = -1 - 8 = -9$.

Since $4 - (5 - 8) \neq (4 - 5) - 8$, subtraction is not associative with respect to the set of integers.

Remember

If an operation is associative, the order in which we take any pair of elements must give the same outcome.

EXAMPLE 11 Is division associative over the set of natural numbers?

SOLUTION $(12 \div 4) \div 6 = \frac{1}{2}$ while $12 \div (4 \div 6) = 12 \div \left(\frac{2}{3}\right) = 18$

Since $(12 \div 4) \div 6 \neq 12 \div (4 \div 6)$, division over the set of natural numbers is not associative.

Distributivity

Let $a, b, c \in A$. For any two operations $*$ and Δ, $*$ distributes over Δ if and only if $a * (b \, \Delta \, c) = (a * b) \, \Delta \, (a * c)$

Try this 2.2 Show that multiplication distributes over addition but addition does not distribute over multiplication, i.e. $a(b + c) = ab + ac$ and $a + (b \times c) \neq (a + b) \times (a + c)$ over the set of real numbers.

EXAMPLE 12 Which of the following operations is distributive over the other for the set of numbers given?

(a) Multiplication over division for the rational numbers

(b) Two binary operations $*$ and Δ defined over the integers as follows:

$x * y = x + y + 2 \qquad x \Delta y = x + y - 2xy$

Is $*$ distributive over Δ?

SOLUTION (a) Let $x = \frac{4}{5}$, $y = \frac{3}{10}$ and $z = \frac{5}{12}$.

We first consider $x \times (y \div z)$:

$$\begin{aligned} x \times (y \div z) &= \frac{4}{5} \times \left(\frac{3}{10} \div \frac{5}{12}\right) \\ &= \frac{4}{5}\left(\frac{3}{10} \times \frac{12}{5}\right) \\ &= \frac{4}{5} \times \frac{18}{25} \\ &= \frac{72}{125} \end{aligned}$$

Now we consider $(x \times y) \div (x \times z)$:

$$\begin{aligned} (x \times y) \div (x \times z) &= \left(\frac{4}{5} \times \frac{3}{10}\right) \div \left(\frac{4}{5} \times \frac{5}{12}\right) \\ &= \frac{6}{25} \div \frac{1}{3} \\ &= \frac{6}{25} \times 3 \\ &= \frac{18}{25} \end{aligned}$$

Since $x \times (y \div z) \neq (x \times y) \div (x \times z)$ multiplication does not distribute over division for the set of rational numbers.

(b) Since we are dealing with the set of integers, we can let $x = 1$, $y = 2$ and $z = 3$.

We first consider $x * (y\Delta z)$.

When $x = 1$, $y = 2$ and $z = 3$:

$$\begin{aligned} y\Delta z &= 2 + 3 - 2(2)(3) \\ &= 5 - 12 \\ &= -7 \end{aligned}$$

$$\begin{aligned} x * (y\Delta z) &= (1) + (-7) + 2 \\ &= -4 \end{aligned}$$

Now we consider $(x * y)\ \Delta\ (x * z)$

$$\begin{aligned} x * y &= (1) + (2) + 2 \\ &= 5 \end{aligned}$$

$$\begin{aligned} x * z &= (1) + (3) + 2 \\ &= 6 \end{aligned}$$

$$\begin{aligned} (x * y)\Delta(x * z) &= (5) + (6) - 2(5)(6) \\ &= 11 - 60 \\ &= -49 \end{aligned}$$

Since we have found three integers for which $x * (y\Delta z) \neq (x * y)\ \Delta\ (x * z)$, $*$ does not distribute over Δ.

An alternative method is to choose any three integers $a, b, c \in Z$.

We find $b\Delta c = b + c - 2bc$.

Then $a * (b\Delta c) = a * (b + c - 2bc) = a + b + c - 2bc + 2$.

Now we consider $(a * b)\Delta(a * c)$

$(a * b) = a + b + 2 \qquad (a * c) = a + c + 2$

$$\begin{aligned} (a * b)\Delta(a * c) &= (a + b + 2)\Delta(a + c + 2) \\ &= (a + b + 2) + (a + c + 2) - 2(a + b + 2)(a + c + 2) \\ &= 2a + b + c + 4 - [2a^2 + 2ac + 4a + 2ab + 2bc + 4b + 4a + 4c + 8] \\ &= -6a - 3b - 3c - 4 - 2a^2 - 2ac - 2ab - 2bc \end{aligned}$$

Since $a * (b\Delta c) \neq (a * b)\Delta(a * c)$, $*$ does not distribute over Δ.

Identity

Let $*$ be a binary operation on a non-empty set A. If there exists an element $e \in A$ such that $e * a = a * e = a$, for all $a \in A$, then e is called the identity element in A.

For any real number $a \in \mathbb{R}$, $a + 0 = 0 + a = a$. Hence, 0 is the identity for addition of the real numbers.

The identity for multiplication is 1, since for any $a \in \mathbb{R}$, $1 \times a = a \times 1 = a$.

Note

Whenever the identity element operates on an element in A, the result is the element in A.

EXAMPLE 13

Let $*$ be a binary operation defined over the real numbers as follows.

$a * b = a + b - 6$, for $a, b \in \mathbb{R}$.

Is there an identity element?

SOLUTION

Let $e \in \mathbb{R}$ for any $a \in \mathbb{R}$. If e is the identity then $a * e = a$.

Since $a * b = a + b - 6$, replacing b by e, we have:

$$a * e = a + e - 6$$

Now $a * e = a$.

Therefore, $a + e - 6 = a$

$$e - 6 = 0$$

$$e = 6$$

Hence, the identity element is 6.

> **Note**
>
> If the identity exists, there will only be one identity element in a set. The identity is said to be a unique element in that set.

Try these 2.3

(a) Is there an identity element for division of the set of natural numbers? Justify your answer.

(b) Let $*$ be a binary operation defined over the real numbers as follows.

$a * b = a + 2b + 4$, for $a, b \in \mathbb{R}$

Is there an identity element? Show your working clearly.

Inverse

Let $*$ be a binary operation on a non-empty set A. Let $a, b, e \in A$. a is the inverse of b and b the inverse of a if and only if $a * b = b * a = e$ where e is the identity element in A.

> **Note**
>
> (i) An element operated on its inverse is equal to the identity.
> (ii) Since $e * e = e * e = e$, the identity is its own inverse. Any element which is its own inverse is called a **self-inverse**.

EXAMPLE 14

Show that $(-a)$ is the inverse for any $a \in \mathbb{R}$ for the operation of addition of real numbers.

SOLUTION

Let $a \in \mathbb{R}$. Since $a + (-a) = (-a) + a = 0$ and 0 is the identity for addition, $(-a)$ is the inverse for any $a \in \mathbb{R}$.

For example, $(2) + (-2) = 2 - 2 = (-2) + (2) = 0$.

The inverse of 2 is (-2).

Whenever we add an element to its inverse we get the identity, which is 0.

EXAMPLE 15 Find the inverse with respect to multiplication of any number $a \in \mathbb{R}$. Identify any exceptions.

SOLUTION Let $b \in \mathbb{R}$ be the inverse of $a \in \mathbb{R}$.

Since an element multiplied by its inverse must give the identity, $a(b) = 1$.

$\Rightarrow b = \frac{1}{a}, a \neq 0$

Every element $a \in \mathbb{R}$ has an inverse $\frac{1}{a} \in \mathbb{R}$, except when $a = 0$.

Note

The multiplicative inverse of 0 does not exist.

Try this 2.4 Can you find any element(s) that has an inverse in the set of integers with respect to the operation of multiplication?

EXAMPLE 16 The binary operation $*$ is defined on the set $\{a, b, c, d\}$ as shown in the table below.

$*$	a	b	c	d
a	d	a	b	c
b	a	b	c	d
c	b	c	d	a
d	c	d	a	b

(a) Is this operation commutative?

(b) Name the identity element or explain why none exists.

(c) For each element having an inverse, name the element and its inverse.

(d) Show that $(a * b) * c = a * (b * c)$.

SOLUTION

(a) Since the table is symmetric along the leading diagonal, the operation is commutative.

(b) The identity is b. From the table $a * b = a$, $b * a = a$, $c * b = c$, $b * c = c$, $d * b = b * d = d$

(Any element operated on the identity gives back the element itself.)

(c) Any element operated on its inverse gives the identity. The inverse of a is c and the inverse of b is b, the inverse of c is a and the inverse of d is d.

Element	Inverse
a	c
b	b
c	a
d	d

Note

b and d are called self-inverses.

Note

When proving any of closure, associativity, commutativity, distributivity etc. we need to prove that they hold for all elements and not just for one set.

(d) We have to show that $(a * b) * c = a * (b * c)$.

We first find $(a * b) * c$.

From the table, $a * b = a$ and $a * c = b$.

Therefore, $(a * b) * c = b$

Now we find $a * (b * c)$.

From the table, $b * c = c$ and $a * c = b$.

Therefore, $a * (b * c) = b$

Hence, $(a * b) * c = a * (b * c)$.

Constructing simple proofs in mathematics

Proof by exhaustion

This method of proof has limited capability. We prove the statement true for every term. We list all values and show the statement to be true. However, although listing all values works for small numbers of terms, if the number of terms is increased, listing may become cumbersome and tedious.

EXAMPLE 17 Prove that $n + 1 > n$ for all positive integers less than 10.

SOLUTION

$n = 1, 1 + 1 = 2 > 1$

$n = 2, 2 + 1 = 3 > 2$

$n = 3, 3 + 1 = 4 > 3$

$n = 4, 4 + 1 = 5 > 4$

$n = 5, 5 + 1 = 6 > 5$

$n = 6, 6 + 1 = 7 > 6$

$n = 7, 7 + 1 = 8 > 7$

$n = 8, 8 + 1 = 9 > 8$

$n = 9, 9 + 1 = 10 > 9$

Therefore, for all integers less than 10, $n + 1 > n$.

Direct proof

A direct proof can be thought of as a flow of implications beginning with 'P' and ending with 'Q'. Small steps are taken and justified at each stage until the proof is derived.

Here are two theorems and their proofs. We assume that P is true, and show that Q must be true.

Theorem

If a divides b and b divides c, then a divides c.

PROOF

By definition of divisibility:

a divides $b \Rightarrow \frac{b}{a} = k_1 \in \mathbb{Z}$

$\Rightarrow a = \frac{b}{k_1}$

b divides $c \Rightarrow \frac{c}{b} = k_2 \in \mathbb{Z}$

$\Rightarrow b = \frac{c}{k_2}$

Substitute $b = \frac{c}{k_2}$ into $a = \frac{b}{k_1}$.

$\Rightarrow a = \frac{\frac{c}{k_2}}{k_1}$

$= \frac{c}{k_1 k_2}$

$\Rightarrow \frac{c}{a} = k_1 k_2$ (Since $k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$, then $k_1 k_2 \in \mathbb{Z}$)

Hence, a divides c.

We take small steps until the proof has been derived.

Theorem

Every odd integer is the difference of two perfect squares.

PROOF

Let a be an integer.

$(a + 1)^2 = a^2 + 2a + 1$

$\Rightarrow (a + 1)^2 - a^2 = 2a + 1$

Since $a \in \mathbb{Z}$

$\Rightarrow 2a$ is an even integer.

Therefore, $2a + 1$ is an odd integer.

Since $(a + 1)^2 - a^2 = 2a + 1$, every odd integer is the difference of two perfect squares.

Q.E.D.

EXAMPLE 18 Prove that if $a > b$ then $a + c > b + c$.

SOLUTION Since $a > b$

$\Rightarrow a - b > 0$

Since $c - c = 0$

$\Rightarrow a - b + c - c > 0$

Rearrange:

$a + c - (b + c) > 0$

$\Rightarrow a + c > b + c$ Q.E.D.

We have a series of logical steps taking us to the solution.

EXAMPLE 19 Prove that the product of any two odd integers is odd.

SOLUTION Let $m = 2p + 1$ and $n = 2q + 1$, where m, n, p and q are integers.

Now

$$\begin{aligned} mn &= (2p + 1)(2q + 1) \\ &= 4pq + 2p + 2q + 1 \\ &= 2(2pq + p + q) + 1 \end{aligned}$$

Since p and q are integers, $2pq$ is also an integer and therefore $2pq + p + q$ is an integer.

Hence, $2(2pq + p + q) + 1$ is an odd integer.

Therefore, the product of two odd integers is an odd integer. Q.E.D.

EXAMPLE 20 Prove that if m is an odd integer, then m^2 is also an odd integer.

SOLUTION Let $m = 2p + 1$, where p is an integer.

$$\begin{aligned} m^2 &= (2p + 1)(2p + 1) \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \end{aligned}$$

Since p is an integer, $2p^2 + 2p$ is an integer and $2(2p^2 + 2p) + 1$ is an odd integer.

Therefore, if m is an odd integer, then m^2 is also an odd integer.

Proof by contradiction

Use proof by contradiction when you wish to prove that something is impossible. The assumption is made that your statement is possible and then you reach a contradiction.

EXAMPLE 21 Prove that $\sqrt{2}$ is irrational.

SOLUTION Let us assume that $\sqrt{2}$ is rational.

$\Rightarrow \sqrt{2} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{Z}$

$\therefore 2 = \frac{a^2}{b^2}$

$a^2 = 2b^2$

Hence, a^2 is an even number.

Since a^2 is an even number, then a is even.

Let $a = 2c$.

$\Rightarrow (2c)^2 = 2b^2$

$\therefore 4c^2 = 2b^2$

$\Rightarrow b^2 = 2c^2$

Hence, b^2 is an even number.

And hence, b is an even number.

Therefore, both a and b are even numbers, which means they have a common factor of 2.

This contradicts what we said above: 'a and b have no common factors'.

Therefore, this contradicts the assumption that $\sqrt{2}$ is rational.

$\therefore \sqrt{2}$ is irrational. Q.E.D.

Proof by counter example

Consider for all $x \in \mathbb{R}$, if P then Q. To prove this statement false find a value of x in $\mathbb{R}$ for which P is true and Q is false. Thus x is a counter example.

Given a universal statement, if we can find a single statement which is not true, then you will disprove your universal statement. You can disprove something by finding a single counter example. However, you cannot prove something by finding only one example.

EXAMPLE 22 Find a counter example to show that this statement is untrue.

For all real numbers a and b, if $b^2 > a^2$ then $b > a$.

SOLUTION Let $a = 3, b = -4$.

$\Rightarrow (-4)^2 > (3)^2, b^2 > a^2$

$16 > 9$

But:

$-4 < 3$

$\Rightarrow b < a$

Hence, the statement is false.

We have proved this by finding one case for which the statement is false.

EXAMPLE 23 Is the following statement true?

For all real numbers x, y and z, if $x > y$ then $xz > yz$.

SOLUTION Let $x = 10$, $y = 8$, $z = -3$.

$10 > 8$ is true.

$xz = 10(-3) = -30$

$yz = 8(-3) = -24$

Since $-30 < -24$

$\Rightarrow xz < yz$

Therefore, the statement is false.

EXAMPLE 24 Is the following statement true?

If n is divisible by 2 or 3, then $n(n + 1)$ is divisible by 6.

SOLUTION Let $n = 4$, which is divisible by 2.

Then $n(n + 1) = 4(4 + 1) = 20$.

20 is not divisible by 6.

Therefore, the statement is false.

EXAMPLE 25 Is the following statement true?

If n is prime, then $n + 1$ is not square.

SOLUTION Let $n = 3$, which is a prime number.

Then $n + 1 = 3 + 1 = 4$.

4 is square.

Therefore, the statement is false, since we have one example for which the statement will not hold.

Using the properties of closure, associativity, commutativity, distributivity, identity and inverse we can prove other theorems of real numbers.

Here are some examples of proofs of theorems.

Theorem

For any $a, b \in \mathbb{R}$, $a \times 0 = 0$.

PROOF

$a \times 1 = a$

$\Rightarrow a \times (1 + 0) = a$

$\Rightarrow a \times 1 + a \times 0 = a$ (Distributive law)

$\Rightarrow a + a \times 0 = a$

$\Rightarrow a \times 0 = 0$

Theorem

For any $a \in \mathbb{R}$, $-a = (-1)a$.

PROOF

We know $a \times 0 = 0$.

Therefore, $a \times (1 + (-1)) = 0$

$\Rightarrow a \times 1 + a \times (-1) = 0$ (Distributive law)

$\Rightarrow a + a \times (-1) = 0$

$\Rightarrow a \times (-1) = -a$

Theorem

For any $a, b \in \mathbb{R}$, $-(a + b) = (-a) + (-b)$.

PROOF

$-(a + b) = (-1)(a + b)$

$\Rightarrow -(a + b) = (-1) \times a + (-1) \times b$ (Distributive law)

$\Rightarrow -(a + b) = (-a) + (-b)$

Theorem

For $a \in \mathbb{R}$ and $a \neq 0$, then $\dfrac{1}{\frac{1}{a}} = a$.

PROOF

First, we show that $\frac{1}{a} \neq 0$.

We know that $1 = a \times \frac{1}{a}$.

Let $\frac{1}{a} = 0$.

$\Rightarrow 1 = a \times 0$.

This is false.

Hence, $\frac{1}{a} \neq 0$.

Since the multiplicative inverse of a is $\frac{1}{a}$ we have $\frac{1}{\frac{1}{a}} = a$.

Ordering axioms

DEFINITION

A non-empty subset of $\mathbb{R}$ is a set of strictly positive numbers ($\mathbb{R}^+$) if the following conditions are satisfied.

(i) If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$.
(ii) If $a, b \in \mathbb{R}^+$, then $a \times b \in \mathbb{R}^+$.
(iii) If $a, b \in \mathbb{R}^+$, then $a + b \in \mathbb{R}^+$ and $a - b$ or $b - a \in \mathbb{R}^+$ or $a - b = 0$ and $b - a = 0$.

Theorem

If $a > b$ and $b > c$, then $a > c$ for any $a, b, c \in \mathbb{R}$.

PROOF

$a > b \Rightarrow a - b \in \mathbb{R}^+$

$b > c \Rightarrow b - c \in \mathbb{R}^+$

$\Rightarrow a - b + b - c \in \mathbb{R}^+$

$\Rightarrow a - c \in \mathbb{R}^+$

$\Rightarrow a - c > 0$

$\Rightarrow a > c$

Hence, if $a > b$ and $b > c$, then $a > c$ for $a, b, c \in \mathbb{R}$.

Theorem

For all $a, b, c \in \mathbb{R}$, if $a > b$, then $a + c > b + c$.

PROOF

$a > b \Rightarrow a - b > 0$

Since $a - b = a - b + c - c$

$a - b = a + c - (b + c)$

$\Rightarrow a + c - (b + c) > 0$

$\Rightarrow a + c > b + c$

Hence, for all $a, b, c \in \mathbb{R}$, if $a > b$, then $a + c > b + c$.

SUMMARY

The real number system

Real numbers ($\mathbb{R}$)

Rational numbers ($\mathbb{Q}$)

Integers ($\mathbb{Z}$)

Whole numbers ($\mathbb{W}$)

Natural numbers ($\mathbb{N}$)

Irrational numbers ($\overline{\mathbb{Q}}$)

Binary operations

A binary operation * on a non-empty set A is a function *: $A \times A \to A$.

Closure: * is closed with respect to A if $\forall$ $a, b \in A$, $a * b \in A$. Addition and multiplication are closed with respect to the set of real numbers.

Commutativity: Let $a, b \in A$. * is commutative iff $a * b = b * a$.

Associativity: Let $a, b, c \in A$. * is associative iff $a * (b * c) = (a * b) * c$.

Distributivity: Let $a, b, c \in A$. For any two binary operations * and Δ, * distributes over Δ iff $a * (b \Delta c) = (a * b) \Delta (a * c)$.
Multiplication distributes over addition for real numbers.

Identitiy: Let $e, a \in A$. e is the identity element in A if $e * a = a * e = a$. The identity element is unique to the set.
The identity for addition is 0 and the identity for multiplication is 1.

Inverse: Let $a, b \in A$. a is the inverse of b and b the inverse of a iff $a * b = b * a = e$.
For any element a with respect to addition of real numbers, the inverse of a is $-a$ since $a + (-a) = (-a) + (a) = 0$.
For multiplication of real numbers, the inverse of a is $\frac{1}{a}$ except when $a = 0$.
The multiplicative inverse of 0 does not exist.

Mathematical proofs

Proof by exhaustion: Prove a statement is true for every value.

Direct proof: A flow of implication beginning with P and ending with Q. We assume that P is true and show that Q must be true.

Proof by contradiction: Assume a statement is possible and reach a contradiction.

Proof by counterexample: Given a universal statement, find a **single statement** for which the universal statement is **not true** and you will disprove the universal statement. **You can disprove something by finding a single counter example, but you cannot prove something by finding one example.**

Checklist

Can you do these?

- Define a binary operation.
- Perform binary operations.
- Define closure, commutativity, associativity, distributivity, identity and inverse.

- **Use the concepts of closure, commutativity, associativity, distributivity, identity and inverse, with simple binary operations.**
- **Construct simple proofs: direct proofs, proofs by counter example.**
- **Use the axioms of the number system, including the non-existence of the multiplicative inverse of zero.**

Review Exercise 2

1 All prime numbers are odd. Prove or disprove this statement.

2 Prove that if x is an integer, divisible by 4, then x is the difference of two perfect squares.

3 If the product of two numbers is even, then the two numbers must be even. Prove or disprove this statement.

4 Prove that if x and y are real numbers, then $x^2 + y^2 \geq 2xy$.

5 The square root of a real number x is always less than x. Is this statement true?

6 Prove that for $a, b \in \mathbb{R}$, $ab = 0 \Leftrightarrow a = 0$ or $b = 0$.

7 Show by counter example that $\dfrac{1}{(a+b)^2}$ and $\dfrac{1}{a^2} + \dfrac{1}{b^2}$ are not equivalent.

8 The binary operation $*$ defined on the set of real numbers is $a * b = a + b + 5$.

(a) Show that the set of real numbers is closed with respect to $*$.

(b) Find the identity element.

(c) Given any element a, find its inverse.

9 Two binary operations $*$ and $\blacksquare$ are defined on the real numbers:

$a * b = 3(a + b)$

$a \blacksquare b = 2ab$

Show that one of these operations is associative.

10 Is the following statement true?

Every number of the form $2^n + 3$ is a prime number.

11 The binary operation $*$ on the set $A = \{a \in \mathbb{R}, a \geq 0]$ is defined as $a * b = |a - b|$, for $a, b \in A$.

(a) Show that A is closed with respect to $*$.

(b) There is an identity in A for $*$.

(c) Every element has an inverse with respect to $*$.

(d) $*$ is not associative in A.

($|x|$ is the positive value of x.)

12 The binary operation Δ is defined on $\mathbb{R}^+$ as $a\Delta b = a^{\ln b}$. Show the following.

(a) The operation Δ is associative.

(b) There is a unique identity in $\mathbb{R}^+$.

(c) Δ is distributive over multiplication.

13 The set X consist of all numbers of the form $x + y\sqrt{3}$ where x and y are integers. Show the following.

(a) X is closed under addition and multiplication of real numbers.

(b) There is an identity in X for addition.

(c) There is an identity in X for multiplication.

(d) Not every element of X has an inverse with respect to multiplication.

14 The binary operation Δ is defined by

$x\Delta y = +\sqrt{x^2 + y^2}$.

Show the following.

(a) The operation Δ is associative.

(b) There is an identity element with respect to Δ.

(c) The operation Δ is not distributive over Δ.

15 The binary operation $*$ is defined on the real numbers as follows.

$a * b = a + b - ab$ where $a, b \in \mathbb{R}$.

(a) Show that there is an identity element with respect to $*$.

(b) Find the inverse for each element.

(c) Show that $*$ is commutative.

(d) Solve $a * (a * 2) = 10$.

16 The set $S = \{a, b, c, d\}$ and the operation Δ is defined by the following table.

Δ	***a***	***b***	***c***	***d***
a	*a*	*b*	*c*	*d*
b	*b*	*d*	*a*	*c*
c	*c*	*a*	*d*	*b*
d	*d*	*c*	*b*	*a*

(a) Is S closed with respect to Δ?

(b) Find the identity in S.

(c) Find the inverse of each element in S.

17 The set $S = \{p, q, r, s\}$ and the operation $*$ is defined by the following table.

$*$	p	q	r	s
p	s	p	q	r
q	p	q	r	S
r	q	r	s	p
s	r	s	p	q

Find the following.

(a) The identity in S

(b) The inverse of each element in S

CHAPTER 3

Principle of Mathematical Induction

At the end of this chapter you should be able to:

- Write a series using sigma notation
- List a series by expanding the sigma form
- Use the summation laws
- Use the standard results for $\sum_{1}^{n} r$, $\sum_{1}^{n} r^2$, $\sum_{1}^{n} r^3$
- Prove by the principle of mathematical induction that a statement is true for a summation
- Prove by the principle of mathematical induction that a statement is true for divisibility

KEYWORDS/TERMS

sigma notation • series • sequence • principle of mathematical induction • divisibility

DEFINITION

A sequence is a set of terms in a well-defined order.

Sequences and series

The following are examples of sequences:

2, 4, 6, 8, 10, 12 . . .

3, 6, 12, 24, 48 . . .

5, 10, 15, 20, 25 . . .

DEFINITION

A series is the sum of the terms of a sequence.

The following are examples of series:

$1 + 2 + 3 + 4 + 5 + \ldots + 20$

$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20 + 22 + 24$

$2 + 4 + 8 + 16 + 32 + 64 + 128 + \ldots$

A series can be written as a list or using the sigma notation. To write a series in sigma notation we need to find the **general term** of the series. The terms of a series are separated by an addition or subtraction sign.

Look at this example of a series.

$1 + 2 + 3 + 4 + 5 + 6 + 7 + \ldots + 20$

The first term is 1, the second term is 2, and the third term is 3 etc.

The general term of a series is represented by t_n or u_n.

Finding the general term of a series

EXAMPLE 1 Find the general term of the following series.

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$

SOLUTION To find the general term of the series, we look for a pattern relating the subscript of the term with the value of the term of the series. Let us see how this works.

The series is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

The first term is 1. Therefore, $u_1 = 1$.

Notice that the subscript and the value are the same. Let us see if this is true for the other terms.

$u_2 = 2$

Again the subscript and the value are the same.

Looking at each term, we see that $u_3 = 3$, $u_4 = 4$ and so on for the other terms of the series.

We can generalise and state that the general term of the series or the rth term is:

$u_r = r$

EXAMPLE 2 Find the general term of the following series.

$2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$

SOLUTION The series is $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 + 18 + 20$.

The first term is 2, $u_1 = 2 = 2(1)$.

The second term is 4, $u_2 = 4 = 2(2)$.

The third term is 6, $u_3 = 6 = 2(3)$.

At this stage, we can see that the subscript of each term is multiplied by 2. The general term of this series is:

$u_r = 2r$

We can check our result by substituting values for r to confirm that $u_r = 2r$ represents our series.

EXAMPLE 3 Find the general term of the following series.

$2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$

SOLUTION The series is $2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$.

The first term is $u_1 = 2 = 2^1$.

The second term is $u_2 = 4 = 2^2$.

The third term is $u_3 = 8 = 2^3$.

The general term of the series is $u_r = 2^r$.

By the third or fourth term you will be able to recognise a pattern and write down a general term, which you can confirm by substituting other values for r.

EXAMPLE 4 Find the general term of the following series.

$3 + 7 + 11 + 15 + 19 + 23 + 27 + 31$

SOLUTION Notice the terms of the series $3 + 7 + 11 + 15 + 19 + 23 + 27 + 31$ increases by a constant value 4.

When the terms of a series increases by a constant term the general term of the series is of the form

$u_r = mr + c$

Where m is the constant difference and c can be found using the first term.

In this series $m = 4$ and $u_1 = 3$, substituting $m = 4$, $r = 1$ and $u_1 = 3$ into

$u_r = mr + c$,

We get

$3 = 4(1) + c$

Therefore

$c = 3 - 4 = -1$

And

$u_r = 4r - 1$.

Sigma notation

EXAMPLE 5 Write the following using sigma notation.

$1 \times 3 + 2 \times 7 + 3 \times 11 + 4 \times 15 + 5 \times 19 + 6 \times 23 + 7 \times 27 + 8 \times 31$

SOLUTION We find the general term of the series which is $u_r = r(4r - 1)$.

The first term of the series occurs when $r = 1$ and the last term when $r = 8$.

The series can be written as

$$\sum_{r=1}^{r=8} r(4r - 1) = 1 \times 3 + 2 \times 7 + 3 \times 11 + 4 \times 15 + 5 \times 19 + 6 \times 23 + 7 \times 27 + 8 \times 31$$

This can also be written as $\sum_{r=1}^{8} r(4r - 1)$ or even as $\sum_{1}^{8} r(4r - 1)$.

EXAMPLE 6 Write the following using sigma notation.

$1 + 3 + 5 + 7 + 9 + 11 + \ldots$

SOLUTION This series is infinite. Therefore, the upper limit can be left out.

The terms of the series go up by a constant difference 2, the general term is of the form $u_r = 2r + c$.

Since $u_1 = 1$, $1 = 2(1) + c$.

Hence, $c = -1$.

And $u_r = 2r - 1$.

The series can be written as $\sum_{r=1}^{\infty} (2r - 1)$.

Expansion of a series

EXAMPLE 7 Expand the series $\sum_{r=1}^{5} (r + 2)$.

SOLUTION Substituting $r = 1$ into $(r + 2)$ gives $1 + 2 = 3$

$r = 2$ into $(r + 2)$ gives $2 + 2 = 4$

$r = 3$ into $(r + 2)$ gives $3 + 2 = 5$

$r = 4$ into $(r + 2)$ gives $4 + 2 = 6$

$r = 5$ into $(r + 2)$ gives $5 + 2 = 7$

Therefore, $\sum_{r=1}^{5} (r + 2) = 3 + 4 + 5 + 6 + 7$

EXAMPLE 8 Given the series $\sum_{r=1}^{n}(r^2 + 2)$, identify the nth term of the series and the 12th term of the series.

SOLUTION The nth term can be found by replacing $r = n$ into $(r^2 + 2)$.

The nth term $T_n = (n^2 + 2)$

Substituting $n = 12$.

$T_{12} = (12^2 + 2) = 146$

$\therefore$ twelfth term is 146.

Standard results

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

$$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$$

Note: These results can only be used directly when the lower limit is 1.

Use of the standard results to find the sum of a series

Remember: A series is the sum of a sequence.

EXAMPLE 9 Find $\sum_{r=1}^{15} r$.

SOLUTION Using $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ with $n = 15$ gives:

$$\sum_{r=1}^{15} r = \frac{15(15+1)}{2} = 15 \times 8 = 120$$

EXAMPLE 10 Evaluate $\sum_{r=1}^{36} r$.

SOLUTION Using $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ with $n = 36$ gives:

$$\sum_{r=1}^{36} r = \frac{36(36+1)}{2} = \frac{36(37)}{2} = 666$$

EXAMPLE 11 Find $\sum_{r=10}^{48} r$.

SOLUTION We need to convert the series so that the lower limit is 1.

If we find the sum from 1 to 48 and subtract the sum from 1 to 9, we will get the sum from 10 to 48.

$$\therefore \sum_{r=10}^{48} r = \sum_{r=1}^{48} r - \sum_{r=1}^{9} r$$

$$\sum_{r=1}^{48} r = \frac{48(48+1)}{2} = \frac{48(49)}{2} = 1176$$

$$\sum_{r=1}^{9} r = \frac{9(9+1)}{2} = \frac{90}{2} = 45$$

$$\therefore \sum_{r=10}^{48} r = 1176 - 45 = 1131$$

EXAMPLE 12 Find the value of $\sum_{r=8}^{25} r^2$.

SOLUTION We can write this sum as:

$$\sum_{r=8}^{25} r^2 = \sum_{r=1}^{25} r^2 - \sum_{r=1}^{7} r^2$$

(Our standard results only work when the lower limit is 1.)

We can use $\sum_{r=1}^{n} r^2 = \frac{n(2n+1)(n+1)}{6}$.

When $n = 25$, $\sum_{r=1}^{25} r^2 = \frac{25(25+1)(2(25)+1)}{6} = \frac{25(26)(51)}{6} = 5525$

When $n = 7$, $\sum_{r=1}^{7} r^2 = \frac{7(7+1)(2(7)+1)}{6} = \frac{7(8)(15)}{6} = 140$

$$\therefore \sum_{r=8}^{25} r^2 = 5525 - 140 = 5385$$

Try these 3.1

(a) Evaluate $\sum_{r=1}^{12} r^2$.

(b) Find the value of $\sum_{r=10}^{30} r^2$.

Summation results

Result 1

Let c be a constant.

$$\sum_{r=1}^{n} c = n \times c$$

EXAMPLE 13 Find $\sum_{r=1}^{20} 5$.

SOLUTION $\sum_{r=1}^{20} 5 = 20 \times 5 = 100$

EXAMPLE 14 Find $\sum_{r=1}^{100} 8$.

SOLUTION $\sum_{r=1}^{100} 8 = 8 \times 100 = 800$

Result 2

Let u_r and v_r be two sequences.

$\sum(u_r + v_r) = \sum u_r + \sum v_r$ (We can separate the sum of two series.)

EXAMPLE 15 Find $\sum_{r=1}^{20}(r^2 + r)$.

SOLUTION Since $\sum_{r=1}^{20}(r^2 + r) = \sum_{r=1}^{20} r^2 + \sum_{r=1}^{20} r$

$$\begin{aligned}\sum_{r=1}^{20}(r^2 + r) &= \frac{20(20 + 1)(2(20) + 1)}{6} + \frac{20(20 + 1)}{2}\\ &= \frac{20(21)(41)}{6} + \frac{20(21)}{2}\\ &= 2870 + 210\\ &= 3080\end{aligned}$$

Result 3

$\sum cu_r = c\sum u_r$, where c is a constant

EXAMPLE 16 Evaluate $\sum_{r=1}^{24}(4r)$.

SOLUTION
$$\begin{aligned}\sum_{r=1}^{24}(4r) &= 4\sum_{r=1}^{24} r\\ &= \frac{4(24)(24 + 1)}{2}\\ &= 2(24)(25)\\ &= 1200\end{aligned}$$

EXAMPLE 17 Find $\sum_{r=1}^{n}(3r^2)$.

SOLUTION

$$\sum_{r=1}^{n} 3r^2 = 3\sum_{r=1}^{n} r^2$$
$$= \frac{3n(n+1)(2n+1)}{6}$$
$$= \frac{n(n+1)(2n+1)}{2}$$

EXAMPLE 18 What is $\sum_{r=1}^{n} 4r(r-1)$?

SOLUTION

$$\sum_{r=1}^{n} 4r(r-1) = \sum_{r=1}^{n}(4r^2 - 4r) \qquad \text{(We expand the brackets.)}$$
$$= 4\sum_{r=1}^{n} r^2 - 4\sum_{r=1}^{n} r$$
$$= 4\frac{n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2}$$
$$= \tfrac{2}{3}n(n+1)(2n+1) - 2n(n+1)$$
$$= 2n(n+1)\left[\tfrac{1}{3}(2n+1) - 1\right]$$
$$= 2n(n+1)\left[\tfrac{1}{3}(2n+1-3)\right]$$
$$= 2n(n+1)\left(\tfrac{1}{3}\right)(2n-2)$$
$$= \frac{2n(n+1)(2)(n-1)}{3}$$
$$= \frac{4n(n^2-1)}{3}$$

Remember

There is no rule for the product of two functions of r. Do not write:

$$\sum_{r=1}^{n} 4r(r-1) = \sum_{r=1}^{n}(4r) \times \sum_{r=1}^{n}(r-1)$$

Try these 3.2

(a) Evaluate $\sum_{r=1}^{20} r(r+3)$.

(b) Find the value of $\sum_{r=10}^{25} 2r(r+1)$.

(c) Find in terms of n, $\sum_{r=1}^{n} r(r^2+2r)$. Simplify your answer.

EXERCISE 3A

In questions **1** to **3**, find the nth term of each sequence.

1 $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{343}, \_\ \_\ \_$

2 $16, 13, 10, 7, 4, \_\ \_\ \_,$

3 $\frac{1}{2 \times 5}, \frac{1}{3 \times 7}, \frac{1}{4 \times 9}, \frac{1}{6 \times 11}, \_\ \_\ \_$

In questions **4** to **6**, write the series using sigma notation:

4 $8 + 16 + 32 + 64 + 128 + 256 + 512$

5 $9 + 12 + 15 + 18 + 21 + 24 + 27 + 30$

6 $4 \times 5 + 5 \times 6 + 6 \times 7 + 7 \times 8 + 8 \times 9 + 9 \times 10 + 10 \times 11$

In questions **7** to **11**, find the nth term of each series.

7 $\sum_{r=1}^{n} (6r - 5)$

8 $\sum_{r=1}^{n} (4r^2 - 3)$

9 $\sum_{r=1}^{2n} (r^3 + r^2)$

10 $\sum_{r=1}^{4n} (6r^3 + 2)$

11 $\sum_{r=1}^{n+2} (3^{2r-1})$

In questions **12** to **14**, identify the given terms of each series.

12 $\sum_{r=1}^{50} (7r + 3)$, the 16th term

13 $\sum_{r=2}^{25} (3r^2 - 1)$, the 8th term

14 $\sum_{r=5}^{42} \left(\frac{1}{4r - 2}\right)$, the 10th term

In questions **15** to **19**, find the value of each sum.

15 $\sum_{r=1}^{r=25} (r - 2)$

16 $\sum_{r=1}^{r=30} (6r + 3)$

17 $\sum_{r=1}^{r=50} r(r + 2)$

18 $\sum_{r=1}^{r=10} r^2(r + 4)$

19 $\displaystyle\sum_{r=1}^{r=45} 6r(r+1)$

In questions **20** to **23**, evaluate each sum.

20 $\displaystyle\sum_{r=5}^{r=12} (r+4)$

21 $\displaystyle\sum_{r=10}^{r=25} (r^2-3)$

22 $\displaystyle\sum_{r=15}^{r=30} r(3r-2)$

23 $\displaystyle\sum_{r=9}^{r=40} (2r+1)(5r+2)$

In questions **24** to **27**, find and simplify each sum.

24 $\displaystyle\sum_{r=1}^{r=n} (r+4)$

25 $\displaystyle\sum_{r=1}^{r=n} 3r(r+1)$

26 $\displaystyle\sum_{r=1}^{r=n} 4r(r-1)$

27 $\displaystyle\sum_{r=1}^{r=n} r^2(r+3)$

In questions **28** to **30**, find and simplify each sum.

28 $\displaystyle\sum_{r=n+1}^{r=2n} 2r(r-1)$

29 $\displaystyle\sum_{r=n+1}^{r=2n} r(r+4)$

30 $\displaystyle\sum_{r=n+1}^{r=2n} (r+1)(r-1)$

Mathematical induction

The principle of mathematical induction is a method for proving that statements involving $\mathbb{N}$ (the set of natural numbers, 1, 2, 3, 4, . . .) or $\mathbb{W}$ (the set of whole numbers) are true for all natural numbers. The procedure for a proof by mathematical induction is as follows.

Step 1

Verify that when $n = 1$, the statement is true.

Step 2

Assume that the statement is true for $n = k$.

Step 3

Show that the statement is true for $n = k + 1$, using the statement in Step 2.

Step 4

When Step 3 has been completed, it is stated that the statement is true for all positive integers $n \in \mathbb{N}$.

Let us see how induction works: when the statement is true for $n = 1$, then according to Step 3, it will also be true for $n = 2$. That implies it will be true for $n = 3$, which implies it will be true for $n = 4$, and so on. It will be true for every natural number.

EXAMPLE 19 Prove by the principle of mathematical induction that the following is true.

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n(n+1)}{2}$$

SOLUTION

PROOF

Let $P(n)$ be the statement $1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$

Step 1

Let us prove that $P(1)$ is true.

When $n = 1$, substituting into $P(n)$ gives:

left-hand side $= 1$

right-hand side $= \dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1.$

Since left-hand side $=$ right-hand side, $P(n)$ is true for $n = 1$.

Step 2

Let us assume that $P(n)$ is true for $n = k$, and so, that $P(k)$ is true. Let us write clearly what our assumption is:

$$1 + 2 + 3 + 4 + \ldots + k = \frac{k(k+1)}{2}$$

We replace n by k on both sides of $P(n)$. We must use $P(k)$ in our next step.

Step 3

We need to prove that $P(n)$ is true when $n = k + 1$, that is, that $P(k + 1)$ is true. Let us write out clearly what we need to prove.

Using $n = k + 1$ in $1 + 2 + 3 + 4 + \ldots + n = \dfrac{n(n+1)}{2}$, we need to prove:

$$1 + 2 + 3 + 4 + \ldots + k + 1 = \frac{(k+1)(k+1+1)}{2}$$

Starting with the left-hand side:

$$1 + 2 + 3 + 4 + 5 + \ldots + (k+1) = [1 + 2 + 3 + 4 + 5 + \ldots + k] + (k+1)$$

The terms in the square brackets represent $P(k)$.

Since we assume $P(k)$ is true, we can now substitute $1 + 2 + 3 + 4 + \ldots + k = \frac{k(k+1)}{2}$ and we get:

$$
\begin{aligned}
1 + 2 + 3 + 4 + 5 + \ldots + (k + 1) &= [1 + 2 + 3 + 4 + 5 + \ldots + k] + (k + 1) \\
&= \frac{k(k+1)}{2} + (k+1) \\
&= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

Hence, $P(k + 1)$, is true whenever $P(k)$ is true.

Thus, $P(n)$ is true whenever n is a positive integer.

Let us try another.

EXAMPLE 20 Prove by mathematical induction that $\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

SOLUTION

PROOF

LET $P(n)$ be the statement $\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$.

Step 1

Substituting $n = 1$ into $P(1)$, we get:

left-hand side $= 1^2 = 1$

$$
\begin{aligned}
\text{right-hand side} &= \frac{1(1+1)(2(1)+1)}{6} \\
&= \frac{1 \times 2 \times 3}{6} \\
&= \frac{\cancel{6}}{\cancel{6}} \\
&= 1
\end{aligned}
$$

Since left-hand side = right-hand side, $P(n)$ is true for $n = 1$.

Assume that $P(n)$ is true for $n = k$, that is $P(k)$ is true. We write clearly our assumption, replacing n by k:

$$\sum_{1}^{k} r^2 = \frac{k(k+1)(2k+1)}{6}.$$

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)\text{th term}$$

The terms of the series are obtained by replacing r in the function r^2. Therefore, the $(k + 1)$th term of this series is $(k + 1)^2$.

Step 3

We need to prove that $P(n)$ is true when $n = k + 1$ that is $P(k + 1)$ is true. Let us write out clearly what we need to prove. Using $n = k + 1$ in $P(n)$, we get:

$$\sum_{1}^{k+1} r^2 = \frac{(k + 1)(k + 1 + 1)(2(k + 1) + 1)}{6}$$

$$\sum_{1}^{k+1} r^2 = \sum_{1}^{k} r^2 + (k + 1)^2$$

$$= \frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 \qquad \left(\text{From Step 2, } \sum_{1}^{k} r^2 = \frac{k(k + 1)(2k + 1)}{6}\right)$$

$$= \frac{k(k + 1)(2k + 1)}{6} + \frac{6(k + 1)^2}{6} \qquad \left(\text{Factorising } \frac{6(k + 1)^2}{6} = (k + 1)^2\text{, we can factor out } \frac{k + 1}{6}\right)$$

$$= \frac{(k + 1)}{6}[k(2k + 1) + 6(k + 1)]$$ (Remember to multiply 6 by $(k + 1)$ instead of 6 by k)

$$= \frac{(k + 1)}{6}[2k^2 + k + 6k + 6]$$

$$= \frac{(k + 1)}{6}[2k^2 + 7k + 6]$$

$$= \frac{(k + 1)(2k + 3)(k + 2)}{6}$$

$$= \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

$$= \frac{(k + 1)(k + 1 + 1)(2(k + 1) + 1)}{6}$$

Hence, $P(k + 1)$, is true whenever $P(k)$ is true.

Thus, $P(n)$ is true whenever n is a positive integer

Let us try another, this time without the detailed explanation.

EXAMPLE 21 Using mathematical induction, show that the following formula is true for all natural numbers $\mathbb{N}$.

$$\sum_{1}^{n} \frac{1}{(3r - 2)(3r + 1)} = \frac{n}{3n + 1}$$

SOLUTION

PROOF

Let $P(n)$ be the statement $\displaystyle\sum_{1}^{n} \frac{1}{(3r - 2)(3r + 1)} = \frac{n}{3n + 1}$.

Substituting $n = 1$ into $P(n)$ gives:

left-hand side $= \dfrac{1}{1 \times 4} = \dfrac{1}{4}$

right-hand side $= \dfrac{1}{3(1) + 1} = \dfrac{1}{4}$

Since left-hand side = right-hand side, $P(n)$ is true for $n = 1$.

Assume that $P(n)$ is true when $n = k$, i.e. $P(k)$ is true.

Replacing n by k we have:

$$\sum_{1}^{k} \frac{1}{(3r-2)(3r+1)} = \frac{k}{3k+1}$$

We need to prove that $P(n)$ is true when $n = k + 1$ that is $P(k + 1)$ is true. Let us write out clearly what we need to prove. Using $n = k + 1$ in $P(n)$, we get:

$$\sum_{1}^{k+1} \frac{1}{(3r-2)(3r+1)} = \frac{k+1}{3(k+1)+1}$$

The $(k + 1)$th term of the series is obtained by substituting $r = k + 1$ into $\frac{1}{(3r-2)(3r+1)}$, i.e. $\frac{1}{(3(k+1)-2)(3(k+1)+1)}$.

$$\sum_{1}^{k+1} \frac{1}{(3r-2)(3r+1)} = \sum_{1}^{k} \frac{1}{(3r-2)(3r+1)} + \frac{1}{(3(k+1)-2)(3(k+1)+1)}$$

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)}$$ (Induction step, this is where we replace (k))

$$= \frac{1}{3k+1}\left(k + \frac{1}{3k+4}\right)$$ (Factorising)

$$= \frac{1}{3k+1}\left(\frac{k(3k+4)+1}{3k+4}\right)$$

$$= \frac{1}{3k+1}\left(\frac{3k^2+4k+1}{3k+4}\right)$$

$$= \frac{1}{3k+1}\left(\frac{(3k+1)(k+1)}{3k+4}\right)$$

$$= \frac{k+1}{3k+4}$$

$$= \frac{k+1}{3(k+1)+1}$$

Hence, $P(k + 1)$, is true whenever $P(k)$ is true.

Thus, $P(n)$ is true whenever n is a positive integer.

Divisibility tests and mathematical induction

EXAMPLE 22 Using mathematical induction, show that $9^n - 1$ is divisible by 8 for all natural numbers n.

SOLUTION Let $P(n)$ be the statement $9^n - 1$ is divisible by 8.

We can rewrite this as $P(n)$ is $9^n - 1 \equiv 8A$ where A is some integer.

When $n = 1$, $9^1 - 1 = 9 - 1 = 8 = 8(1)$ which is divisible by 8.

Therefore $P(1)$ is true.

Assume that $P(n)$ is true for some integer $n = k$ that is $P(k)$ is true.

Writing in equation form: $9^k - 1 = 8A$.

We need to prove that $P(n)$ is true when $n = k + 1$, that is $P(k + 1)$ is true. Let us write out clearly what we need to prove. Using $n = k + 1$ in $P(n)$, we get:

PROOF

$9^{k+1} - 1 = 8B$, where B is some integer

$$\begin{aligned} 9^{k+1} - 1 &= 9^{k+1} + 8A - 9^k \qquad \text{(From our assumption, } 9^k - 1 = 8A. \text{ Therefore } -1 = 8A - 9^k\text{)} \\ &= 9^{k+1} - 9^k + 8A \\ &= 9^k \times 9^1 - 9^k + 8A \\ &= 9^k(9 - 1) + 8A \\ &= 9^k(8) + 8A \\ &= 8(9^k + A) \\ &= 8B, \text{ where } B \text{ is some integer } B = 9^k + A \end{aligned}$$

Hence, $P(k + 1)$, is true whenever $P(k)$ is true.

Thus, $P(n)$ is true whenever n is a natural number.

EXAMPLE 23 Using mathematical induction, show that $22 + 3^{4n+6} + 17^{n+2}$ is divisible by 16 for all natural numbers n.

SOLUTION

Let $P(n)$ be the statement $22 + 3^{4n+6} + 17^{n+2}$ is divisible by 16.

We can write this as $P(n)$ is $22 + 3^{4n+6} + 17^{n+2} = 16A$, where A is some integer.

When $n = 1$, $22 + 3^{4(1)+6} + 17^{(1)+2} = 22 + 3^{10} + 17^3 = 63\,984 = 3999(16)$.

The answer is divisible by 16.

Therefore, $P(1)$ is true.

Assume that $P(n)$ is true for some integer $n = k$, that is $P(k)$ is true.

We can write this as $22 + 3^{4k+6} + 17^{k+2} = 16A$.

We need to prove that $P(n)$ is true for $n = k + 1$, that is $P(k + 1)$ is true.

We need to prove that $22 + 3^{4(k+1)+6} + 17^{(k+1)+2} = 16B$, where B is some integer.

PROOF

$$\begin{aligned} 22 + 3^{4(k+1)+6} + 17^{(k+1)+2} &= 22 + 3^{4k+10} + 17^{k+3} \qquad \text{(Induction step: } 22 = 16A - 3^{4k+6} + 17^{k+2}\text{)} \\ &= (16A - 3^{4k+6} - 17^{k+2}) + 3^{4k+10} + 17^{k+3} \\ &= 16A + 3^{4k+10} - 3^{4k+6} + 17^{k+3} - 17^{k+2} \\ &= 16A + (3^{4k+6} \times 3^4) - 3^{4k+6} + (17^{k+2} \times 17) - 17^{k+2} \\ &= 16A + 3^{4k+6} \times (3^4 - 1) + 17^{k+2} \times (17 - 1) \\ &= 16A + 3^{4k+6} \times (80) + 17^{k+2} \times (16) \end{aligned}$$

$= 16(A + 3^{4k+6} \times (5) + 17^{k+2})$

$= 16B$, where $B = A + 3^{4k+6} \times (5) + 17^{k+2}$, which is an integer.

Hence, $P(k + 1)$ is true.

Thus, $P(n)$ is true whenever n is a natural number.

EXERCISE 3B

In questions **1** to **10**, prove the equalities by mathematical induction.

1 $\displaystyle\sum_{r=1}^{r=n}(3r - 2) = \frac{1}{2}n(3n - 1)$

2 $\displaystyle\sum_{r=1}^{r=n}(4r - 3) = n(2n - 1)$

3 $\displaystyle\sum_{r=1}^{r=n}(2r - 1)2r = \frac{1}{3}n(n + 1)(4n - 1)$

4 $\displaystyle\sum_{r=1}^{r=n}(r^2 + r^3) = \frac{n(n + 1)(3n + 1)(n + 2)}{12}$

5 $\displaystyle\sum_{r=1}^{r=n}r^3 = \frac{n^2(n + 1)^2}{4}$

6 $\displaystyle\sum_{r=1}^{r=n}\frac{1}{r(r + 1)} = \frac{n}{(n + 1)}$

7 $\displaystyle\sum_{r=1}^{r=n}(-1)^{r+1}r^2 = (-1)^{n+1}\frac{n(n + 1)}{2}$

8 $\displaystyle\sum_{r=1}^{r=n}\frac{1}{r(r + 1)(r + 2)} = \frac{n(n + 3)}{4(n + 1)(n + 2)}$

9 $\displaystyle\sum_{r=1}^{r=n}\frac{1}{(3r - 1)(3r + 2)} = \frac{n}{6n + 4}$

10 Prove, using the principle of mathematical induction, that for any integer ≥ 1, $3^{4n} - 1$ is divisible by 16.

11 Prove, using the principle of mathematical induction, that for any integer ≥ 1, $n^4 + 3n^2$ is divisible by 4.

SUMMARY

Principle of mathematical induction

Sequences

Series: sum of the terms of a sequence

$\sum_{r=1}^{n} r$: sum all numbers from 1 to n

$\sum_{r=1}^{n} c = nc$

$\sum cu_r = c\sum u_r$

$\sum (u_r + v_r) = \sum u_r + \sum v_r$

$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

$\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

$\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

Proof by mathematical induction

Prove the statement true for $n = 1$

Assume the statement true for $n = k$

Prove the statement true for $n = k + 1$

Hence, by mathematical induction, the statement is true for all n

Application of proof by mathematical induction

- Series
- Divisibility tests

Checklist

Can you do these?

- Write a series using sigma notation.
- List a series by expanding the sigma form.
- Use the summation laws.
- Use the standard results for $\sum_{1}^{n} r$, $\sum_{1}^{n} r^2$, $\sum_{1}^{n} r^3$.
- Prove by the principle of mathematical induction that a statement is true for a summation.
- Prove by the principle of mathematical induction that a statement is true for divisibility.

Review Exercise 3

1 Here is a series: $6 \times 7 + 8 \times 10 + 10 \times 13 \ldots$

(a) Find the nth term of the series.

(b) Find the sum of the first n terms of the series, simplifying your answer as far as possible.

2 (a) Find and simplify $\sum_{r=1}^{r=n} r(3r-2)$.

(b) Hence, find:

(i) $\sum_{r=1}^{r=20} r(3r-2)$

(ii) $\sum_{r=21}^{r=100} r(3r-2)$

3 Prove by induction that $\sum_{1}^{n} 2r(r-5) = \dfrac{2n(n+1)(n-7)}{3}$.

4 Given that $a_n = 3^{2n-1} + 1$, show that $a_{n+1} - a_n = 8(3^{2n-1})$. Hence, prove by induction that a_n is divisible by 4 for all positive integers n.

5 Find and simplify $\sum_{1}^{n} 2r(r^2 - 1)$. Hence, prove your result by induction.

6 Given that $a_n = 5^{2n+1} + 1$, show that $a_{n+1} - a_n = 24(5^{2n+1})$. Hence, prove by induction that a_n is divisible by 6 for all non-negative integers n.

7 Find the sum of the series $\sum_{r=1}^{n} (6r^3 + 2)$, simplifying your answer as far as possible. Prove by induction that your summation is correct.

8 Prove by induction that $\sum_{r=1}^{n} (r+4) = \frac{1}{2}n(n+9)$.

9 Use the principle of mathematical induction to prove that $\sum_{r=1}^{n} 4r(r-1) = \dfrac{4n(n+1)(n-1)}{3}$.

10 Prove by induction that $\sum_{r=1}^{n} \dfrac{1}{r(r+1)} = \dfrac{n}{n+1}$.

11 Prove that $\sum_{r=1}^{n} 3(2^{r-1}) = 3(2^n - 1)$.

12 Prove that $n(n^2 + 5)$ is divisible by 6 for all positive integers n.

13 Prove by mathematical induction that $n^5 - n$ is divisible by 5 for all positive integers n.

14 Prove by induction that $\sum_{n=1}^{k} \dfrac{1}{(n+4)(n+3)} = \dfrac{1}{4} - \dfrac{1}{k+4}$.

CHAPTER 4

Polynomials

At the end of this chapter you should be able to:

- Identify a polynomial
- Identify the order of a polynomial
- Identify the terms of a polynomial
- Add, subtract, multiply and divide polynomials
- Evaluate polynomials
- Compare polynomials
- Use the remainder theorem
- Factorise polynomials using the factor theorem
- Solve equations involving polynomials
- Factorise $x^n - y^n$ where $n = 2, 3, 4, 5, 6$

KEYWORDS/TERMS

polynomial • order of a polynomial • terms • evaluate • remainder theorem • factor theorem • factorise • solve

DEFINITION

A polynomial is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ where $a_0, a_1, a_2, \dots a_{n-1}, a_n$ are constants and $n \in \mathbb{W}$.

Review of polynomials

Degree or order of polynomials

The degree or order of a polynomial is the highest power of x in the polynomial.

For example, $6x^5 + 5x^2 - 3x + 1$ is a polynomial of degree 5.

Algebra of polynomials

EXAMPLE 1

Given that $P(x) = 4x^3 - 3x^2 + 2x + 1$ and $Q(x) = 6x^2 - 2x + 2$, find the following.

(a) $P(x) + Q(x)$

(b) $P(x) - Q(x)$

(c) $P(x) \times Q(x)$

(d) $\dfrac{P(x)}{Q(x)}$

SOLUTION

(a)
$$\begin{aligned} P(x) + Q(x) &= 4x^3 - 3x^2 + 2x + 1 + 6x^2 - 2x + 2 \\ &= 4x^3 - 3x^2 + 6x^2 + 2x - 2x + 1 + 2 \\ &= 4x^3 + 3x^2 + 3 \end{aligned}$$

(b)
$$\begin{aligned} P(x) - Q(x) &= 4x^3 - 3x^2 + 2x + 1 - (6x^2 - 2x + 2) \\ &= 4x^3 - 9x^2 + 4x - 1 \end{aligned}$$

(c)
$$\begin{aligned} P(x) \times Q(x) &= (4x^3 - 3x^2 + 2x + 1) \times (6x^2 - 2x + 2) \\ &= (4x^3)(6x^2 - 2x + 2) - (3x^2)(6x^2 - 2x + 2) \\ &\quad + (2x)(6x^2 - 2x + 2) + (1)(6x^2 - 2x + 2) \\ &= 24x^5 - 8x^4 + 8x^3 - 18x^4 + 6x^3 - 6x^2 + 12x^3 - 4x^2 + 4x \\ &\quad + 6x^2 - 2x + 2 \\ &= 24x^5 - 26x^4 + 26x^3 - 4x^2 + 2x + 2 \end{aligned}$$

(d)
$$\begin{array}{r@{}l} & \frac{2}{3}x - \frac{5}{18} \\ 6x^2 - 2x + 2 \big) & \overline{4x^3 - 3x^2 + 2x + 1} \\ & -\left(4x^3 - \frac{4}{3}x^2 + \frac{4}{3}x\right) \\ \hline & -\frac{5}{3}x^2 + \frac{2}{3}x + 1 \\ & -\left(\frac{-5}{3}x^2 + \frac{10}{18}x - \frac{10}{18}\right) \\ \hline & \frac{1}{9}x + \frac{14}{9} \end{array}$$

$$\therefore \frac{4x^3 - 3x^2 + 2x + 1}{6x^2 - 2x + 2} \equiv \frac{2}{3}x - \frac{5}{18} + \frac{\frac{1}{9}x + \frac{14}{9}}{6x^2 - 2x + 2}$$

Evaluating polynomials

EXAMPLE 2 Let $f(x) = 3x^3 - 2x^2 + x - 4$.

Evaluate the following.

(a) $f(1)$

(b) $f(-2)$

(c) $f(3)$

SOLUTION

(a) Substituting $x = 1$ into $f(x)$ gives:

$$\begin{aligned} f(x) &= 3x^3 - 2x^2 + x - 4 \\ f(1) &= 3(1)^3 - 2(1)^2 + (1) - 4 \\ &= 3 - 2 + 1 - 4 \\ &= -2 \end{aligned}$$

(b) Substituting $x = -2$ into $f(x)$ gives:

$$\begin{aligned} f(-2) &= 3(-2)^3 - 2(-2)^2 + (-2) - 4 \\ &= -24 - 8 - 2 - 4 \\ &= -38 \end{aligned}$$

(c) Substituting $x = 3$ into $f(x)$ gives:

$$\begin{aligned} f(3) &= 3(3)^3 - 2(3)^2 + (3) - 4 \\ &= 81 - 18 + 3 - 4 \\ &= 62 \end{aligned}$$

Rational expressions

A rational expression is an expression of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials in x and $Q(x) \neq 0$. We can add, subtract, multiply and divide rational expressions in the same way that we can with numbers.

EXAMPLE 3 Solve the equation $\frac{x + 1}{x - 2} = 0$.

SOLUTION

$$\frac{x + 1}{x - 2} = 0$$

Multiplying by $x - 2$ gives:

$$(x - 2) \times \frac{x + 1}{x - 2} = 0$$

$$x + 1 = 0$$

$$x = -1$$

Hence, $x = -1$.

EXAMPLE 4

(a) Write the expression $\frac{1}{x - 1} - \frac{4}{x + 2}$ as a single fraction.

(b) Hence, solve the equation $\frac{1}{x - 1} - \frac{4}{x + 2} = 0$.

SOLUTION

(a) $\dfrac{1}{x-1} - \dfrac{4}{x+2} = \dfrac{x+2-4(x-1)}{(x-1)(x+2)}$ (Lowest common multiple of the denominators is $(x-1)(x+2)$)

$= \dfrac{-3x+6}{(x-1)(x+2)}$

(b) $\dfrac{1}{x-1} - \dfrac{4}{x+2} = 0$

We use $\dfrac{-3x+6}{(x-1)(x+2)} = 0$

Multiplying by $(x-1)(x+2)$ gives:

$-3x + 6 = 0$

$\Rightarrow 3x = 6$

$\Rightarrow x = 2$

Hence, $x = 2$.

EXAMPLE 5

(a) Simplify $\dfrac{x+1}{2x-1} + \dfrac{3x+2}{x-3}$.

(b) Hence, solve $\dfrac{x+1}{2x-1} + \dfrac{3x+2}{x-3} = 3$, giving the exact values of x.

SOLUTION

(a) $\dfrac{x+1}{2x-1} + \dfrac{3x+2}{x-3} = \dfrac{(x+1)(x-3)+(3x+2)(2x-1)}{(2x-1)(x-3)}$

$= \dfrac{x^2-2x-3+6x^2+x-2}{(2x-1)(x-3)}$

$= \dfrac{7x^2-x-5}{(2x-1)(x-3)}$

(b) $\dfrac{x+1}{2x-1} + \dfrac{3x+2}{x-3} = 3$

$\Rightarrow \dfrac{7x^2-x-5}{(2x-1)(x-3)} = 3$

$7x^2 - x - 5 = 3(2x-1)(x-3)$

$7x^2 - x - 5 = 6x^2 - 21x + 9$

$x^2 + 20x - 14 = 0$

$x = \dfrac{-20 \pm \sqrt{20^2 - 4(-14)}}{2}$

$= \dfrac{-20 \pm \sqrt{456}}{2}$

$= \dfrac{-20 \pm 2\sqrt{114}}{2}$

$= -10 \pm \sqrt{114}$

$\therefore x = -10 + \sqrt{114}, \quad x = -10 - \sqrt{114}$

Comparing polynomials

Two polynomials are equal if and only if their corresponding coefficients are equal.

EXAMPLE 6 Find a if $x^2 + 4x + 2 \equiv x^2 + ax + 2$.

SOLUTION Since $x^2 + 4x + 2 \equiv x^2 + ax + 2$, the corresponding coefficients on both sides must be identical. Comparing coefficients of x gives:

$4 = a$

EXAMPLE 7 Given that $3x^3 + ax^2 + bx + 2 \equiv (3x^2 + 1)(x + 2)$, where a and b are constants, find a and b.

SOLUTION Expanding the right-hand side gives:

$(3x^2 + 1)(x + 2) = 3x^3 + 6x^2 + x + 2$

$\therefore\ 3x^3 + ax^2 + bx + 2 = 3x^3 + 6x^2 + x + 2$

Comparing coefficients of x^2 gives:

$a = 6$

Comparing coefficients of x gives:

$b = 1$

EXAMPLE 8 Find the constants A, B, C and D such that $(x + 2)(2x + 3)(x - 1) \equiv Ax^3 + Bx^2 + Cx + D$.

SOLUTION Expanding the brackets on the left-hand side gives:

$$\begin{aligned}(x + 2)(2x + 3)(x - 1) &= (2x^2 + 3x + 4x + 6)(x - 1)\\ &= (2x^2 + 7x + 6)(x - 1)\\ &= 2x^3 + 7x^2 + 6x - 2x^2 - 7x - 6\\ &= 2x^3 + 5x^2 - x - 6\end{aligned}$$

Since $(x + 2)(2x + 3)(x - 1) \equiv Ax^3 + Bx^2 + Cx + D$

$\Rightarrow 2x^3 + 5x^2 - x - 6 \equiv Ax^3 + Bx^2 + Cx + D$

Comparing like terms:

$Ax^3 = 2x^3 \Rightarrow A = 2$

$Bx^2 = 5x^2 \Rightarrow B = 5$

$Cx = -x \Rightarrow C = -1$

$D = -6$

Hence, $A = 2$, $B = 5$, $C = -1$, $D = -6$.

Alternative solution:

Since $(x + 2)(2x + 3)(x - 1) \equiv Ax^3 + Bx^2 + Cx + D$, the two sides are equivalent for all values of x.

Substituting the values of x for which each factor is zero gives:

$x + 2 = 0, x = -2$

$2x + 3 = 0, x = \frac{-3}{2}$

$x - 1 = 0, x = 1$

Therefore, when $x = 1, \Rightarrow 0 = A + B + C + D$ [1]

When $x = -2 \Rightarrow 0 = -8A + 4B - 2C + D$ [2]

When $x = -\frac{3}{2} \Rightarrow 0 = -\frac{27}{8}A + \frac{9}{4}B - \frac{3}{2}C + D$ [3]

When $x = 0 \Rightarrow (2)(3)(-1) = D$

$D = -6$

Equating coefficients of x^3:

$x \times 2x \times x = Ax^3$

$2x^3 = Ax^3$

$A = 2$

Substituting $A = 2$ into [1] and [2] gives:

$2 + B + C - 6 = 0 \Rightarrow B + C = 4$ [4]

$-16 + 4B - 2C - 6 = 0 \Rightarrow 4B - 2C = 22$

$2B - C = 11$ [5]

$\Rightarrow 3B = 15$ [4] + [5]

$B = \frac{15}{3} = 5$

Substituting $B = 5$ into [4] gives:

$5 + C = 4$

$C = -1$

$\therefore\ A = 2, B = 5, C = -1$ and $D = -6$

EXAMPLE 9

Given that $x^3 + ax^2 + x + 6 \equiv (x + 1)(x - 2)(bx + c)$ for all values of x, find the values of a, b and c.

SOLUTION

$x^3 + ax^2 + x + 6 \equiv (x + 1)(x - 2)(bx + c)$

Substituting $x = 2$ into the identity gives:

$2^3 + a(2)^2 + 2 + 6 = (2 + 1)(2 - 2)(2b + c)$

$8 + 4a + 8 = 0$

$4a = -16$

$a = -4$

Expanding the right-hand side:

$$\begin{aligned}(x + 1)(x - 2)(bx + c) &= (x^2 - x - 2)(bx + c)\\ &= bx^3 - bx^2 - 2bx + cx^2 - cx - 2c\\ &= bx^3 - x^2(b - c) - x(2b + c) - 2c\end{aligned}$$

$\therefore\ x^3 + ax^2 + x + 6 = bx^3 - x^2(b - c) - x(2b + c) - 2c$

Comparing coefficients of x^3 gives:

$1 = b$

Comparing the constants gives:

$6 = -2c$

$c = -3$

$\therefore\ a = -4,\ b = 1$ and $c = -3$

EXAMPLE 10 Given that $3x^2 - 5x + 4 \equiv \lambda(x - 2)^2 + \mu(x - 2) + \phi$, for all values of x, find the values of λ, μ and ϕ.

SOLUTION

$3x^2 - 5x + 4 \equiv \lambda(x - 2)^2 + \mu(x - 2) + \phi$

Substituting $x = 2$ gives:

$3(2)^2 - 2(5) + 4 \equiv \lambda(2 - 2)^2 + \mu(2 - 2) + \phi$

$12 - 10 + 4 \equiv \phi$

$\phi = 6$

$\therefore\ 3x^2 - 5x + 4 = \lambda(x - 2)^2 + \mu(x - 2) + 6$

Expanding the right-hand side gives:

$3x^2 - 5x + 4 \equiv \lambda(x^2 - 4x + 4) + \mu x - 2\mu + 6$

$3x^2 - 5x + 4 \equiv \lambda x^2 - 4\lambda x + 4\lambda + \mu x - 2\mu + 6$

$3x^2 - 5x + 4 \equiv \lambda x^2 + x(-4\lambda + \mu) + 4\lambda - 2\mu + 6$

Equating coefficients of x^2:

$3 = \lambda$

Equating coefficients of x:

$-5 = -4\lambda + \mu$

Substituting $\lambda = 3$ gives:

$-5 = -12 + \mu$

$\mu = 7$

$\therefore\ \lambda = 3,\ \mu = 7$ and $\phi = 6$

Remember

There are two ways of solving problems of this type: we can expand and equate coefficients of 'like' terms or we can substitute values of x on the two sides of the identity, form equations and solve the equations simultaneously.

Try these 4.1

(a) Given that $(x - 2)(x + 1)(x + 3) \equiv Ax^3 + Bx^2 + Cx + D$, find A, B, C and D.

(b) Given that $2x^2 - 3x + 1 \equiv A(x + 1)^2 + Bx + C$, find A, B and C.

EXERCISE 4A

In questions **1** to **10**, find the values of the unknown constants.

1 $x^2 + x + b \equiv (x + b)(x - 2) + a$

2 $4x^2 + 6x + 1 \equiv p(x + q)^2 + r$

3 $8x^3 + 27x^2 + 49x + 15 = (ax + 3)(x^2 + bx + c)$

4 $x^3 + px^2 - 7x + 6 = (x - 1)(x - 2)(qx + r)$

5 $2x^3 + 7x^2 - 7x - 30 = (x - 2)(ax^2 + bx + c)$

6 $x^3 - 3x^2 + 4x + 2 \equiv (x - 1)(x^2 - 2x + a) + b$

7 $4x^3 + 3x^2 + 5x + 2 = (x + 2)(ax^2 + bx + c)$

8 $2x^3 + Ax^2 - 8x - 20 = (x^2 - 4)(Bx + c)$

9 $ax^3 + bx^2 + cx + d = (x + 2)(x + 3)(x + 4)$

10 $ax^3 + bx^2 + cx + d = (4x + 1)(2x - 1)(3x + 2)$

11 Given that $f(x) = 4x^3 - 3x^2 + 2x + 1$, find the quotient and remainder when $f(x)$ is divided by $x - 2$.

12 Given that $\dfrac{5x^3 - 6x^2 + 2x + 1}{x - 2} \equiv Ax^2 + Bx + C + \dfrac{21}{x - 2}$, find A, B and C.

13 Find the quotient and remainder when $x^5 - 2x^4 - x^3 + x^2 + x + 1$ is divided by $x^2 + 1$.

14 Express each of the following in the form $\dfrac{f(x)}{g(x)}$.

(a) $\dfrac{2}{x + 1} + \dfrac{3}{x - 2}$

(b) $\dfrac{x + 1}{x + 3} - \dfrac{2x + 1}{2x - 4}$

(c) $\dfrac{x}{x^2 + 2x + 1} - \dfrac{x - 1}{x + 2}$

(d) $\dfrac{3x + 4}{x - 1} - \dfrac{x}{x + 1} + \dfrac{x + 2}{2x + 1}$

(e) $\dfrac{x^2}{2 - x} - \dfrac{x^2}{3 - x}$

Remainder theorem

EXAMPLE 11 Find the remainder when $3x^3 - 2x^2 + 4x + 1$ is divided by $x + 1$.

SOLUTION Using long division gives:

$$
\begin{array}{r}
3x^2 - 5x + 9 \\
x + 1 \overline{)\,3x^3 - 2x^2 + 4x + 1} \\
-(3x^3 + 3x^2) \\
\hline
-5x^2 + 4x \\
-\ (-5x^2 - 5x) \\
\hline
9x + 1 \\
-(9x + 9) \\
\hline
-8
\end{array}
$$

$\dfrac{3x^3}{x} = 3x^2$ which is the first term in the quotient.

$\dfrac{-5x^2}{x} = -5x$ is the second term of the quotient.

$\dfrac{9x}{x} = 9$

When $3x^3 - 2x^2 + 4x + 1$ is divided by $x + 1$ the quotient is $3x^2 - 5x + 9$ and the remainder is -8.

We can rewrite this as:

$$\frac{3x^3 - 2x^2 + 4x + 1}{x + 1} = 3x^2 - 5x + 9 - \frac{8}{x + 1}$$

We can also multiply both sides by $(x + 1)$ and write it as:

$$3x^3 - 2x^2 + 4x + 1 \equiv (3x^2 - 5x + 9)(x + 1) - 8$$

The remainder theorem

When a polynomial $f(x)$ is divided by a linear expression $(x - \lambda)$, the remainder is $f(\lambda)$.

PROOF

When $f(x)$ is divided by $x - \lambda$, we get a quotient $Q(x)$ and a remainder R.

$$\therefore \frac{f(x)}{x-\lambda} = Q(x) + \frac{R}{x-\lambda}$$

$\Rightarrow f(x) = (x - \lambda)\, Q(x) + R$ (Multiplying both sides by $(x - \lambda)$)

Substituting $x = \lambda$ gives:

$f(\lambda) = (\lambda - \lambda)Q(\lambda) + R \Rightarrow R = f(\lambda)$

The remainder is $f(\lambda)$.

Let us use the remainder theorem on Example 11, where we wanted to find the remainder when $f(x) = 3x^3 - 2x^2 + 4x + 1$ is divided by $x + 1$.

Since we are dividing by $x + 1$, $x = -1$ when $x + 1 = 0$.

By the remainder theorem, the remainder is $f(-1)$.

Substituting $x = -1$ into $f(x)$ gives:

$$\begin{aligned} f(-1) &= 3(-1)^3 - 2(-1)^2 + 4(-1) + 1 \\ &= -3 - 2 - 4 + 1 \\ &= -8 \end{aligned}$$

This is the same answer as when we used long division.

EXAMPLE 12 Find the remainder when $f(x) = 4x^3 - x^2 + x - 2$ is divided by

(a) $x - 1$

(b) $x + 2$

(c) $2x + 1$

SOLUTION

(a) When $x - 1 = 0$, $x = 1$. By the remainder theorem, when $f(x)$ is divided by $x - 1$, the remainder is $f(1)$.

$$\begin{aligned} \therefore f(1) &= 4(1)^3 - (1)^2 + (1) - 2 \\ &= 4 - 1 + 1 - 2 \\ &= 2 \end{aligned}$$

(b) When $x + 2 = 0$, $x = -2$. By the remainder theorem, when $f(x)$ is divided by $x + 2$ the remainder is $f(-2)$.

$$\begin{aligned} f(-2) &= 4(-2)^3 - (-2)^2 + (-2) - 2 \\ &= -32 - 4 - 2 - 2 \\ &= -40 \end{aligned}$$

(c) When $2x + 1 = 0$, $x = -\frac{1}{2}$. By the remainder theorem, when $f(x)$ is divided by $2x + 1$, the remainder is $f\left(\frac{-1}{2}\right)$.

$$f\left(-\frac{1}{2}\right) = 4\left(\frac{-1}{2}\right)^3 - \left(\frac{-1}{2}\right)^2 + \left(\frac{-1}{2}\right) - 2$$
$$= \frac{-1}{2} - \frac{1}{4} - \frac{1}{2} - 2$$
$$= -3\frac{1}{4}$$

EXAMPLE 13 The remainder when $f(x) = 4x^3 + ax^2 + 2x + 1$ is divided by $3x - 1$ is 4. Find the value of a.

SOLUTION When $3x - 1 = 0$, $x = \frac{1}{3}$. By the remainder theorem $f\left(\frac{1}{3}\right) = 4$.

Since $f(x) = 4x^3 + ax^2 + 2x + 1$, substituting $x = \frac{1}{3}$ gives:

$$f\left(\frac{1}{3}\right) = 4\left(\frac{1}{3}\right)^3 + a\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) + 1$$
$$= \frac{4}{27} + \frac{1}{9}a + \frac{2}{3} + 1$$
$$= \frac{1}{9}a + \frac{49}{27}$$

Since $f\left(\frac{1}{3}\right) = 4$

$$\frac{1}{9}a + \frac{49}{27} = 4$$
$$\frac{1}{9}a = 4 - \frac{49}{27}$$
$$\frac{1}{9}a = \frac{59}{27}$$
$$a = \frac{59}{27} \times 9$$
$$a = \frac{59}{3}$$

EXAMPLE 14 The expression $6x^3 - 4x^2 + ax + b$ leaves a remainder of 5 when divided by $x - 1$ and a remainder of 1 when divided by $x + 1$. Find the values of a and b.

SOLUTION Let $f(x) = 6x^3 - 4x^2 + ax + b$.

When dividing by $x - 1$ the remainder is 5.

$\Rightarrow f(1) = 5$

Now $f(1) = 6(1)^2 - 4(1)^2 + a(1) + b$
$= 6 - 4 + a + b$
$= 2 + a + b$

$\therefore\ 2 + a + b = 5$

$a + b = 3$ [1]

When dividing by $x + 1$ the remainder is 1.

$\Rightarrow f(-1) = 1$

$f(-1) = 6(-1)^3 - 4(-1)^2 + a(-1) + b$
$= -10 - a + b$

$\therefore\ -10 - a + b = 1$

$-a + b = 11$ [2]

Solving the equations simultaneously gives:

$2b = 14$ [1] + [2]

$b = 7$

Substituting $b = 7$ into [1] gives:

$a + 7 = 3, a = -4$

Hence, $a = -4$ and $b = 7$.

EXAMPLE 15 The expression $4x^3 - x^2 + ax + 2$ leaves a remainder of b when divided by $x + 1$ and when the same expression is divided by $x - 2$ the remainder is $2b$. Find the values of a and b.

SOLUTION Let $f(x) = 4x^3 - x^2 + ax + 2$.

When $x = -1, f(-1) = 4(-1)^3 - (-1)^2 + a(-1) + 2$

$= -4 - 1 - a + 2$

$= -a - 3$

By the remainder theorem, $f(-1) = b$.

$\Rightarrow -a - 3 = b$

$a + b = -3$ [1]

When $x = 2, f(2) = 4(2)^3 - (2)^2 + a(2) + 2$

$= 32 - 4 + 2a + 2$

$= 2a + 30$

By the remainder theorem, $f(2) = 2b$.

$\therefore 2a + 30 = 2b$

$\therefore a + 15 = b$

$-a + b = 15$ [2]

$a + b - a + b = -3 + 15$ [1]+[2]

$\Rightarrow 2b = 12$

$b = 6$

Substituting into [1] gives:

$a + 6 = -3$

$a = -9$

Hence, $a = -9$ and $b = 6$.

Try these 4.2

(a) Find the remainder when $6x^3 - 3x^2 + x - 2$ is divided by the following.

(i) $x - 2$

(ii) $x + 1$

(iii) $2x - 1$

(b) When the expression $x^4 + ax^2 - 2x + 1$ is divided by $x - 1$ the remainder is 4. Find the value of a.

(c) When the expression $x^3 - 4x^2 + ax + b$ is divided by $2x - 1$ the remainder is 1. When the same expression is divided by $(x - 1)$ the remainder is 2. Find the values of a and b.

EXERCISE 4B

1 By using the remainder theorem, find the remainder when:

(a) $ax^4 + 3x^2 - 2x + 1$ is divided by $x - 1$

(b) $3x^3 + 6x^2 - 7x + 2$ is divided by $x + 1$

(c) $x^5 + 6x^2 - x + 1$ is divided by $2x + 1$

(d) $(4x + 2)(3x^2 + x + 2) + 7$ is divided by $x - 2$

(e) $x^7 + 6x^2 + 2$ is divided by $x + 2$

(f) $4x^3 - 3x^2 + 5$ is divided by $2x + 3$

(g) $3x^4 - 4x^3 + x^2 + 1$ is divided by $x - 3$

2 When the expression $x^2 - ax + 2$ is divided by $x - 2$, the remainder is a. Find a.

3 The expression $5x^2 - 4x + b$ leaves a remainder of 2 when divided by $2x + 1$. Find the value of b.

4 The expression $3x^3 + ax^2 + bx + 1$ leaves a remainder of 2 when divided by $x - 1$ and a remainder of 13 when divided by $x - 2$. Find the values of a and of b.

5 The expression $x^3 + px^2 + qx + 2$ leaves a remainder -3 when divided by $x + 1$ and a remainder of 54 when divided by $x - 2$. Find the numerical value of the remainder when the expression is divided by $2x + 1$.

6 Given that $f(x) = 2x^3 - 3x^2 - 4x + 1$ has the same remainder when divided by $x + a$ and by $x - a$, find the possible values of a.

7 Given that the remainder when $f(x) = 2x^3 - x^2 - 2x - 1$ is divided by $x - 2$, is twice the remainder when divided by $x - 2a$, show that $32a^3 - 8a^2 - 8a - 9 = 0$.

8 The remainder when $2x^3 - 5x^2 - 4x + b$ is divided by $x + 2$ is twice the remainder when it is divided by $x - 1$. Find the value of b.

9 The sum of the remainder when $x^3 + (\lambda + 5)x + \lambda$ is divided by $x - 1$ and by $x + 2$ is 0. Find the value of λ.

10 The remainder when $3x^3 + kx^2 + 15$ is divided by $x - 3$ is one-third the remainder when the same expression is divided by $3x - 1$. Find the value of k.

11 When the expression $3x^3 + px^2 + qx + 2$ is divided by $x^2 + 2x + 3$, the remainder is $x + 5$. Find the values of p and q.

12 The expression $8x^3 + px^2 + qx + 2$ leaves a remainder of $3\frac{1}{2}$ when divided by $2x - 1$ and a remainder of -1 when divided by $x + 1$. Find the values of p and the value of q.

13 When the expression $6x^5 + 4x^3 - ax + 2$ is divided by $x + 1$, the remainder is 15. Find the numerical value of a. Hence, find the remainder when the expression is divided by $x - 2$.

The factor theorem

The factor theorem

$x - \lambda$ is a factor of $f(x)$ if and only if $f(\lambda) = 0$.

PROOF

$$\frac{f(x)}{x-\lambda} = Q(x) + \frac{R}{x-\lambda}$$

$\Rightarrow \quad f(x) = (x-\lambda)Q(x) + R \quad$ (Multiplying both sides by $(x - \lambda)$)

Since $x - \lambda$ is a factor of $f(x) \Rightarrow R = 0$.

When $x = \lambda$:

$f(\lambda) = (\lambda - \lambda)Q(x) + 0$

$\therefore \quad f(\lambda) = 0$

EXAMPLE 16 Determine whether or not each of the following is a factor of the expression $x^3 + 2x^2 + 2x + 1$.

(a) $x - 1$

(b) $x + 1$

(c) $3x - 2$

SOLUTION Let $f(x) = x^3 + 2x^2 + 2x + 1$.

(a) When $x - 1 = 0$, $x = 1$.

If $x - 1$ is a factor of $f(x)$, then $f(1) = 0$

$f(1) = 1^3 + 2(1)^2 + 2(1) + 1 = 1 + 2 + 2 + 1 = 6$

Since $f(1) \neq 0$, $x - 1$ is not a factor of $f(x)$

(b) When $x + 1 = 0$, $x = -1$

$$\begin{aligned} f(-1) &= (-1)^3 + 2(-1)^2 + 2(-1) + 1 \\ &= -1 + 2 - 2 + 1 \\ &= -3 + 3 \\ &= 0 \end{aligned}$$

Since $f(-1) = 0 \Rightarrow x + 1$ is a factor of $f(x)$.

(c) When $3x - 2 = 0$, $x = \frac{2}{3}$

$$\begin{aligned} f\left(\frac{2}{3}\right) &= \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 + 2\left(\frac{2}{3}\right) + 1 \\ &= \frac{8}{27} + \frac{8}{9} + \frac{4}{3} + 1 \\ &= \frac{8 + 24 + 36 + 27}{27} \\ &= \frac{95}{27} \end{aligned}$$

Since $f\left(\frac{2}{3}\right) \neq 0 \Rightarrow 3x - 2$ is not a factor of $f(x)$.

EXAMPLE 17 For what value of k is $f(x) = 2x^3 - 2x^2 + kx + 1$ exactly divisible by $x - 2$?

SOLUTION Let $f(x) = 2x^3 - 2x^2 + kx + 1$.

Since $f(x)$ is divisible by $x - 2$, by the factor theorem $f(2) = 0$.

Substituting into $f(x)$ gives;

$$f(2) = 2(2)^3 - 2(2)^2 + k(2) + 1$$
$$= 16 - 8 + 2k + 1$$
$$= 9 + 2k$$

$f(2) = 0$

$\Rightarrow 9 + 2k = 0$

$2k = -9$

$k = \frac{-9}{2}$

EXAMPLE 18 The polynomial $2x^3 + 9x^2 + ax + 3$ has a factor $x + 3$.

(a) Find a.

(b) Show that $(x + 1)$ is also a factor and find the third factor.

SOLUTION

(a) Let $f(x) = 2x^3 + 9x^2 + ax + 3$.

Since $x + 3$ is a factor of (x), by the factor theorem $f(-3) = 0$.

$\therefore 2(-3)^3 + 9(-3)^2 + a(-3) + 3 = 0$

$\Rightarrow -54 + 81 - 3a + 3 = 0$

$\Rightarrow 3a = 30$

$\Rightarrow a = 10$

$\therefore f(x) = 2x^3 + 9x^2 + 10x + 3$

(b) If $x + 1$ is a factor then $f(-1) = 0$

$$f(-1) = 2(-1)^3 + 9(-1)^2 + 10(-1) + 3$$
$$= -2 + 9 - 10 + 3$$
$$= -12 + 12$$
$$= 0$$

$\therefore x + 1$ is a factor of $f(x)$.

Now we find the third factor.

Since $x + 1$ and $x + 3$ are factors, then $(x + 1)(x + 3)$ is a factor.

$\therefore (x + 1)(x + 3) = x^2 + 4x + 3$ is a factor of $f(x)$.

To find the third factor we can divide:

$$\begin{array}{r}
2x + 1 \\
x^2 + 4x + 3 \overline{\big)\, 2x^3 + 9x^2 + 10x + 3} \\
-\underline{(2x^3 + 8x^2 + 6x)} \\
x^2 + 4x + 3 \\
-\underline{(x^2 + 4x + 3)} \\
0
\end{array}$$

$\therefore$ $2x + 1$ is the third factor.

Alternative method to find the third factor:

Since $f(x) = 2x^3 + 9x^2 + 10x + 3$ and $x + 1$ and $x + 3$ are factors of $f(x)$:

$2x^3 + 9x^2 + 10x + 3 = (x + 1)(x + 3)(cx + d)$

To find c and d we can compare coefficients

Coefficients of x^3: $2 = 1 \times 1 \times c$

$\therefore c = 2$

Comparing constants:

$3 = 1 \times 3 \times d$

$3d = 3$

$d = 1$

$\therefore$ the third factor is $2x + 1$

EXAMPLE 19 The expression $6x^3 + px^2 + qx + 2$ is exactly divisible by $2x - 1$ and leaves a remainder of 2 when divided by $x - 1$. Find the values of p and q.

SOLUTION Let $f(x) = 6x^3 + px^2 + qx + 2$.

Since $2x - 1$ is a factor of $f(x) \Rightarrow f\left(\frac{1}{2}\right) = 0$.

$$f\left(\frac{1}{2}\right) = 6\left(\frac{1}{2}\right)^3 + p\left(\frac{1}{2}\right)^2 + q\left(\frac{1}{2}\right) + 2$$

$$= \frac{3}{4} + \frac{1}{4}p + \frac{1}{2}q + 2$$

$$= \frac{1}{4}p + \frac{1}{2}q + \frac{11}{4}$$

Since $f\left(\frac{1}{2}\right) = 0$

$\Rightarrow \frac{1}{4}p + \frac{1}{2}q + \frac{11}{4} = 0$

$\Rightarrow p + 2q + 11 = 0$ (Multiplying by 4)

$p + 2q = -11$ [1]

Using the remainder theorem, $f(1) = 2$.

Since there is a remainder of 2 when $f(x)$ is divided by $x - 1$:

$f(1) = 6(1)^3 + p(1)^2 + q(1) + 2$

$= p + q + 8$

Since $f(1) = 2$

$\Rightarrow p + q + 8 = 2$

$p + q = -6$ [2]

Solving simultaneously, and subtracting [2] from [1] gives:

$p + 2q - p - q = -11 - (-6)$

$\Rightarrow q = -5$

Substituting $q = -5$ into [2] gives:

$p - 5 = -6$

$p = -6 + 5$

$p = -1$

Hence, $p = -1$ and $q = -5$.

Try these 4.3

(a) Determine whether or not each of the following is a factor of the expression $3x^3 - x^2 - 3x + 1$.

(i) $x - 1$

(ii) $2x + 1$

(iii) $3x - 1$

(b) The expression $4x^3 + px^2 - qx - 6$ is exactly divisible by $4x + 1$ and leaves a remainder of -20 when divided by $x - 1$. Find the values of p and q.

Factorising polynomials and solving equations

A combination of the factor theorem and long division can be used to factorise polynomials. Descartes' rule of signs can assist in determining whether a polynomial has positive or negative roots and can give an idea of how many of each type of roots.

(a) To find the number of positive roots in a polynomial, we count the number of times the consecutive terms of the function changes sign and then subtract multiples of 2. For example, if $f(x) = 4x^3 - 3x^2 + 2x + 1$, then $f(x)$ changes sign two times consecutively. $f(x)$ has either 2 positive roots or 0 positive roots.

(b) To identify the number of negative roots, count the number of sign changes in $f(-x)$. The number of sign changes or an even number fewer than this represents the number of negative roots of the function.

If $f(x) = 4x^3 - 3x^2 + 2x + 1$, then $f(-x) = 4(-x)^3 - 3(-x)^2 + 2(-x) + 1$

$\Rightarrow f(-x) = -4x^3 - 3x^2 - 2x + 1$

Since there is 1 sign change, there is 1 negative root to the equation.

When identifying the roots of a polynomial we try to find roots by trial and error first. The rational root test can give us some values to try first.

Let $f(x) = 2x^3 + 7x^2 + 7x + 2 = 0$.

All the factors of the coefficient of the leading polynomial x^3 are $\pm 1, \pm 2$. All the factors of the constant are $\pm 1, \pm 2$. Therefore, possible roots of this equation are:

$\frac{\pm 1}{+1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm 1, \pm\frac{1}{2}, \pm 2$

Since all the coefficients of the terms are positive, we try only the negative values.

$f(-1) = 2(-1)^3 + 7(-1)^2 + 7(-1) + 2 = -2 + 7 - 7 + 2 = 0$

$\therefore x = -1$ is a root of $f(x) = 0$ and $x + 1$ is a factor of $f(x)$.

$$f\left(\frac{-1}{2}\right) = 2\left(\frac{-1}{2}\right)^3 + 7\left(\frac{-1}{2}\right)^2 + 7\left(\frac{-1}{2}\right) + 2$$
$$= \frac{-1}{4} + \frac{7}{4} - \frac{7}{2} + 2 = \frac{-15}{4} + \frac{15}{4} = 0$$

$\therefore x = \frac{-1}{2}$ is a root of $f(x) = 0$, and $2x + 1$ is a factor of $f(x)$.

EXAMPLE 20 Factorise $f(x) = 4x^3 - 15x^2 + 17x - 6$.

SOLUTION By trial and error:

$$\begin{aligned} f(1) &= 4(1)^3 - 15(1)^2 + 17(1) - 6 \\ &= 4 - 15 + 17 - 6 \\ &= 21 - 21 \\ &= 0 \end{aligned}$$

Since $f(1) = 0$

$\Rightarrow x - 1$ is a factor of $f(x)$

We can divide to get a quadratic factor:

$$\begin{array}{r} 4x^2 - 11x + 6 \\ x - 1\overline{)4x^3 - 15x^2 + 17x - 6} \\ -\underline{(4x^3 - 4x^2)} \\ -11x^2 + 17x \\ -\underline{(-11x^2 + 11x)} \\ 6x - 6 \\ -\underline{(6x - 6)} \\ 0 \end{array}$$

$$\begin{aligned} \therefore f(x) &= (x - 1)(4x^2 - 11x + 6) \\ &= (x - 1)(x - 2)(4x - 3) \end{aligned}$$

(By factorising: $4x^2 - 11x + 6$
$= 4x^2 - 8x - 3x + 6$
$= 4x(x - 2) - 3(x - 2)$
$= (4x - 3)(x - 2)$)

EXAMPLE 21 Find all the real factors of $f(x) = x^4 + x^3 - x - 1$.

SOLUTION By trial and error:

$$\begin{aligned} f(1) &= 1^4 + 1^3 - 1 - 1 \\ &= 2 - 2 \\ &= 0 \end{aligned}$$

$\therefore\ x - 1$ is a factor of $f(x)$

$$\begin{aligned} f(-1) &= (-1)^4 + (-1)^3 - (-1) - 1 \\ &= 1 - 1 + 1 - 1 \\ &= 0 \end{aligned}$$

$\therefore\ x + 1$ is a factor of $f(x)$

Since $(x - 1)$ and $(x + 1)$ are factors of $f(x)$, a quadratic factor is:

$(x - 1)(x + 1) = x^2 - 1$

By division:

$$\begin{array}{r l}
 & x^2 + x + 1 \\
x^2 - 1 \big) & \overline{x^4 + x^3 - x - 1} \\
 & -(x^4 - x^2) \\
 & \overline{\quad x^3 + x^2 - x} \\
 & \quad -(x^3 - x) \\
 & \quad \overline{\quad x^2 - 1} \\
 & \quad\quad -(x^2 - 1) \\
 & \quad\quad \overline{\qquad 0}
\end{array}$$

$\therefore\ f(x) = (x - 1)(x + 1)(x^2 + x + 1)$

EXAMPLE 22 Solve the equation $x^3 + 3x^2 - 10x - 24 = 0$.

SOLUTION Let $f(x) = x^3 + 3x^2 - 10x - 24$.

By trial and error:

$$\begin{aligned} f(-2) &= (-2)^3 + 3(-2)^2 - 10(-2) - 24 \\ &= -8 + 12 + 20 - 24 \\ &= -32 + 32 \\ &= 0 \end{aligned}$$

Since $f(-2) = 0 \Rightarrow x + 2$ is a factor of $f(x)$

By division:

$$\begin{array}{r l}
 & x^2 + x - 12 \\
x + 2 \big) & \overline{x^3 + 3x^2 - 10x - 24} \\
 & -(x^3 + 2x^2) \\
 & \overline{\quad x^2 - 10x} \\
 & \quad -(x^2 + 2x) \\
 & \quad \overline{\quad -12x - 24} \\
 & \quad\quad -(-12x - 24) \\
 & \quad\quad \overline{\qquad 0}
\end{array}$$

$\therefore f(x) = (x + 2)(x^2 + x - 12)$
$= (x + 2)(x + 4)(x - 3)$

Hence, $(x + 2)(x + 4)(x - 3) = 0$

$x + 2 = 0, x = -2$

$x + 4 = 0, x = -4$

$x - 3 = 0, x = 3$

$\therefore$ the roots are $x = -2, -4, 3$.

EXAMPLE 23 Factorise the expression $10x^3 + x^2 - 8x - 3$. Hence, find the roots of the equation $10x^3 + x^2 - 8x - 3 = 0$.

SOLUTION Let $f(x) = 10x^3 + x^2 - 8x - 3$.

By trial and error:

$$
\begin{aligned}
f(1) &= 10(1)^3 + (1)^2 - 8(1) - 3 \\
&= 10 + 1 - 8 - 3 \\
&= 11 - 11 \\
&= 0
\end{aligned}
$$

$\therefore x - 1$ is a factor of $f(x)$

By division:

$$
\begin{array}{r}
10x^2 + 11x + 3 \\
x - 1 \overline{\smash{\big)}\, 10x^3 + x^2 - 8x - 3} \\
-\underline{(10x^3 - 10x^2)} \\
11x^2 - 8x \\
-\underline{(11x^2 - 11x)} \\
3x - 3 \\
-\underline{(3x - 3)} \\
0
\end{array}
$$

$\therefore f(x) = (x - 1)(10x^2 + 11x + 3)$
$= (x - 1)(2x + 1)(5x + 3)$

Since $f(x) = 0 \Rightarrow (x - 1)(2x + 1)(5x + 3) = 0$

$\Rightarrow x - 1 = 0, x = 1$

$2x + 1 = 0, x = -\frac{1}{2}$

$5x + 3 = 0, x = -\frac{3}{5}$

The roots are $x = 1$, $x = -\frac{1}{2}$ and $x = -\frac{3}{5}$.

EXAMPLE 24 Find the exact roots of the equation $x^3 - x^2 - 4x - 2 = 0$.

SOLUTION Let $f(x) = x^3 - x^2 - 4x - 2$.

By trial and error:

$$
\begin{aligned}
f(-1) &= -1 - 1 + 4 - 2 \\
&= -4 + 4 \\
&= 0
\end{aligned}
$$

$\therefore x + 1$ is a factor of $f(x)$.

By long division:

$$\begin{array}{r} x^2 - 2x - 2 \\ x+1 \overline{) x^3 - x^2 - 4x - 2} \\ -(x^3 + x^2) \\ \hline -2x^2 - 4x \\ -(-2x^2 - 2x) \\ \hline -2x - 2 \\ -(-2x - 2) \\ \hline 0 \end{array}$$

$\therefore f(x) = (x + 1)(x^2 - 2x - 2)$

$f(x) = 0 \Rightarrow (x + 1)(x^2 - 2x - 2) = 0$

$\Rightarrow x + 1 = 0, x = -1$

$\Rightarrow x^2 - 2x - 2 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)} \qquad \text{(Using the quadratic formula)}$$

$$= \frac{2 \pm \sqrt{12}}{2}$$

$$= \frac{2 \pm 2\sqrt{3}}{2}$$

$$= 1 \pm \sqrt{3}$$

$\therefore x = -1, 1 + \sqrt{3}, 1 - \sqrt{3}$

Try these 4.4

(a) Show that $x - 1$ is a factor of $2x^3 + 5x^2 - 4x - 3$. Hence, solve the equation $2x^3 + 5x^2 - 4x - 3 = 0$

(b) Factorise $f(x) = x^3 - 2x^2 - 5x + 6$. Hence, solve $f(x) = 0$.

EXERCISE 4C

1 Factorise each of the following.

(a) $x^3 + 2x^2 - x - 2$

(b) $x^3 + 6x^2 + 11x + 6$

(c) $x^3 - 7x + 6$

(d) $x^3 - 4x^2 + x + 6$

(e) $x^3 - 7x - 6$

(f) $6x^3 + 31x^2 + 3x - 10$

2 Solve the following equations.

(a) $3x^3 + x^2 - 20x + 12 = 0$

(b) $2x^3 + 13x^2 + 17x - 12 = 0$

(c) $2x^3 - 11x^2 + 3x + 36 = 0$

(d) $3x^3 + 10x^2 + 9x + 2 = 0$

3 Decide whether each of these statements is true.

(a) $x + 1$ is a factor of $x^3 + 2x^2 + 2x + 1$

(b) $x - 1$ is a factor of $2x^4 + 3x^2 - x + 2$

(c) $x - 2$ is a factor of $x^5 - 4x^4 + 3x^3 - 2x^2 + 4$

(d) $2x + 1$ is a factor of $2x^3 + 9x^2 + 10x + 3$

4 Find the value of k for which $x - 1$ is a factor of $x^3 - 3x^2 + kx + 2$.

5 Show that $x - 2$ is a factor of $f(x) = x^3 - 12x + 16$. Hence, factorise $f(x)$ completely.

6 Given that $f(x) = 4x^3 - 3x^2 + 5x + k$ is exactly divisible by $x + 1$, find the value of k.

7 Solve the equation $x^3 + 3x^2 - 6x - 8 = 0$.

8 Given that $f(x) = 3x^3 - kx^2 + 5x + 2$ is exactly divisible by $2x - 1$, find the value of k.

9 Find the exact solutions of $x^3 + 5x^2 + 3x - 1 = 0$.

10 The expression $4x^3 + ax^2 + bx + 3$ is exactly divisible by $x + 3$ but leaves a remainder of 165 when divided by $x - 2$.

(a) Find the values of a and b.

(b) Factorise $4x^3 + ax^2 + bx + 3$.

11 Given that $x^2 - 3x + 2$ is a factor of $f(x)$, where $f(x) = x^4 + x^3 + ax^2 + bx + 10$, find the following.

(a) The value of a and of b

(b) The other quadratic factor of $f(x)$.

12 Given that $6x + 1$ and $3x - 2$ are factors of $(x) = 36x^3 + ax^2 + bx - 2$, find the values of a and b. Hence, find the third factor of $f(x)$.

13 The expression $4x^3 + ax^2 + bx + 3$ is divisible by $x + 3$ but leaves a remainder of -12 when divided by $x - 1$. Calculate the value of a and of b.

14 Given that $x - 3$ and $x + 1$ are both factors of $g(x) = 4x^4 + px^3 - 21x^2 + qx + 27$, find the values of p and q. Given also that $g(x) = (x - 3)(x + 1)(ax^2 - b)$, find a and b, and hence, factorise $g(x)$ completely.

Factorising $x^n - y^n$

EXAMPLE 25 Factorise $x^2 - y^2$.

SOLUTION $x^2 - y^2 = (x - y)(x + y)$ by the difference of two squares.

EXAMPLE 26 Factorise $x^3 - y^3$.

SOLUTION Let $f(x) = x^3 - y^3$.

When $x = y$, $f(y) = y^3 - y^3 = 0$

By the factor theorem, $x - y$ is a factor of $x^3 - y^3$.

By division:

$$\begin{array}{r l}
 & x^2 + xy + y^2 \\ \cline{2-2}
x - y \big) & x^3 + 0x^2y + 0xy^2 - y^3 \\
 & -(x^3 - x^2y) \\ \cline{2-2}
 & x^2y + 0xy^2 \\
 & -(x^2y - xy^2) \\ \cline{2-2}
 & xy^2 - y^3 \\
 & -(xy^2 - y^3) \\ \cline{2-2}
 & 0
\end{array}$$

$\therefore x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

Using $y \equiv -y$ in $(x - y)(x^2 + xy + y^2)$, we get:

$x^3 - (-y)^3 = (x - (-y))(x^2 + x(-y) + (-y)^2)$

$\Rightarrow x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Hence, $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

and $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

EXAMPLE 27 Factorise $8m^3 - 27$.

SOLUTION We can write $8m^3 - 27$ as $(2m)^3 - 3^3$, using $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

With $x = 2m$ and $y = 3$, we have:

$(2m)^3 - 3^3 = (2m - 3)((2m)^2 + (2m)(3) + (3)^2)$

Hence, $8m^3 - 27 = (2m - 3)(4m^2 + 6m + 9)$

EXAMPLE 28 Factorise $x^4 - y^4$.

SOLUTION

$$\begin{aligned} x^4 - y^4 &= (x^2 - y^2)(x^2 + y^2) \quad \text{(Difference of two squares)} \\ &= (x - y)(x + y)(x^2 + y^2) \end{aligned}$$

EXAMPLE 29 Find all the real factors of $x^4 - 16$.

SOLUTION

$$\begin{aligned} x^4 - 16 &= x^4 - 2^4 \\ &= (x^2 - 2^2)(x^2 + 2^2) \\ &= (x - 2)(x + 2)(x^2 + 4) \end{aligned}$$

$\therefore$ the real factors of $x^4 - 16$ are $x - 2$, $x + 2$ and $x^2 + 4$.

EXAMPLE 30 Factorise $x^5 - y^5$.

SOLUTION Let $f(x) = x^5 - y^5$.

When $x = y$, $f(y) = y^5 - y^5$

$= 0$

$\Rightarrow x - y$ is a factor of $x^5 - y^5$

By division:

$$\begin{array}{r}
x^4 + x^3y + x^2y^2 + xy^3 + y^4 \\
x - y \overline{)\, x^5 + 0x^4y + 0x^3y^2 + 0x^2y^3 + 0xy^4 - y^5} \\
-\underline{(x^5 - x^4y)} \qquad\qquad\qquad\qquad\qquad\qquad \\
x^4y + 0x^3y^2 \qquad\qquad\qquad\qquad \\
-\underline{(x^4y - x^3y^2)} \qquad\qquad\qquad\qquad \\
x^3y^2 + 0x^2y^3 \qquad\qquad\quad \\
-\underline{(x^3y^2 - x^2y^3)} \qquad\qquad\quad \\
x^2y^3 + 0xy^4 \qquad \\
-\underline{(x^2y^3 - xy^4)} \qquad \\
xy^4 - y^5 \\
-\underline{(xy^4 - y^5)} \\
0
\end{array}$$

$\therefore x^5 - y^5 = (x^4 + x^3y + x^2y^2 + xy^3 + y^4)(x - y)$

Using our solution to Example 30, we can replace $y \equiv -y$ and obtain:

$x^5 - (-y)^5 = (x^4 + x^3(-y) + x^2(-y)^2 + x(-y)^3 + (-y)^4)(x - (-y))$

$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

Notice the pattern in the factors. As the powers of x decrease the powers of y increases in each term.

EXAMPLE 31 Find the factors of $x^5 - 32$.

SOLUTION $x^5 - 32 = x^5 - 2^5$

Using $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ with $x \equiv x$, $y = 2$ gives:

$x^5 - 2^5 = (x - 2)(x^4 + x^3(2) + x^2(2)^2 + x(2)^3 + (2)^4)$

$x^5 - 32 = (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)$

Try these 4.5 Factorise these.

(a) $27x^3 - 64$

(b) $81x^4 - 16$

(c) $x^6 - y^6$

SUMMARY

Polynomials

Degree of a polynomial

Remainder theorem

When a polynomial $f(x)$ is divided by $x - \lambda$, the remainder is $f(\lambda)$.

Factor theorem

If $x - \lambda$ is a factor of $f(x)$, then $f(\lambda) = 0$.

$x^2 - y^2 = (x - y)(x + y)$

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$x^4 - y^4 = (x - y)(x + y)(x^2 + y^2)$

$x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$

$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4)$

$x^6 - y^6 = (x - y)(x + y)(x^2 + xy + y^2)(x^2 - xy + y^2)$

Checklist

Can you do these?

- Identify a polynomial.
- Identify the order of a polynomial.
- Identify the terms of a polynomial.
- Add, subtract, multiply and divide polynomials.
- Evaluate polynomials.
- Compare polynomials.
- Use the remainder theorem.
- Factorise polynomials using the factor theorem.
- Solve equations involving polynomials.
- Factorise $x^n - y^n$ where $n = 2, 3, 4, 5, 6$.

Review Exercise 4

1 Find all the real factors of $x^6 - 64$.

2 Factorise in full $32x^5 - 243$.

3 Factorise the following.

(a) $6x^3 + x^2 - 5x - 2$

(b) $3x^3 - 2x^2 - 7x - 2$

(c) $4x^3 + 9x^2 - 10x - 3$

4 Solve the following equations.

(a) $x^3 - 2x^2 - 4x + 8 = 0$

(b) $2x^3 - 7x^2 - 10x + 24 = 0$

5 Find the remainder when

(a) $7x^3 - 5x^2 + 2x + 1$ is divided by $3x - 2$

(b) $6x^3 + 7x + 1$ is divided by $x + 3$

(c) $x^4 + 2$ is divided by $2x + 5$

6 It is given that $f(x) = x^3 - ax^2 + 2ax - 8$, where a is a constant. Use the factor theorem to find a linear factor of $f(x)$, and find the set of values of a for which all the roots of the equation $f(x) = 0$ are real.

7 It is given that $f(x) = (3x + 2)(x - 1)(x - 2)$.

(a) Express $f(x)$ in the form $Ax^3 + Bx^2 + Cx + D$, giving the values of A, B, C and D.

(b) Hence, find the value of the constant b such that $x + 2$ is a factor of $f(x) + bx$.

8 (a) Show that $(x - \sqrt{2})$ and $(x + \sqrt{2})$ are factors of $x^4 + 5x^3 + 4x^2 - 10x - 12$. Hence, write down one quadratic factor of $x^4 + 5x^3 + 4x^2 - 10x - 12$ and find a second quadratic factor of this polynomial.

(b) Solve $x^4 + 5x^3 + 4x^2 - 10x - 12 = 0$.

9 The polynomial $q(x)$, where $q(x) = x^4 - ax^3 + bx^2 + x + 6$, has factors $(x - 2)$ and $(x - 3)$. Show that $a = 4$ and $b = 2$.

10 Given that $Ax^3 + Bx^2 + Cx + D = (2x + 1)(x - 2)(3x + 4)$, find the values of A, B, C and D.

11 The polynomial $P(x) = 3x^4 + Bx^3 + Cx^2 + Dx + 2$ can be written as $P(x) = (3x^2 + 2x + 1)(x^2 - 4x + 2)$. Find the values of A, B, C and D.

12 The expression $ax^3 - bx^2 + 8x + 2$ leaves a remainder of -110 when divided by $x + 2$ and a remainder of 13 when divided by $x - 1$. Find the remainder when the same expression is divided by $3x + 2$.

13 Show that $(x + 1)$ is a factor of $f(x) = 2x^3 - x^2 - 5x - 2$. Hence, write $f(x)$ as the product of three linear factors.

14 Solve the equation $6x^3 - 5x^2 - 13x - 2 = 0$.

15 Show that $x + 3$ is a factor of $f(x) = x^4 + x^3 - 11x^2 - 27x - 36$. Factorise $f(x)$ completely. Hence, solve the equation $f(x) = 0$.

16 In 2006, a hurricane struck a country causing over \$100 million dollars in damage. The following data represents the number of major hurricanes striking the country for each decade from 1921 to 2000.

Decade (t)	1	2	3	4	5	6	7	8
Major hurricanes striking the country (h)	5	8	10	8	6	4	5	5

(a) Plot this data on graph paper.

(b) The cubic function that best fits this data is $h(t) = 0.16t^3 - 2.3t^2 + 9.3t - 2.2$. Use this function to predict the number of major hurricanes that struck the country between 1971 and 1980.

(c) Plot the graph of $h(t)$ by using a table of values. Draw the graph on the same graph paper as in **(a)**.

(d) Is $h(t)$ a good function to investigate the given data?

(e) After 2000, for the next five years, five major hurricanes struck this country, does this support the model in **(b)**?

17 An open, rectangular box is made from a piece of cardboard 24 cm by 16 cm, by cutting a square of side x cm from each corner and folding up the sides.

(a) Show that the resulting volume is given by $V = 4x^3 - 80x^2 + 384x$.

(b) Find the height, x, of the box when the volume is 512 cm^3.

18 Let $f(x) = 2x^3 - 7x^2 - 10x + 24$.

(a) Factorise $f(x)$.

(b) Solve the equation $f(x) = 0$.

CHAPTER 5

Indices, Surds and Logarithms

At the end of this chapter you should be able to:

- Simplify a surd
- Use the rules of surds
- Rationalise a surd
- Define an exponential function $y = a^x$ for $x \in \mathbb{R}$
- Sketch the graph of exponential functions
- Identify the properties of the exponential functions from the graph
- Define a logarithmic function
- Investigate the properties of the logarithmic function
- Define the function $y = e^x$
- Define the inverse of the exponential function as the logarithmic function
- Use the fact that $y = e^x \leftrightarrow x = \log_e y$
- Use the laws of logs to solve equations
- Solve equations of the form $a^x = b$
- Change of base of logarithms
- Solve problems based on the applications of exponentials and logarithms

KEYWORDS/TERMS

exponential functions • base • index • logarithm • naperian logarithm • inverse • domain • range

Indices

In an expression such as a^b, the base is a and b is called the power or index. The properties of indices which we recall here were developed in *Mathematics for CSEC® Examinations* (CSEC Mathematics).

Laws of indices

Law 1

$a^m x a^n = a^{m+n}$

Law 2

$\dfrac{a^m}{a^n} = a^{m-n}$

Law 3

$a^{-m} = \dfrac{1}{a^m}$

Law 4

$a^0 = 1$

Law 5

$(a^m)^n = a^{mn}$

Law 6

$a^{\frac{1}{m}} = \sqrt[m]{a}$

Law 7

$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$ or $a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

EXAMPLE 1 Simplify the expressions, expressing all answers with positive exponents.

(a) $x^{\frac{1}{5}}\ x^{\frac{3}{5}}$

(b) $a^{\frac{5}{6}}a^{-\frac{1}{3}}$

(c) $\dfrac{y^{\frac{-2}{5}}}{y^{\frac{-3}{10}}}y^2$

(d) $(27x^3y^6)^{\frac{1}{3}}$

SOLUTION

(a) $x^{\frac{1}{5}}x^{\frac{3}{5}} = x^{\frac{1}{5}+\frac{3}{5}} = x^{\frac{4}{5}}$

(b) $a^{\frac{5}{6}}\ a^{-\frac{1}{3}} = a^{\frac{5}{6}-\frac{1}{3}} = a^{\frac{1}{2}}$

(c) $\dfrac{y^{\frac{-2}{5}}}{y^{\frac{-3}{10}}}y^2 = y^{\frac{-2}{5}+2-\left(\frac{-3}{10}\right)} = y^{\frac{19}{10}}$

(d) $(27x^3y^6)^{\frac{1}{3}} = 27^{\frac{1}{3}}x^{3\times\frac{1}{3}}y^{6\times\frac{1}{3}} = 3xy^2$

EXAMPLE 2 Simplify the following expressions.

(a) $\dfrac{(x^2+4)^{\frac{1}{2}} - x^2(x^2+4)^{\frac{-1}{2}}}{x^2+4}$

(b) $\dfrac{(x+4)^{\frac{1}{2}} - 2x\,(x+4)^{\frac{-1}{2}}}{x+4}$

SOLUTION

(a)
$$\frac{(x^2+4)^{\frac{1}{2}} - x^2(x^2+4)^{\frac{-1}{2}}}{x^2+4} = \frac{(x^2+4)^{\frac{1}{2}} - \dfrac{x^2}{(x^2+4)^{\frac{1}{2}}}}{x^2+4}$$
$$= \frac{\dfrac{(x^2+4)^{\frac{1}{2}}(x^2+4)^{\frac{1}{2}} - x^2}{(x^2+4)^{\frac{1}{2}}}}{x^2+4}$$
$$= \frac{\dfrac{x^2+4-x^2}{(x^2+4)^{\frac{1}{2}}}}{x^2+4}$$
$$= \frac{\dfrac{4}{(x^2+4)^{\frac{1}{2}}}}{x^2+4} = \frac{4}{(x^2+4)^{\frac{3}{2}}}$$

(b)
$$\frac{(x+4)^{\frac{1}{2}} - 2x(x+4)^{\frac{-1}{2}}}{x+4} = \frac{(x+4)^{\frac{1}{2}} - \dfrac{2x}{(x+4)^{\frac{1}{2}}}}{x+4}$$
$$= \frac{(x+4) - 2x}{(x+4)(x+4)^{\frac{1}{2}}}$$
$$= \frac{4-x}{(x+4)^{\frac{3}{2}}}$$

EXERCISE 5A (Revision)

1 Find the values of the following.

(a) $125^{\frac{1}{3}}$ (b) $81^{\frac{1}{4}}$

(c) $16^{\frac{1}{4}}$ (d) $(4^6)^{\frac{2}{3}}$

(e) $8^{\frac{-1}{3}}$ (f) $\left(\frac{1}{4}\right)^{-2}$

(g) $(121)^{\frac{3}{2}}$ (h) $81^{\frac{-1}{4}}$

2 Simplify the following.

(a) $a^{\frac{3}{4}}a^{\frac{1}{4}}$

(b) $a^{\frac{-2}{3}}a^{\frac{3}{4}}$

(c) $y^{\frac{-1}{4}} \div y^{\frac{5}{4}}$

(d) $\dfrac{y^2y^4}{y^3}$

3 Find the values of the following.

(a) $\dfrac{64^{\frac{1}{3}} \times 216^{\frac{1}{3}}}{8}$

(b) $\dfrac{4^6 \times 5^4}{20^7}$

(c) $\dfrac{32^{\frac{1}{6}} \times 16^{\frac{1}{12}}}{8^{\frac{1}{6}} \times 4^{\frac{1}{3}}}$

4 Simplify these.

(a) $8^{2x} \times 2^{5x} \div 4^{6x}$

(b) $27^{3x} \times 9^{3x} \div 81^{5x}$

(c) $\left(10^{\frac{1}{3}x} \times 20^{\frac{2}{3}x}\right) \div 2^{\frac{2}{3}x}$

(d) $\left(16^{\frac{3}{4}x} \div 8^{\frac{5}{3}x}\right) \times 4^{x+1}$

5 Simplify these.

(a) $\dfrac{(1-x)^{\frac{3}{2}}}{(1-x)^{\frac{1}{2}}}$

(b) $\dfrac{(1+x)^{\frac{1}{3}} + (1+x)^{\frac{3}{4}}}{(1+x)^{\frac{1}{3}}}$

6 Evaluate $\sqrt{\dfrac{3^{10} + 3^9}{3^{11} + 3^{12}}}$.

In questions **7** to **9**, write each expression as a single quotient. Simplify as far as possible.

7 $\dfrac{x}{(1+x)^{\frac{1}{2}}} + 2(1+x)^{\frac{1}{2}}$

8 $(x+1)^{\frac{1}{3}} + \dfrac{x}{3(x+1)^{\frac{2}{3}}}$

9 $\dfrac{\sqrt{4x+3}}{2\sqrt{x-5}} + \dfrac{\sqrt{x-5}}{5\sqrt{4x+3}}$

In questions **10** to **12**, factorise and simplify each of the expressions.

10 $(x+1)^{\frac{3}{2}} + \frac{3}{2}x(x+1)^{\frac{1}{2}}$

11 $3(x^2+4)^{\frac{4}{3}} + 8x^2(x^2+4)^{\frac{1}{3}}$

12 $2x(3x+4)^{\frac{4}{3}} + 4x^2(3x+4)^{\frac{1}{3}}$

Surds

The symbol $\sqrt[n]{x}$ is called a radical. The integer n is called the **index**, and x is called the **radicand**. $\sqrt{x}$ represents the square root of x. Recall that an irrational number is a number that cannot be expressed as a fraction of two integers.

An irrational number involving a root is called a **surd**. Examples of surds are $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{7}$ etc.

Rules of surds

Rule 1

$\sqrt{xy} = \sqrt{x}\,\sqrt{y}$

The root of a product is equal to the product of the roots.

EXAMPLE 3 Simplify $\sqrt{12}$.

SOLUTION

$$\sqrt{12} = \sqrt{4 \times 3} = \sqrt{4}\,\sqrt{3}$$
$$= 2\sqrt{3}$$

EXAMPLE 4 Simplify $\sqrt{75}$.

SOLUTION

$$\sqrt{75} = \sqrt{25 \times 3}$$
$$= \sqrt{25}\sqrt{3}$$
$$= 5\sqrt{3}$$

Rule 2

$\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$

The root of a quotient is the quotient of the separate roots.

EXAMPLE 5 Simplify $\sqrt{\dfrac{72}{25}}$.

SOLUTION

$$\sqrt{\frac{72}{25}} = \frac{\sqrt{36 \times 2}}{\sqrt{25}} = \frac{\sqrt{36} \times \sqrt{2}}{5} = \frac{6}{5}\sqrt{2}$$

EXAMPLE 6 Simplify $\sqrt{\dfrac{96}{5}}$.

SOLUTION

$$\sqrt{\frac{96}{5}} = \frac{\sqrt{96}}{\sqrt{5}}$$
$$= \frac{\sqrt{16 \times 6}}{\sqrt{5}} = \frac{\sqrt{16}\sqrt{6}}{\sqrt{5}} = \frac{4\sqrt{6}}{\sqrt{5}}$$

Note

$\sqrt{x + y} \neq \sqrt{x} + \sqrt{y}$

One of the most common errors in surds is to separate the sum or difference of a surd by writing $\sqrt{x + y} = \sqrt{x} + \sqrt{y}$. Remember that this is incorrect.

Simplifying surds

EXAMPLE 7 Simplify this as far as possible.

$\sqrt{80} + 3\sqrt{180} - \sqrt{320}$

SOLUTION We rewrite all terms, using $\sqrt{5}$.

$$\begin{aligned}\sqrt{80} &= \sqrt{16 \times 5}\\ &= \sqrt{16}\sqrt{5}\\ &= 4\sqrt{5}\end{aligned}$$

$$\begin{aligned}3\sqrt{180} &= 3\sqrt{36 \times 5}\\ &= 3\sqrt{36}\sqrt{5}\\ &= 3 \times 6\sqrt{5}\\ &= 18\sqrt{5}\end{aligned}$$

$$\begin{aligned}\sqrt{320} &= \sqrt{64 \times 5}\\ &= \sqrt{64}\sqrt{5}\\ &= 8\sqrt{5}\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{80} + 3\sqrt{180} - \sqrt{320} &= 4\sqrt{5} + 18\sqrt{5} - 8\sqrt{5}\\ &= 22\sqrt{5} - 8\sqrt{5}\\ &= 14\sqrt{5}\end{aligned}$$

EXAMPLE 8 Simplify $\sqrt{320x^3} + \sqrt{125x^3}$.

SOLUTION

$$\begin{aligned}\sqrt{320x^3} = \sqrt{320} \times \sqrt{x^3} &= \sqrt{64 \times 5} \times \sqrt{x^2 \times x}\\ &= \sqrt{64}\sqrt{5} \times \sqrt{x^2}\sqrt{x}\\ &= 8\sqrt{5} \times x\sqrt{x}\\ &= 8x\sqrt{5x}\end{aligned}$$

$$\begin{aligned}\sqrt{125x^3} &= \sqrt{125}\sqrt{x^3}\\ &= \sqrt{25 \times 5}\sqrt{x^2 \times x}\\ &= \sqrt{25}\sqrt{5} \times \sqrt{x^2}\sqrt{x}\\ &= 5\sqrt{5} \times x\sqrt{x}\\ &= 5x\sqrt{5}\sqrt{x}\\ &= 5x\sqrt{5x}\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{320x^3} + \sqrt{125x^3} &= 8x\sqrt{5x} + 5x\sqrt{5x}\\ &= 13x\sqrt{5x}\end{aligned}$$

Try these 5.1 Simplify the following surds.

(a) $\sqrt{75} + \sqrt{48} - 2\sqrt{675}$

(b) $4\sqrt{288} - 3\sqrt{882}$

(c) $5\sqrt{80} + 3\sqrt{20} - \sqrt{125}$

Conjugate surds

$\sqrt{x} + \sqrt{y}$ and $\sqrt{x} - \sqrt{y}$ are called conjugate surds. When a surd and its conjugate are multiplied together the product is a rational number.

EXAMPLE 9 Find $(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$.

SOLUTION

$$\begin{aligned}(\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y}) &= \sqrt{x}\sqrt{x} - \cancel{\sqrt{x}\sqrt{y}} + \cancel{\sqrt{x}\sqrt{y}} - \sqrt{y}\sqrt{y}\\ &= x - y\end{aligned}$$

EXAMPLE 10 Find the value of $(\sqrt{5} + 2)(\sqrt{5} - 2)$.

SOLUTION Expanding the brackets gives:

$$\begin{aligned}(\sqrt{5} + 2)(\sqrt{5} - 2) &= \sqrt{5}\sqrt{5} - \cancel{2\sqrt{5}} + \cancel{2\sqrt{5}} - 2 \times 2\\ &= 5 - 4\\ &= 1\end{aligned}$$

EXAMPLE 11 Simplify $(\sqrt{x} + 2)(\sqrt{x} - 2)$.

SOLUTION Expanding the brackets gives:

$$\begin{aligned}(\sqrt{x} + 2)(\sqrt{x} - 2) &= \sqrt{x} \times \sqrt{x} - \cancel{2\sqrt{x}} + \cancel{2\sqrt{x}} - 4\\ &= x - 4\end{aligned}$$

EXAMPLE 12 Show that $(4 + \sqrt{x + 2})(4 - \sqrt{x + 2}) = 14 - x$.

SOLUTION We start with the left-hand side.

Expanding the brackets gives:

$$\begin{aligned}(4 + \sqrt{x+2})(4 - \sqrt{x+2}) &= 16 - 4\sqrt{x+2} + 4\sqrt{x+2} - (\sqrt{x+2})(\sqrt{x+2})\\ &= 16 - (x + 2)\\ &= 14 - x\end{aligned}$$

Rationalising the denominator

EXAMPLE 13 Rationalise $\frac{4}{\sqrt{5}}$.

SOLUTION

$$\begin{aligned}\frac{4}{\sqrt{5}} &= \frac{4}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}\\ &= \frac{4\sqrt{5}}{5}\end{aligned}$$

EXAMPLE 14 Rationalise $\frac{2}{\sqrt{x}}$.

SOLUTION

$$\frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}} \times \frac{\sqrt{x}}{\sqrt{x}}$$
$$= \frac{2\sqrt{x}}{x}$$

EXAMPLE 15 Rationalise the denominator of $\frac{1}{\sqrt{2}+3}$.

SOLUTION We multiply the numerator and denominator by the conjugate of the denominator.

The conjugate of $\sqrt{2}+3$ is $\sqrt{2}-3$.

$$\therefore \frac{1}{\sqrt{2}+3} = \frac{1}{\sqrt{2}+3} \times \frac{\sqrt{2}-3}{\sqrt{2}-3}$$
$$= \frac{\sqrt{2}-3}{\sqrt{2}\sqrt{2}-3\sqrt{2}+3\sqrt{2}-9} \quad \text{(Expanding the denominator)}$$
$$= \frac{\sqrt{2}-3}{2-9} \quad (\sqrt{2}\times\sqrt{2}=2)$$
$$= \frac{\sqrt{2}-3}{-7}$$
$$= \frac{3-\sqrt{2}}{7}$$

EXAMPLE 16 Rationalise the denominator of $\frac{4}{1-\sqrt{5}}$.

SOLUTION We multiply the numerator and denominator by the conjugate of the denominator.

The conjugate of $1-\sqrt{5}$ is $1+\sqrt{5}$.

$$\frac{4}{1-\sqrt{5}} = \frac{4}{1-\sqrt{5}} \times \frac{1+\sqrt{5}}{1+\sqrt{5}}$$
$$= \frac{4(1+\sqrt{5})}{1+\sqrt{5}-\sqrt{5}-5}$$
$$= \frac{4(1+\sqrt{5})}{-4}$$
$$= -(1+\sqrt{5})$$

EXAMPLE 17 Rationalise the denominator of $\frac{2+\sqrt{2}}{3-\sqrt{2}}$.

SOLUTION The conjugate of $3-\sqrt{2}$ is $3+\sqrt{2}$.

Multiplying the numerator and denominator by $3+\sqrt{2}$, we get:

$$\frac{2+\sqrt{2}}{3-\sqrt{2}} = \frac{2+\sqrt{2}}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$$
$$= \frac{6+2\sqrt{2}+3\sqrt{2}+2}{9+3\sqrt{2}-3\sqrt{2}-2}$$
$$= \frac{8+5\sqrt{2}}{7}$$

Try these 5.2 Rationalise the denominator of these.

(a) $\dfrac{2}{1-\sqrt{3}}$

(b) $\dfrac{4}{2+\sqrt{7}}$

(c) $\dfrac{1}{\sqrt{2}+\sqrt{5}}$

EXAMPLE 18 Rationalise the denominator of $\dfrac{1}{1+\sqrt{x}}$.

SOLUTION The conjugate of $1+\sqrt{x}$ is $1-\sqrt{x}$.

Multiplying the numerator and denominator by $1-\sqrt{x}$ gives:

$$\frac{1}{1+\sqrt{x}} = \frac{1}{1+\sqrt{x}} \times \frac{1-\sqrt{x}}{1-\sqrt{x}}$$
$$= \frac{1-\sqrt{x}}{1-\sqrt{x}+\sqrt{x}-x}$$
$$= \frac{1-\sqrt{x}}{1-x}$$

EXAMPLE 19 Rationalise the denominator of $\dfrac{2}{1-\sqrt{x+1}}$.

SOLUTION The conjugate of $1-\sqrt{x+1}$ is $1+\sqrt{x+1}$.

Multiplying the numerator and denominator by $1+\sqrt{x+1}$ gives:

$$\frac{2}{1-\sqrt{x+1}} = \frac{2}{1-\sqrt{x+1}} \times \frac{1+\sqrt{x+1}}{1+\sqrt{x+1}}$$
$$= \frac{2(1+\sqrt{x+1})}{(1+\sqrt{x+1})(1-\sqrt{x+1})}$$
$$= \frac{2(1+\sqrt{x+1})}{1+\cancel{\sqrt{x+1}}-\cancel{\sqrt{x+1}}-\sqrt{x+1}\sqrt{x+1}}$$
$$= \frac{2(1+\sqrt{x+1})}{1-(x+1)}$$
$$= \frac{2(1+\sqrt{x+1})}{-x}$$

EXAMPLE 20 (a) Rationalise the denominator of $\dfrac{2-\sqrt{x}}{2+3\sqrt{x}}$.

(b) Hence, show that $\left(\dfrac{2-\sqrt{x}}{2+3\sqrt{x}}\right)(4-9x) = 4-8\sqrt{x}+3x$.

SOLUTION (a) The conjugate of the denominator is $2-3\sqrt{x}$.

Multiplying the numerator and denominator by $2-3\sqrt{x}$ gives:

$$\frac{2-\sqrt{x}}{2+3\sqrt{x}} = \frac{2-\sqrt{x}}{2+3\sqrt{x}} \times \frac{2-3\sqrt{x}}{2-3\sqrt{x}}$$
$$= \frac{4-6\sqrt{x}-2\sqrt{x}+3x}{4-6\sqrt{x}+6\sqrt{x}-9x}$$
$$= \frac{4-8\sqrt{x}+3x}{4-9x}$$

(b) Since $\dfrac{2-\sqrt{x}}{2+3\sqrt{x}} = \dfrac{4-8\sqrt{x}+3x}{4-9x}$

$$\left(\frac{2-\sqrt{x}}{2+3\sqrt{x}}\right)(4-9x) = \frac{4-8\sqrt{x}+3x}{\cancel{4-9x}} \times (\cancel{4-9x})$$

$$= 4 - 8\sqrt{x} + 3x$$

Try these 5.3

Rationalise the denominator of these.

(a) $\dfrac{1}{2+\sqrt{x}}$

(b) $\dfrac{3}{4-\sqrt{x+1}}$

(c) $\dfrac{2+\sqrt{x+1}}{3-\sqrt{x}}$

EXERCISE 5B

1 Simplify each of the following surds.

(a) $\sqrt{1083}$ (b) $\sqrt{1445}$

(c) $\sqrt{1058}$

2 Simplify each of the following surds.

(a) $\sqrt{147} - 5\sqrt{192} + \sqrt{108}$

(b) $2\sqrt{847} + 4\sqrt{576} - 4\sqrt{1008}$

3 Rationalise the denominators of the following.

(a) $\dfrac{2-\sqrt{5}}{2+3\sqrt{5}}$ (b) $\dfrac{1}{\sqrt{5}-4}$

(c) $\dfrac{4}{2-\sqrt{7}}$ (d) $\dfrac{12}{4-\sqrt{11}}$

(e) $\dfrac{\sqrt{3}-1}{2\sqrt{3}+3}$ (f) $\dfrac{1}{2\sqrt{3}+5}$

4 Simplify the following.

(a) $\dfrac{1}{\sqrt{3}+\sqrt{7}} + \dfrac{1}{\sqrt{3}-\sqrt{7}}$

(b) $\dfrac{1+\sqrt{5}}{\sqrt{5}+2} - \dfrac{1-\sqrt{5}}{\sqrt{5}-2}$

(c) $\dfrac{1}{(1-\sqrt{3})^2} + \dfrac{1}{(1+\sqrt{3})^2}$

5 Show that $\dfrac{1}{2+\sqrt{2}} = 1 - \dfrac{\sqrt{2}}{2}$.

6 Rationalise the denominators of the following.

(a) $\dfrac{x}{\sqrt{x}-2}$ (b) $\dfrac{1}{2+\sqrt{x}}$

(c) $\dfrac{x}{5 - 4\sqrt{x}}$ (d) $\dfrac{1}{\sqrt{x+2} - 3}$

(e) $\dfrac{x+1}{\sqrt{x+1} - 4}$ (f) $\dfrac{6}{\sqrt{2x-3} + 4}$

7 Show that $\dfrac{x - 13}{\sqrt{x+3} + 4} = \sqrt{x+3} - 4$.

8 Prove that $\sqrt{2x+6} - 3 = \dfrac{2x - 3}{\sqrt{2x+6} + 3}$.

9 Show that $\dfrac{2x - 17}{\sqrt{2x-1} + 4} = \sqrt{2x-1} - 4$.

10 Express $\dfrac{1 + \tan 30°}{1 - \tan 30°}$ in the form $a + b\sqrt{3}$ where a and b are integers.

$\left[\tan 30° = \dfrac{1}{\sqrt{3}}\right]$

DEFINITION

An **exponential function** is a function of the form $f(x) = a^x$ where a is a positive real number, $a \neq 1$.

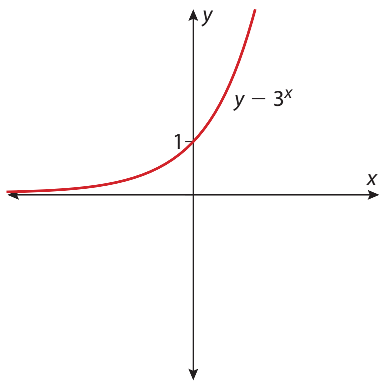
Exponential functions

Graphs of exponential functions

EXAMPLE 21 Sketch the graph of $y = 3^x$.

SOLUTION

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



From the graph we can see:

$3^x > 0$ for all x, therefore the range of the function is $(0, \infty)$. The graph has no x-intercept and lies above the x-axis.

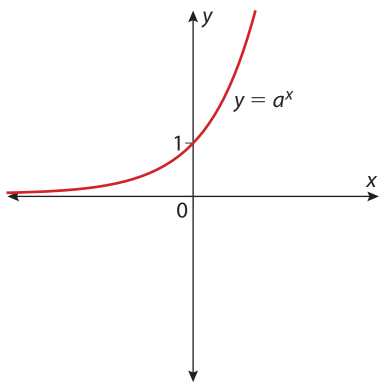
As $x \to -\infty$, $3^x \to 0$ (as x gets more negative the function approaches 0).

As $x \to \infty$, $3^x \to \infty$ (as x gets larger the function also gets larger).

The function is an increasing function and therefore one–one. We can generalise for the exponential graph as follows:

Properties of $y = a^x$, $a > 1$

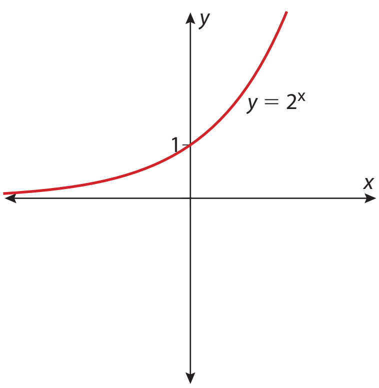
- The domain of a^x is $x \in \mathbb{R}$ and the range is $y > 0$.
- The graph does not cut the x-axis.
- The line $y = 0$ is a horizontal asymptote as $x \to -\infty$.
- a^x is an increasing function and is one–one.
- The graph of y is smooth and continuous.



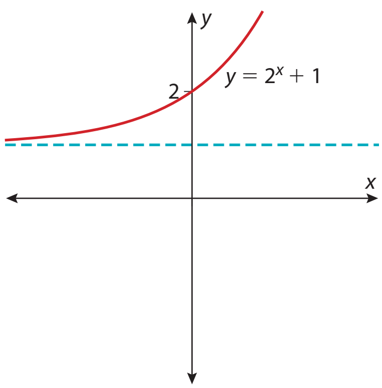
EXAMPLE 22 Sketch the graph of **(a)** $y = 2^x$ **(b)** $y = 2^x + 1$.

SOLUTION

(a)



(b)



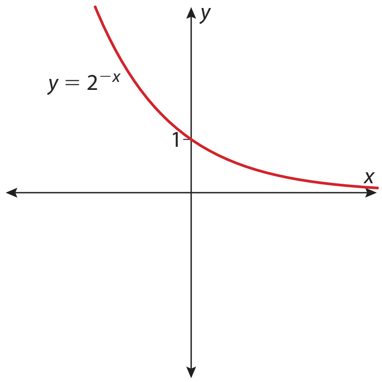
> **Note**
>
> $y = 2^x + 1$ approaches the line $y = 1$.

The graph of $y = 2^x + 1$ is a shift in the graph of $y = 2^x$ upwards by 1 unit.

EXAMPLE 23 Sketch the graph of $y = 2^{-x}$.

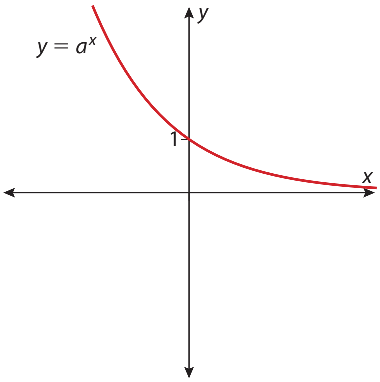
SOLUTION

x	-2	-1	0	1	2	3
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



Properties of $y = a^x$, $0 < a < 1$

- The domain of a^x is $x \in R$ and the range is $y > 0$.
- The graph does not cut the x-axis.
- The line $y = 0$ is a horizontal asymptote as $x \to \infty$.
- a^x is a decreasing function and is one–one.
- The graph of y is smooth and continuous.



DEFINITION

The number e is defined as the $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

The number e

We can investigate the number e by looking at a table of values as follows:

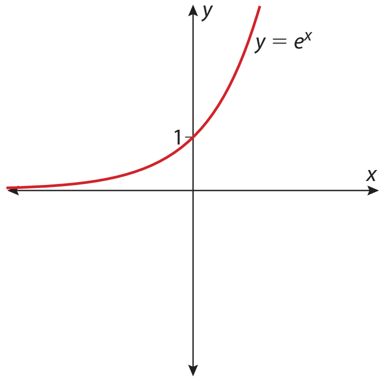
n	$f(n) = \left(1 + \frac{1}{n}\right)^n$
1	2
2	2.25
8	2.565784514
64	2.697344953
100	2.704813829
1000	2.716923932
10 000	2.718145927
100 000	2.718268237
1 000 000	2.718280469

As n gets larger $f(n) = \left(1 + \frac{1}{n}\right)^n$ approaches 2.718 (to 3 decimal places). This number is given the symbol e.

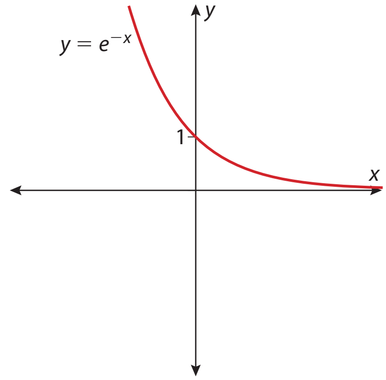
EXAMPLE 24 Sketch the graph of **(a)** $y = e^x$ **(b)** $y = e^{-x}$.

SOLUTION

(a)

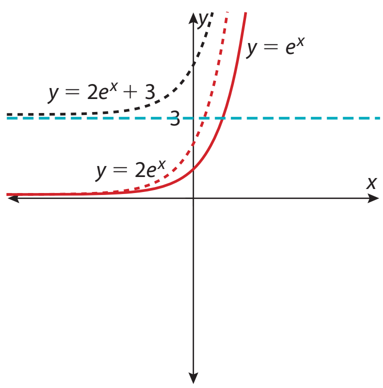


(b)



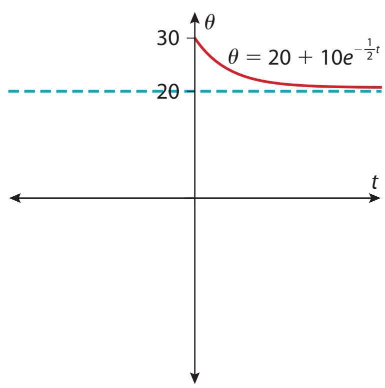
EXAMPLE 25 Using the graph of $f(x) = e^x$, sketch the graph of $g(x) = 2e^x + 3$. State the transformations which map $f(x)$ onto $g(x)$.

SOLUTION



The graph of $g(x)$ can be obtained from $f(x)$ by a stretch along the y-axis by factor 2, followed by a translation of 3 units upwards.

EXAMPLE 26 Sketch the graph of $\theta = 20 + 10e^{-\frac{1}{2}t}$, for $t \geq 0$. Identify clearly the value of θ when $t = 0$ and the value of θ as $t \to \infty$.



SOLUTION

When $t = 0$, $\theta = 20 + 10e^{-\frac{1}{2}(0)}$

$$\theta = 20 + 10e^0$$

Since $e^0 = 1$, $\theta = 30$.

As $t \to \infty$, $e^{-\frac{1}{2}t} \to 0$

$\therefore\ \theta \to 20 + 10(0)$

Hence $\theta \to 20$.

Try these 5.4 Sketch the graph of

(a) $y = e^x - 3$

(b) $x = \dfrac{4000}{1 + 3999e^{-0.2t}}$.

EXERCISE 5C

1 Sketch the graph of $y = e^x$. Using the graph of $y = e^x$, sketch

(a) $y = e^{3x}$, (b) $y = e^x + 2$, (c) $y = 2e^x + 3$.

2 Given that $x = 3 + 4e^{-2t}$, write down the value of x when $t = 0$ and the value of x as $t \to \infty$. Hence sketch the graph of $x = 3 + 4e^{-2t}$.

3 Describe the transformations which map the graph of $\theta = e^t$ onto $\theta = 2e^{-t} + 5$. Hence sketch the graph of $\theta = 2e^{-t} + 5$.

4 Sketch the graphs of $y = 9e^{-x}$ and $y = 2x + 1$. How many solutions are there to the equation $9 - 2xe^x - e^x = 0$?

5 Given that $x = \dfrac{5}{4 + 10e^{-t}}$, find the value of x when (a) $t = 0$, (b) $t \to \infty$. Sketch the graph of x against t for $t \geq 0$.

6 A colony of bacteria grows according to the law

$x = 400e^{0.05t}$

where t is the time in years and x is the amount of bacteria present in time t.

(a) What is the value of x when $t = 0$?

(b) What happens to the population of the bacteria as t increases?

(c) Sketch the graph of x against t.

7 A liquid cools from its original temperature of 70 °C to a temperature θ °C in t minutes.

Given that $\theta = 70e^{-0.02t}$, find the value of

(a) θ when $t = 4$ minutes,

(b) θ when $t = 6$ minutes.

(c) Sketch the graph of θ against t.

8 Given that $M = M_0e^{-\lambda t}$, where M_0 and λ are constants, sketch the graph of M against t.

9 Sketch the graphs of $y = 2^x$ and $y = 2^{-x}$. Hence sketch the graphs of $y = 2^{-x} + 3$ and $y = 3(2^x) + 4$.

10 The charge Q on a capacitor is given by $Q = 10(1 - e^{-xt})$ where, x is a constant and t is the time. Sketch the graph of Q against t.

Exponential equations

We can solve exponential equations having the same base by equating the indices on either side of the equation as follows:

If $a^x = a^y$ then $x = y$.

Let us use this to solve equations involving equal indices.

EXAMPLE 27 Solve the equation $4^{2x+1} = 16$.

SOLUTION We can write both sides of the equation with base 4 (since $16 = 4^2$):

$4^{2x+1} = 4^2$

Equating the indices

$2x + 1 = 2$

Solving

$2x = 1$

$x = \frac{1}{2}$

EXAMPLE 28 Solve the equation $e^{x^2} = \frac{1}{e^{3x+2}}$.

SOLUTION Using $\frac{1}{x^n} = x^{-n}$ we can write $\frac{1}{e^{3x+2}} = e^{-3x-2}$.

$e^{x^2} = e^{-3x-2}$

Equating the indices we get the quadratic equation

$x^2 = -3x - 2$

$x^2 + 3x + 2 = 0$

Solving

$(x + 1)(x + 2) = 0$

$x + 1 = 0$ or $x + 2 = 0$

$\therefore\ x = -1$ or $x = -2$

EXAMPLE 29 Solve the equation $e^{4x} = \frac{e^{12}}{e^{x^2}}$.

SOLUTION $e^{4x} = \frac{e^{12}}{e^{x^2}}$

Using the rules of indices to write the right-hand side, as one index, we get

$e^{4x} = e^{12 - x^2}$

Equating the indices on both sides since the base on each side of the equation are the same:

$4x = 12 - x^2$

Solving the quadratic

$x^2 + 4x - 12 = 0$

$(x - 2)(x + 6) = 0$

$x - 2 = 0$ or $x + 6 = 0$

Hence $x = 2$ or $x = -6$.

EXAMPLE 30 Solve the equation $5^x = \frac{1}{125}$.

SOLUTION

$5^x = \frac{1}{125}$

The right-hand side can be written as an index to base 5 as follows

$5^x = 5^{-3}$

Equating indices

$x = -3$

Try these 5.5

(a) Solve the equation $3^{2x} = 243^{x+1}$.

(b) Find the values of x satisfying the equation $e^{x+2} = \frac{1}{e^{3x-1}}$.

Logarithmic functions

$$\underset{\text{index}}{x} = \log_{\underset{\text{base}}{a}} \overset{\text{the number}}{y}$$

$$\overset{\text{the number}}{y} = \underset{\text{base}}{a}^{x \leftarrow \text{index}}$$

A **logarithm** of a number is defined as the power to which a base has to be raised to be equal to the number.

$y = a^x \leftrightarrow x = \log_a y$

(This can be read as the logarithm of y to base a is equal to the index x.)

We can convert an exponent to a logarithm and a logarithm to an exponent using the definition.

Converting exponential expressions to logarithmic expressions

EXAMPLE 31 Change each of the following to a logarithmic expression.

(a) $x = 1.5^6$ **(b)** $y = 2^4$ **(c)** $a^3 = 8$

SOLUTION

(a) $x = 1.5^6$

Using

$y = a^x \leftrightarrow x = \log_a y$

we get

$x = 1.5^6 \rightarrow 6 = \log_{1.5} x$

(The base remains the base in both forms and the index is equal to the log.)

(b) $y = 2^4$

Using

$y = a^x \leftrightarrow x = \log_a y$

we get

$y = 2^4 \rightarrow 4 = \log_2 y$

(The base remains the base in both forms and the index is equal to the log.)

(c) $a^3 = 8$

Using

$y = a^x \leftrightarrow x = \log_a y$

we get

$a^3 = 8 \rightarrow 3 = \log_a 8$

(The base remains the base in both forms and the index is equal to the log.)

Try these 5.6 Convert the following to logarithmic expressions.

(a) $x = a^2$

(b) $16 = 4^2$

(c) $10 = a^x$

Changing logarithms to exponents using the definition of logarithm

EXAMPLE 32 Convert the following to exponents.

(a) $\log_a 6 = 3$ **(b)** $\log_2 4 = x$ **(c)** $\log_4 7 = y$

SOLUTION

(a) $\log_a 6 = 3$

Using

$y = a^x \leftrightarrow x = \log_a y$

Replacing $y = 6, a = a, x = 3$ we get

$\log_a 6 = 3 \rightarrow 6 = a^3$

(b) $\log_2 4 = x$

Using

$y = a^x \leftrightarrow x = \log_a y$

Replacing $y = 4, a = 2, x = x$ we get

$\log_2 4 = x \rightarrow 4 = 2^x$

(c) $\log_4 7 = y$

Using

$y = a^x \leftrightarrow x = \log_a y$

Replacing $y = 7$, $a = 4$, $x = y$ we get

$\log_4 7 = y \rightarrow 7 = 4^y$

Try these 5.7 Convert the following to exponential form.

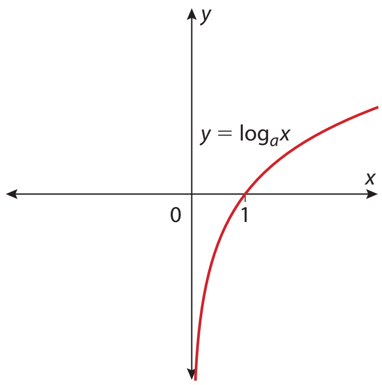
(a) $\log_3 y = 6$

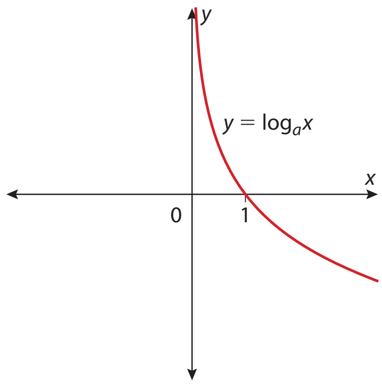
(b) $x = \log_3 10$

(c) $\log_x 6 = 4$

Graph of the logarithmic function

The inverse of $y = a^x$ is $x = \log_a y$ therefore the graph of the logarithmic function can be obtained by reflecting the graph of the exponential function along the line $y = x$.





Properties of the logarithmic function

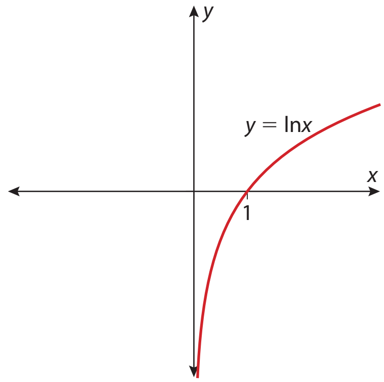
- The domain is $x \in \mathbb{R}^+$, the range is $y \in \mathbb{R}$.
- The y-axis is a vertical asymptote.
- The graph is smooth and continuous.
- A logarithmic function is decreasing if $0 < a < 1$ and increasing if $a > 1$.

Naperian logarithms

Recall that common logarithm is the term used for logarithm to base 10 and is written as $\log_{10} x$ or $\lg x$. Logarithms having a base of e can be written as $\ln x$ or $\log_e x$ and are called naperian or natural logarithms.

The Naperian logarithm and the exponential function with base e are inverses of each other. If $y = \ln x$ then $x = e^y$.

To sketch the graph of $y = \ln x$ we reflect the graph of $y = e^x$ along the line $y = x$ since these functions are inverses of each other.



Properties of logarithms

- $\log_a a = 1$
- $\log_a 1 = 0$
- $\log_a x = \log_a y \Rightarrow x = y$
- $\log_a (xy) = \log_a x + \log_a y$ and in particular when $a = e$, $\ln (xy) = \ln x + \ln y$
- $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ and in particular when $a = e$, $\ln \left(\frac{x}{y}\right) = \ln x - \ln y$
- $\log_a x^n = n \log_a x$ and in particular when $a = e$, $\ln (x^n) = n (\ln x)$
- $\log_a a^{f(x)} = f(x) \log_a a = f(x)$, since $\log_a a = 1$
- $\ln e^{f(x)} = f(x)$
- $e^{\ln f(x)} = f(x)$

EXAMPLE 33 Simplify **(a)** $\ln e^{x^2}$ **(b)** $\ln e^{4x+2}$ **(c)** $\ln e^{\cos x}$

SOLUTION

(a) Using $\ln (x^n) = n (\ln x)$

we get

$\ln e^{x^2} = x^2 \ln e = x^2$ since $\ln e = 1$

(b) $\ln e^{4x+2} = (4x + 2) \ln e = 4x + 2$ since $\ln e = 1$

(c) $\ln e^{\cos x} = (\cos x) \ln e = \cos x$ since $\ln e = 1$

Note

$\ln e^{f(x)} = f(x)$

Solving logarithmic equations

EXAMPLE 34 Solve the equation $\log_2(4x + 2) = 6$.

SOLUTION

$\log_2(4x + 2) = 6$

Converting to an exponent using $y = a^x \leftrightarrow x = \log_a y$, where $a = 2$, $y = 4x + 2$ and $x = 6$

we get

$4x + 2 = 2^6$

Therefore

$4x + 2 = 64$

$4x = 62$

$x = \frac{62}{4}$

$x = \frac{31}{2}$

EXAMPLE 35 Solve the equation $\ln(3x + 2) - \ln 2 = 1$, giving your answer to 3 decimal places.

SOLUTION

Using $\ln x - \ln y = \ln\left(\frac{x}{y}\right)$

we get

$\ln\left(\frac{3x + 2}{2}\right) = 1$

Recall that $\log_a x = y \Rightarrow x = a^y$.

We can use the definition and convert to an exponent

$\frac{3x + 2}{2} = e^1$

$3x + 2 = 2e^1$

$3x + 2 = 5.437$

$3x = 3.437$

$x = \frac{3.437}{3} = 1.146$ (3 d.p.)

EXAMPLE 36 Solve $\log_2(x - 7) + \log_2(x - 8) = 1$.

SOLUTION

$\log_2(x - 7) + \log_2(x - 8) = 1$

Since $\log a + \log b = \log ab$, we get

$\log_2(x - 7)(x - 8) = 1$

Recall that $\log_a x = y \Rightarrow x = a^y$.

Converting to an exponent

$(x - 7)(x - 8) = 2^1$

we get

$x^2 - 15x + 56 = 2$

$x^2 - 15x + 54 = 0$

Factorising

$(x - 6)(x - 9) = 0$

$x - 6 = 0$ or $x - 9 = 0$

Hence, $x = 6$ or $x = 9$

EXAMPLE 37 Solve the equation

$\log_3 (x^2 - 9) - \log_3 (x + 3) = 3.$

SOLUTION Combining the left-hand side under one log

$\log_3\left(\frac{x^2 - 9}{x + 3}\right) = 3$

Converting to an exponent, we get

$\frac{x^2 - 9}{x + 3} = 3^3$

$\frac{(x - 3)(x + 3)}{(x + 3)} = 27$

$x - 3 = 27$

$\therefore x = 30$

EXAMPLE 38 Find the exact value of x satisfying the equation $\ln (x + 1) - \ln x = 2.$

SOLUTION Converting to a single log

$\ln\left(\frac{x + 1}{x}\right) = 2$

Removing ln

$\frac{x + 1}{x} = e^2$

Making x the subject of the formula

$x + 1 = xe^2$

$x - xe^2 = -1$

$x(1 - e^2) = -1$

Dividing by $1 - e^2$

$x = \frac{-1}{1 - e^2}$

$\therefore x = \frac{1}{e^2 - 1}$

EXAMPLE 39 Find the value(s) of x satisfying the equation $2\log_2 x - \log_2(x+1) = 1$.

SOLUTION

$2\log_2 x - \log_2(x+1) = 1$

The left-hand side of the equation can be converted to one log.

$\log_2 x^2 - \log_2(x+1) = 1$

$\log_2\left(\frac{x^2}{x+1}\right) = 1$

$\frac{x^2}{x+1} = 2^1$

$x^2 = 2(x+1)$

$x^2 = 2x + 2$

$x^2 - 2x - 2 = 0$

Using the quadratic formula

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-2)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{12}}{2}$$

$$x = \frac{2 \pm 2\sqrt{3}}{2}$$

$x = 1 \pm \sqrt{3}$

x must be positive for $\log_2 x$ to exist.

Therefore, $x = 1 + \sqrt{3}$

Equations involving exponents

EXAMPLE 40 Solve the equation $2^{x-1} = 9$.

SOLUTION

Taking logs to base e, we get

$\ln 2^{x-1} = \ln 9$

Using $\ln(x^n) = n(\ln x)$ gives

$(x-1)\ln 2 = \ln 9$

$$x - 1 = \frac{\ln 9}{\ln 2}$$

$$x = 1 + \frac{\ln 9}{\ln 2} = 4.170$$

EXAMPLE 41 Solve the equation $e^{4x+1} = 3$.

SOLUTION

Taking logs to base e on both sides (recalling that $\ln e^{f(x)} = f(x)$)

$\ln e^{4x+1} = \ln 3$

Note

We could also take logs to base 10.

$4x + 1 = \ln 3$

$4x = \ln 3 - 1$

$x = \dfrac{\ln 3 - 1}{4}$

$x = 0.025$

EXAMPLE 42 Find the value of x satisfying the equation $4^{x+2} = 5$.

SOLUTION

$4^{x+2} = 5$

Taking logs to base 10

$\lg 4^{x+2} = \lg 5$

$(x + 2) \lg 4 = \lg 5$ (using $\log_a x^n = n \log_a x$)

$x + 2 = \dfrac{\lg 5}{\lg 4}$

$x = -0.839$

Note

We can also take logs to base e:

$\ln(4^{x+2}) = \ln 5$

$(x + 2) \ln 4 = \ln 5$

$x + 2 = \dfrac{\ln 5}{\ln 4}$

$x = \dfrac{\ln 5}{\ln 4} - 2$

$= -0.839$

EXAMPLE 43 Solve the equation $2^{x+1} = 5^{1-2x}$.

SOLUTION

$2^{x+1} = 5^{1-2x}$

Taking logs to base 10

$\lg 2^{x+1} = \lg 5^{1-2x}$

Since $\lg x^n = n \lg^x$

$\lg 2^{x+1} = (x + 1)\lg 2$ and $\lg 5^{1-2x} = (1 - 2x)\lg 5$

$(x + 1) \lg 2 = (1 - 2x) \lg 5$

$(x + 1)\, 0.3010 = (1 - 2x)\, 0.6990$

$0.3010x + 0.3010 = 0.6990 - 1.398x$

$1.699x = 0.398$

$x = 0.234$

EXAMPLE 44 Find the value(s) of x satisfying the equation $e^{2x} + e^x = 12$.

SOLUTION We cannot take logs in this case, let us see what happens if we do.

$\ln(e^{2x} + e^x) = \ln 12$

What do we do next? Did you do this

$\ln e^{2x} + \ln e^x = \ln 12$? This is incorrect since there is no rule of logs that works across the addition sign.

$e^{2x} + e^x = 12$

We can write this equation as $(e^x)^2 + e^x = 12$

This equation is a quadratic in e^x.

Let $y = e^x$

We have

$y^2 + y - 12 = 0$

Factorising

$(y + 4)(y - 3) = 0$

$y + 4 = 0$ or $y - 3 = 0$

$y = -4$ or $y = 3$

Now replacing $y = e^x$

$e^x = -4$ or $e^x = 3$

Since e^x cannot be negative

$e^x = 3$

Taking logs to base e

$x = \ln 3$

EXAMPLE 45 Find the value(s) of x satisfying the equation $12e^{2x} - 13e^x + 3 = 0$.

SOLUTION $12e^{2x} - 13e^x + 3 = 0$

We can write this equation as $12(e^x)^2 - 13e^x + 3 = 0$

This equation is a quadratic in e^x. (Remember that taking logs in an equation like this will not work.)

Let $y = e^x$

We have

$12y^2 - 13y + 3 = 0$

Factorising

$(3y - 1)(4y - 3) = 0$

$3y - 1 = 0$ or $4y - 3 = 0$

$y = \frac{1}{3}$ or $y = \frac{3}{4}$

Now replacing $y = e^x$

$e^x = \frac{1}{3}$ or $e^x = \frac{3}{4}$

Taking logs to base e

$x = \ln\frac{1}{3}$ or $x = \ln\frac{3}{4}$

EXAMPLE 46 Solve the equation $6e^x = 7 - 2e^{-x}$.

SOLUTION

$6e^x = 7 - 2e^{-x}$

$6e^x = 7 - \frac{2}{e^x}$

Multiplying throughout by e^x

$6(e^x)^2 = 7e^x - 2$

$6(e^x)^2 - 7e^x + 2 = 0$

Let $y = e^x$

We have

$6y^2 - 7y + 2 = 0$

Factorising

$(3y - 2)(2y - 1) = 0$

$3y - 2 = 0$ or $2y - 1 = 0$

$y = \frac{2}{3}$ or $y = \frac{1}{2}$

Now replacing $y = e^x$

$e^x = \frac{2}{3}$ or $e^x = \frac{1}{2}$

Taking logs to base e

$x = \ln\frac{2}{3}$ or $x = \ln\frac{1}{2}$

Change of base formula (change to base b from base a)

If $y = \log_a x$, then by definition $x = a^y$.

Taking logs to base b

$\log_b x = \log_b a^y$

Using the properties of logs

$\log_b x = y\log_b a$

Since $y = \log_a x$, we have

$\log_b x = \log_a x \log_b a$

Therefore

$$\log_a x = \frac{\log_b x}{\log_b a}$$

which is our change of base formula.

When $x = b$

$$\log_a b = \frac{\log_b b}{\log_b a}$$

which is

$$\log_a b = \frac{1}{\log_b a} \text{ since } \log_b b = 1.$$

EXAMPLE 47 Convert $\log_4 2$ to the log of a number to base 2.

SOLUTION Using $\log_a b = \frac{1}{\log_b a}$ we get

$$\log_4 2 = \frac{1}{\log_2 4}$$

EXAMPLE 48 Solve the equation $\log_2 x = 4 \log_x 2$.

SOLUTION $\log_2 x = 4 \log_x 2$

Since the bases are different we can use the change of base formula to convert from base x to base 2 as follows.

Since $4\log_x 2 = \frac{4}{\log_2 x} \Rightarrow \log_2 x = \frac{4}{\log_2 x}$

Multiplying by $\log_2 x$ gives

$(\log_2 x)^2 = 4$

Taking square roots

$\log_2 x = \pm 2$

Using the definition

$x = 2^2$ or $x = 2^{-2}$

Hence

$x = 4$ or $x = \frac{1}{4}$

EXAMPLE 49 Given that $\log_2 x = 2 + \log_8 x$, find the value of x satisfying this equation.

SOLUTION $\log_2 x = 2 + \log_8 x$

We can change $\log_8 x$ to base 2 as follows.

Using $\log_a x = \frac{\log_b x}{\log_b a}$, we get

$$\log_8 x = \frac{\log_2 x}{\log_2 8}$$

$$\log_2 8 = \log_2 2^3 = 3\log_2 2 = 3$$

Therefore

$$\log_8 x = \frac{\log_2 x}{3}$$

Hence

$$\log_2 x = 2 + \frac{1}{3}\log_2 x$$

$$\log_2 x - \frac{1}{3}\log_2 x = 2$$

$$\frac{2}{3}\log_2 x = 2$$

$$\log_2 x = 3$$

By definition

$$x = 2^3$$

$$x = 8$$

EXAMPLE 50 Solve the equation $\log_5 x = 9\log_x 5$.

SOLUTION Changing base

$$\log_x 5 = \frac{1}{\log_5 x}$$

Therefore

$$\log_5 x = \frac{9}{\log_5 x}$$

Multiplying by $\log_5 x$

$$(\log_5 x)^2 = 9$$

Taking square roots

$$\log_5 x = \pm 3$$

$$\log_5 x = 3 \text{ or } \log_5 x = -3$$

Using the definition

$$x = 5^3 = 125 \text{ or } x = 5^{-3} = \frac{1}{125}$$

Logarithms and exponents in simultaneous equations

EXAMPLE 51 Solve these simultaneous equations.

$$\log_2(x - 10y) = 2$$

$$\log_3 x - \log_3(y + 1) = 2$$

SOLUTION We can remove the logs from both equations and then solve.

$\log_2(x - 10y) = 2$ [1]

$\Rightarrow \log_2(x - 10y) = 2\log_2 2$ (since $\log_2 2 = 1$)

$\Rightarrow \log_2(x - 10y) = \log_2 2^2$

Therefore, $x - 10y = 2^2$ (we can remove the logs)

Hence, $x - 10y = 4$ [2]

$\log_3 x - \log_3(y + 1) = 2$ [3]

$\Rightarrow \log_3\left(\frac{x}{y+1}\right) = 2$ $\left(\text{since } \log_c a - \log_c b = \log_c\left(\frac{a}{b}\right)\right)$

$\Rightarrow \log_3\left(\frac{x}{y+1}\right) = 2\log_3 3$

$\Rightarrow \log_3\left(\frac{x}{y+1}\right) = \log_3 3^2$

Therefore, $\left(\frac{x}{y+1}\right) = 3^2$

Hence, $x = 9y + 9$ [4]

Subtracting [4] from [2] gives:

$x - 10y - x = 4 - 9y - 9$

$-10y = -9y - 5$

$-y = -5$

$y = 5$

When $y = 5$, $x = 9(5) + 9$ (Substituting $y = 5$ into [4])

$x = 54$

Hence, $x = 54$, $y = 5$

EXAMPLE 52 Solve these simultaneous equations.

$2^x = 4(8)^y$

$\log_2 7 = 1 + \log_2(11y - 2x)$

SOLUTION Write the equations without the indices and logarithms.

Since, $2^x = 4(8)^y$

$2^x = 2^2(2^3)^y$ $(4 = 2^2, 8 = 2^3)$

$\Rightarrow 2^x = 2^2 \times 2^{3y}$

$\Rightarrow 2^x = 2^{2 + 3y}$

Equating indices gives:

$x = 2 + 3y$

$x - 3y = 2$ [1]

Also, $\log_2 7 = 1 + \log_2(11y - 2x)$

$\log_2 7 = \log_2 2 + \log_2(11y - 2x)$ $\quad (1 = \log_2 2)$

$\log_2 7 - \log_2 2 = \log_2(11y - 2x)$

$\Rightarrow \log_2\left(\frac{7}{2}\right) = \log_2(11y - 2x)$

$\Rightarrow \frac{7}{2} = 11y - 2x$ $\quad [2]$

Multiply [1] by 2 gives:

$2(x - 3y) = 2 \times 2$

$2x - 6y = 4$ $\quad [3]$

Adding [2] and [3] gives:

$\frac{7}{2} + 4 = 11y - 2x + 2x - 6y$

$\frac{15}{2} = 5y$

$\Rightarrow y = \frac{3}{2}$

When $y = \frac{3}{2}$, $x - 3\left(\frac{3}{2}\right) = 2$ $\quad$ (Substituting $y = \frac{3}{2}$ into [1])

$x = 2 + \frac{9}{4}$

$= \frac{13}{2}$

Hence, $x = \frac{13}{2}$, $y = \frac{3}{2}$

Application problems

EXAMPLE 53 The length of a bar, x, at a temperature of T°C is given by the equation $x = x_0 e^{\alpha T}$ where x_0 and α are constants. Evaluate T when $x = 2.62$, $x_0 = 2.47$ and $\alpha = 0.002$.

SOLUTION Substituting into $x = x_0 e^{\alpha T}$

We have

$2.62 = 2.47 e^{0.002T}$

$e^{0.002T} = \frac{2.62}{2.47}$

Taking logs to base e

$0.002T = \ln\left(\frac{2.62}{2.47}\right) = 0.058\,96$

$T = 29.478\,°\text{C}$

EXAMPLE 54 A liquid cools from its original temperature of 70 °C to a temperature of θ°C in t minutes. Given that $\theta = 70e^{-0.02t}$, find the value of **(i)** θ when $t = 12$ minutes, **(ii)** t when $\theta = 40$ °C.

SOLUTION

(i) When $t = 12$, $\theta = 70e^{-0.02\,(12)} = 55.06$ °C

(ii) When $\theta = 40$ °C, $40 = 70e^{-0.02t}$

$\therefore\ e^{-0.02t} = \frac{4}{7}$

Taking logs to base e

$-0.02t = \ln\left(\frac{4}{7}\right)$

$t = 27.98$ minutes

EXAMPLE 55 The number of bacteria in a culture at time t was given by $x = \alpha\, e^{\frac{1}{5}t}$. Find **(i)** the number of bacteria present at time $t = 0$, and **(ii)** the value of t when the colony was double its initial size.

SOLUTION

(i) When $t = 0$, $x = \alpha\, e^{0} = \alpha$

(ii) When the colony doubles its initial size, $x = 2\alpha$

$\therefore\ 2\alpha = \alpha\, e^{\frac{1}{5}t}$

$e^{\frac{1}{5}t} = 2$

$\frac{1}{5}t = \ln 2$

$t = 5\ln 2 = 3.47$

Compound interest

The amount of money A after t years when a principal P is invested at an annual interest rate r compounded n times per year is given by $A = P\left(1 + \frac{r}{n}\right)^{nt}$. Let us see how this formula was derived:

We are investing \$$P$ at an interest rate of r compounded n times per year, therefore

$I = P \times r \times t$

$I = P \times r \times \frac{1}{n}$

The amount after one period will be (the principal plus the interest in that period)

$A = P + \frac{Pr}{n} = P\left(1 + \frac{r}{n}\right)$

For the second period

New principal is $P\left(1 + \frac{r}{n}\right)$

Therefore the interest in the second period

$I = P\left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)$

The amount after the second period interest is

$$A = P\left(1 + \frac{r}{n}\right) + P\left(1 + \frac{r}{n}\right)\left(\frac{r}{n}\right)$$

Factoring

$$A = P\left(1 + \frac{r}{n}\right)\left(1 + \frac{r}{n}\right)$$

Therefore

$$A = P\left(1 + \frac{r}{n}\right)^2$$

For the third period

Principal is $P\left(1 + \frac{r}{n}\right)^2$

Interest is

$$I = P\left(1 + \frac{r}{n}\right)^2 \times \frac{r}{n}$$

The amount after the third interest period is

$$A = P\left(1 + \frac{r}{n}\right)^2 + P\left(1 + \frac{r}{n}\right)^2 \times \frac{r}{n}$$

Factoring

$$A = P\left(1 + \frac{r}{n}\right)^2\left(1 + \frac{r}{n}\right)$$

Therefore

$$A = P\left(1 + \frac{r}{n}\right)^3$$

From the pattern we can see that for t years there will be nt periods and $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

EXAMPLE 56 What rate of interest compounded annually should you seek if you want to double your investment in 10 years?

SOLUTION Since $A = P\left(1 + \frac{r}{n}\right)^{nt}$

we need r when $A = 2P$, $n = 1$, $t = 10$.

Therefore

$$2P = P\left(1 + \frac{r}{1}\right)^{1(10)}$$

$$2 = (1 + r)^{10}$$

Taking the tenth root on both sides

$$2^{\frac{1}{10}} = 1 + r$$

$$r = 0.0718 = 7.18\%$$

EXAMPLE 57 What is the amount of money that results from investing \$800.00 at 4% compounded quarterly after a period of 2 years?

SOLUTION Using $A = P\left(1 + \frac{r}{n}\right)^{nt}$

where $P = \$800.00$, $r = 0.04$, $n = 4$ and $t = 2$ we get

$A = 800\left(1 + \frac{0.04}{4}\right)^{4(2)}$

Therefore

$A = 800 \times 1.01^8 = \$866.29$

Continuous compound interest

If the year is divided into n equal intervals, as n tends to infinity the capital for one year is

$A = P\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n$

Since $\lim_{n\to\infty}\left(1 + \frac{r}{n}\right)^n = e^r$

$A = Pe^r$

For t years

$A = P(e^{rt})$

and the present value P is

$P = \frac{A}{e^{rt}}$

i.e. $P = Ae^{-rt}$

EXAMPLE 58 Suppose that \$5000.00 is invested in an account where interest is compounded continuously at a constant annual equivalent rate of 6%. How much money is in the account after **(a)** 5 years, **(b)** 7.5 years?

SOLUTION

(a) Using

$A = P(e^{rt})$

where $P = \$5000.00$, $r = 0.06$ and $t = 5$ years,

we get $A = 5000e^{(0.06)(5)} = \$6749.29$

(b) Using

$A = P(e^{rt})$

where $P = \$5000.00$, $r = 0.06$ and $t = 7.5$ years,

we get $A = 5000e^{(0.06)(7.5)} = \$7841.56$

EXERCISE 5D

1 Express the following in logarithm form.

(a) $10^2 = 100$ (b) $x^0 = 1$ (c) $5^0 = 1$

(d) $a^1 = a$ (e) $x^3 = 8$ (f) $4^2 = 16$

(g) $5^3 = 125$ (h) $y^2 = 9$ (i) $x^{\frac{1}{2}} = 4$

2 Find the value of x in each of the following.

(a) $\log_x 4 = 2$ (b) $\log_x 625 = 4$ (c) $\log_2 64 = x$

(d) $\log_4 x = 2$ (e) $\log_9 x = 0$ (f) $\log_x 8 = 3$

(g) $\log_x 9 = 2$ (h) $\log_x 216 = 3$ (i) $\log_8 x = 2$

3 Find the value of the following.

(a) $\log_8 \frac{1}{8}$ (b) $\log_3 729$ (c) $\log_8 64$

(d) $\log_2 32$ (e) $\log_5 \frac{1}{125}$ (f) $\log_x x^{10}$

(g) $\log_7 \frac{1}{343}$ (h) $\log_8 \sqrt{4096}$ (i) $\log_6 1296$

4 Evaluate the following by changing to base 10.

(a) $\log_3 4$ (b) $\log_6 7$ (c) $\log_5 2$

(d) $\log_9 6$ (e) $\log_5 18$ (f) $\log_2 17$

(g) $\log_9 4$ (h) $\log_5 6$ (i) $\log_4 29$

5 Evaluate the following by converting to logs to base e.

(a) $\log_5 12$ (b) $\log_3 22$ (c) $\log_5 18$ (d) $\log_4 17$

(e) $\log_3 32$ (f) $\log_5 41$ (g) $\log_4 62$ (h) $\log_2 28$

6 Solve the following equations.

(a) $2^x = 8$ (b) $3^x = 81$ (c) $4^x = 3^{x+1}$

(d) $2^{x+2} = 4^{x-3}$ (e) $3^{2x} = 4^{x+1}$

7 Find the values of x satisfying the following equations.

(a) $2^{2x} - 5(2^x) + 4 = 0$

(b) $3^{2x} - 4(3^{x+1}) + 27 = 0$

(c) $e^{2x} - 6e^x + 6 = 0$

8 Solve the following equations.

(a) $8^x = 64$ (b) $9^x = 27$

(c) $4^x = 128$ (d) $2^x\, 2^{x+1} = \frac{1}{2}$

(e) $4^{2x} - 5(2^{2x-1}) + 1 = 0$ (f) $\frac{1}{27}(3^{2x}) - \frac{4}{9}(3^x) + 1 = 0$

(g) $16^{\frac{x}{2}} - 3(2^{x+1}) + 8 = 0$

9 Solve the following equations.

(a) $(1 + i)^{-9} = 0.95$ (b) $(1 + i)^4 = (1.01)^9$

10 Express the following in terms of $\ln x$ and $\ln y$.

(a) $\ln x^3 y^2$ (b) $\ln \sqrt{xy}$

(c) $\ln\left(\frac{x^4}{y^1}\right)$ (d) $\ln\left(\frac{xy^3}{x^6}\right)$

11 Express the following as a single logarithm, simplifying as far as possible.

(a) $\ln 14 - \ln 21 + \ln 8$ (b) $4 \ln 2 + \frac{1}{3} \ln 8$

(c) $2 \ln\left(\frac{4}{9}\right) - \ln\left(\frac{8}{27}\right)$ (d) $3 \ln 4 + 4 \ln 2 - 4 \ln 6$

(e) $4 \ln 5 - \ln 25 + 2 \ln 2$

12 If $y = \log_b x$, find in terms of y:

(a) $\log_b x^2$ (b) $\log_b x^4$ (c) $\log_x b^2$

(d) $\log_x (b^2 x)^3$ (e) $\log_x \left(\frac{b^6}{x^4}\right)$

13 Solve for x:

(a) $\log_4 x + \log_x 16 = 3$

(b) $3 \log_6 x + 2 \log_x 6 = 5$

(c) $\log_2 x = 4 \log_x 2$

(d) $\log_4 (2x) + \log_4 (x + 1) = 1$

(e) $\log_9 x = \log_3 (3x)$

(f) $\log_2 x + \log_x 2 = 2$

14 Find the value of x in the following.

(a) $0.6^x = 9.7$ (b) $(1.5)^{x+2} = 9.6$

(c) $\frac{1}{0.8^{x-2}} = 0.9$ (d) $\left(\frac{1}{0.24}\right)^{2+x} = 1.452^{x+1}$

(e) $2.79^{x-1} = 3.377^x$

15 By using an appropriate substitution solve the following equations.

(a) $2^{2x} + 2^{x+2} - 32 = 0$ (b) $2^{2x} - 9(2^{x-1}) + 2 = 0$

(c) $9^x = 3^{x+2} - 8$ (d) $25^y - 7(5^y) = 8$

16 The population of a colony of bacteria is given by $P = P_0 e^{kt}$ where P is the population after time t hours and P_0 is the initial population.

(a) Given that the population doubles in 7 hours, find k.

(b) Find the time it takes for the population to triple.

17 The model $P = \dfrac{0.8}{1 + 21.7e^{-0.16t}}$ represents the proportion of new and foreign used cars with a CNG kit installed in Trinidad and Tobago. Let $t = 0$ represent 1998.

(a) What proportion of new and foreign used cars had a CNG kit in 1998?

(b) Determine the maximum proportion of new and foreign used cars that have a CNG kit.

(c) Draw a graph of P vs t.

(d) When will 40% of new and foreign used cars have a CNG kit?

18 The value M of a bank account in which \$80 is invested at 4% interest, compounded annually, is given by $M = 800(1.04)^t$.

(a) Find the value of the account when $t = 6$.

(b) Find t in terms of M.

19 The current i (in amperes) in a certain electric circuit is given by $i = 15(1 - e^{-250t})$ where t is the time.

(a) Sketch the graph of i vs t.

(b) Find t in terms of i.

(c) Find the time when the current $i = 2$ amps.

20 Some of the inhabitants of a small village in Trinidad are known to have a highly infectious disease. The number of individuals, x, who have contracted the disease t days after the outbreak is modelled by the equation

$$x = \frac{4000}{1 + 3999e^{-0.2t}}$$

At what time has 40% of the population been infected?

21 A cup of hot chocolate, initially at boiling point, cools so that after t minutes, the temperature θ°C is given by $\theta = 10 + 90e^{-\frac{t}{8}t}$.

(a) Sketch the graph of θ against t.

(b) Find the value of t when the temperature reaches 50 °C.

22 If $2^x \times 4^{2y} = 1$ and $5^{5x} \times 25^y = \dfrac{1}{25}$, calculate the values of x and y.

23 Given that $\dfrac{9^x}{3^y} = 3$ and that $\lg(2x + 2y) = 1$, calculate the values of x and y.

SUMMARY

Indices, surds and logarithms

$\sqrt{xy} = \sqrt{x}\sqrt{y}$

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$

$a^m \times a^n = a^{m+n}$

$\frac{a^m}{a^n} = a^{m-n}$

$a^{-m} = \frac{1}{a^m}$

$a^0 = 1$

$(a^m)^n = a^{mn}$

$a^{\frac{1}{m}} = \sqrt[m]{a}$

$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{\frac{1}{n}}\right)^m$

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

$y = a^x, a > 1$

$y = a^x, 0 < a < 1$

$y = \log_e x = \ln x$

$\ln 1 = 0$

$\ln e = 1$

$\ln x + \ln y = \ln xy$

$\ln x - \ln y = \ln\left(\frac{x}{y}\right)$

$\ln x^n = n \ln x$

$\log_a x = \frac{\log_b x}{\log_b a}$

$e^{\ln f(x)} = f(x)$

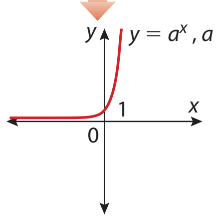
$\ln e^{f(x)} = f(x)$

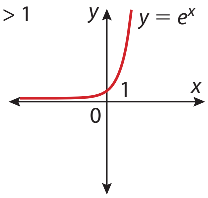
$y = a^x, a > 1$

Domain is $x \in \mathbb{R}$, Range: $y > 0$

$y = 0$ is a horizontal asymptote

a^x is an increasing function and is one-to-one



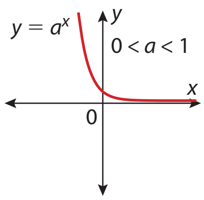


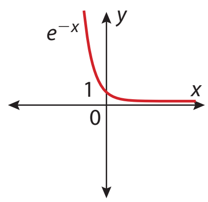
$y = a^x, 0 < a < 1$

Domain is $x \in \mathbb{R}$, Range: $y > 0$

$y = 0$ is a horizontal asymptote

a^x is a decreasing function and is one-to-one





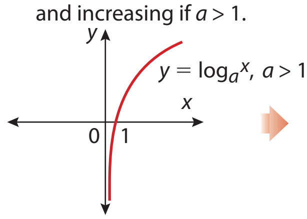
$x = a^y \Leftrightarrow y = \log_a x$

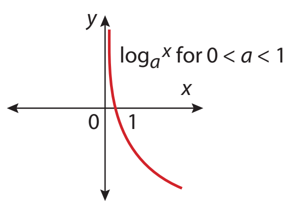
$y = \log_a x$

Domain is $x \in \mathbb{R}^+$, Range: $y \in \mathbb{R}$

y-axis is a vertical asymptote

The function is decreasing if $0 < a < 1$ and increasing if $a > 1$.





Checklist

Can you do these?

- Identify and use the laws of indices.
- Identify and use the laws of surds.
- Simplify surds.
- Identify the conjugate of a surd.
- Rationalise surds.

Review Exercise 5

1 Solve the following equations.

(a) $5^{x+3} = 1$

(b) $2^{x-1} = 4^{x+2}$

(c) $3^{2x}\, 3^{x-1} = 27$

2 Solve the following equations.

(a) $100\,(1 + i)^{-25} = 35$

(b) $800\,(1 - d)^{11} = 500$

3 Prove that if a, b and c are positive real numbers then $(\log_a b)(\log_b c) = \log_a c$.

4 Express the following as a single logarithm, simplifying as for as possible.

(a) $\ln(3x) - 2\ln y + 3\ln x^2$

(b) $\frac{1}{2}\ln(9x^2) + 2\ln x - 3\ln 2x$

(c) $\ln(x - 1) - \ln(x^2 - 1) + \ln(x^2 + 3x + 2)$

5 Find the value(s) of x satisfying the equation $\log_3 x = \log_9 (x + 6)$.

6 Given that $\log_2 x = \log_4 (8x - 16)$, find x.

7 By using an appropriate substitution, solve the following equations.

(a) $3^{2x+1} + 9 = 28(3^x)$ (b) $64^x = 8^{x+1} - 16$

8 A colony of bacteria grows according to the law
$x = 400e^{0.05t}$

where t is the time in years and x is the number of bacteria present.

(a) Find the initial number of bacteria.

(b) Find the population after 5 years.

(c) How long it will take for the population of the bacteria to reach 450?

(d) How long will it take for the population to double its size?

9 A colony of bacteria increases according to the law
$x = x_0 e^{kt}$

where x is the number of bacteria at time t and $k > 0$.

(a) If the number of bacteria triples after 4 days, find the value of k.

(b) How long will it take for the number of bacteria to double?

10 The size N of an insect population at time t days is given by $N = 450e^{0.03t}$.

(a) What is the population after 10 days?

(b) When will the population double?

(c) When will the insect population reach 1000?

11 The proportion (P) of Blue Ray Disc owners in Trinidad and Tobago at time t is given by the model

$$P = \frac{0.6}{1 + 60e^{-0.25t}}$$

Let $t = 0$ represent 2008, $t = 1$ represent 2009, and so on.

(a) Determine the proportion of households owning a Blue Ray Disc player in the year 2008.

(b) Determine the maximum proportion of Blue Ray Disc players owned by Trinidadians.

(c) Determine the year in which the proportion of Blue Ray Disc Players will half its maximum.

12 The number of bacteria in a culture at time t was given by $n = \lambda\, e^{5t}$.

(a) Find the number of bacteria present at time $t = 0$.

(b) Find the value of t when the colony was double its initial size.

13 In 2010, Partap deposited \$7000 in a fixed deposit account which promises interest of 6% compounded yearly. The amount \$$x$ at the end of n years is $x = 7000(1.06)^n$

(a) Find the amount of money Partap will have at the end of 2017.

(b) Find the year in which the amount of money first reached \$18 000.00.

14 A liquid cools from its original temperature of 70 °C to a temperature θ °C in t minutes. Given that $\theta = 70e^{-0.02t}$,

(a) Find the value of θ when $t = 12$ minutes.

(b) Find the value of t when $\theta = 40$ °C.

15 The charge Q on a capacitor is given by $Q = Q_1(1 - e^{-xt})$ where Q_1 is the initial charge, x is a constant and t is the time.

(a) Find an equation for t.

(b) Sketch the graph of Q.

(c) Find t when $Q = \frac{1}{2}Q_1$.

16 Given that $3^p \times 9^q = 2187$ and that $\ln(4p - q) = \ln 2 + \ln 5$, calculate the values of p and q.

17 Given that $\log_4(2x - 3y) = 2$ and that $\log_3 x - \log_3(2y + 1) = 1$, calculate the values of x and y.

18 Show that $\frac{1 + \sqrt{7}}{2 - \sqrt{7}} = -3 - \sqrt{7}$. Hence, find $\frac{1 + \sqrt{7}}{2 - \sqrt{7}} + \frac{2 - \sqrt{7}}{1 + \sqrt{7}}$.

19 Simplify $\frac{(1 - x)^2(2 + x)}{1 - \sqrt{x}}$.

CHAPTER 6

Functions

At the end of this chapter you should be able to:

- Decide whether a relation is a function
- Identify the domain of a function
- Identify the range of a function
- Show that a function is one-to-one (injective)
- Show that a function is onto (surjective)
- Show that a function is bijective (both one-to-one and onto)
- Find the inverse of a function
- Understand the relationship between a function and its inverse
- Find a composite function
- Define functions as a set of ordered pairs
- Define functions as a formula
- Plot and sketch functions and their inverses (if they exist)
- State the geometrical relationship between a function and its inverse
- Perform calculations using functions
- Identify increasing and decreasing functions, using the sign of $\dfrac{f(a) - f(b)}{a - b}$ where $a \neq b$
- Understand the relationship between the graph of: $y = f(x)$ and $y = af(x)$; $y = f(x)$ and $y = f(x) + a$; $y = f(x)$ and $y = f(x + a)$; $y = f(x)$ and $y = f(ax)$; $y = f(x)$ and $y = af(x + b)$; $y = f(x)$ and $y = |f(x)|$
- Sketch the graph of a rational function
- Manipulate piecewise defined functions

KEYWORDS/TERMS

relation • function • domain • range • codomain • one-to-one • injective • onto • surjective • bijective • inverse • composite function • ordered pairs • formula • increasing function • decreasing function • translation • stretch • transformation

Relations and functions

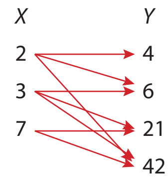
DEFINITION

Let A and B be non-empty sets. A **mapping** from A to B is a rule which associates a unique member of B to each member of A. A is called the **domain** of the mapping and B is called the **codomain** of the mapping.

A **relation** is a set of ordered pairs. There is no special connection between the pairs of numbers in a relation and any pairs of numbers identify a relation.

Let $X = \{(1, 2), (4, 5), (4, 7), (8, 10), (9, 10)\}$. X is a relation since it is a set of ordered pairs.

Let $X = \{2, 3, 7\}$ and $Y = \{4, 6, 21, 42\}$. Consider the relation 'is a factor of'. This relation can be represented using the diagram at right.



The set X is the domain of the mapping and Y is the codomain of the mapping. We can represent the mapping as a set of ordered pairs:

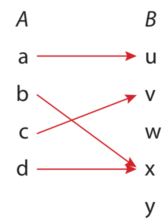
$\{(2, 4), (2, 6), (2, 42), (3, 6), (3, 21), (3, 42), (7, 21), (7, 42)\}$

DEFINITION

A **function** f from a set A to a set B assigns to each element a in set A a single element $f(a)$ in set B. The element in set B is called the **image** of a under f. The set A is the domain of the function and the set B is the codomain of the function. The range of the function is the set of elements that are images of $a \in A$.

When writing the mapping as a set of ordered pairs, the first element in the ordered pair must be an element in the domain and the second element in the ordered pair an element in the codomain.

A function is a mapping in which every element in the domain has a unique image in the codomain. The mapping at the right represents a function.



Notice that one arrow comes out of each element of A. Thus every element is mapped onto a unique element of B.

The domain of this function is $A = \{a, b, c, d\}$.

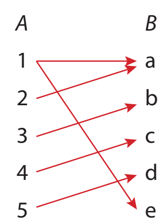
The codomain of the function is $B = \{u, v, w, x, y\}$.

The **range** of the function is the set $= \{u, v, x\}$. These are the images of the elements of A.

Note

The codomain is the set of elements that are possible and the range is the actual set of values that are assigned to an element in the domain.

EXAMPLE 1 Determine whether the mapping at the right is a function.



SOLUTION Notice that there are two arrows coming out of the number 1. In this case the image of 1 is both a and b. For this reason, the mapping is not a function.

EXAMPLE 2 Given $A = \{10, 20, 35, 70\}$ and $B = \{2, 3, 4, 5, 7, 9\}$, consider the mapping 'is a multiple of', and determine

(a) the domain of the mapping

(b) the codomain of the mapping

(c) whether the mapping is a function.

SOLUTION

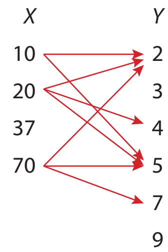
(a) The domain of the mapping is {10, 20, 35, 70}

(b) The codomain of the mapping is {2, 3, 4, 5, 7, 9}

(c) This mapping is not a function since elements of A map on to more than one element of B, for example:

$10 \rightarrow 2$

$10 \rightarrow 5$



Describing a function

There are many ways to describe a function. Here are some examples.

(i) Functions can be given by a formula, for example $f(x) = x + 1$ or $f: x \rightarrow x + 1$, $x \in \mathbb{R}$.

(ii) A function can be given as a graph.

(iii) A function can be given by listing its values: $f(1) = 3, f(2) = 5, f(4) = 7$.

(iv) A function can be given as a set of ordered pairs.

(v) A function can be represented by an arrow diagram.

EXAMPLE 3

Is $f(x) = \sqrt{x}$ a function?

SOLUTION

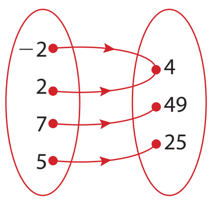
If the codomain of $\sqrt{x}$ is the set of real numbers, then $\sqrt{x}$ is not a function since one value of x maps onto two different values of y. A function *must* be single valued. We cannot get two or more values for the same input.

If the codomain is the set of non-negative real numbers, then $f(x) = \sqrt{x}$ is a function.

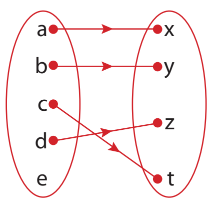
Try these 6.1

Which of the following relations is not a function? Give a reason for your answer.

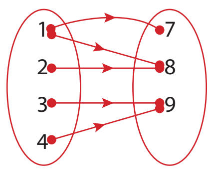
(a)



(b)



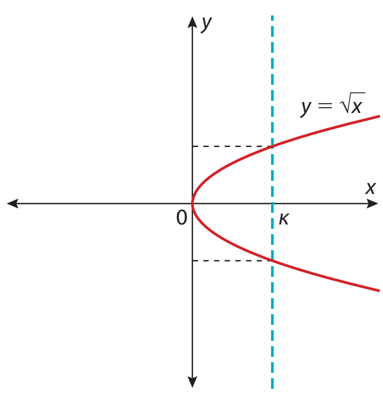
(c)



The vertical line test

This method of testing whether a mapping is a function or not involves looking at the graph of the function. If you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. This is called the vertical line test.

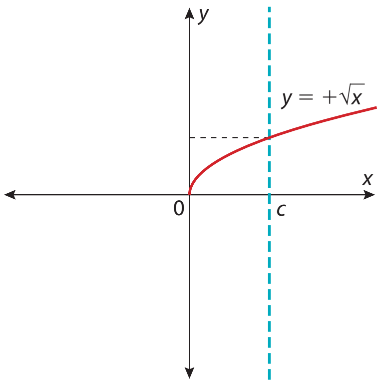
(i) The graph of $y = \sqrt{x}$ with codomain all real numbers is shown at right.



A line drawn parallel to the y-axis cuts the graph twice. This means that for one value of x there will be two values of y. Therefore, $y = \sqrt{x}$ is not a function.

(ii) The graph of $y = +\sqrt{x}$, which has codomain the set of non-negative real numbers, is shown at right.

A line drawn parallel to the y-axis cuts the graph at most once. This means that for every x-value there will be at most one y-value. Hence, $f(x) = +\sqrt{x}$ is a function.



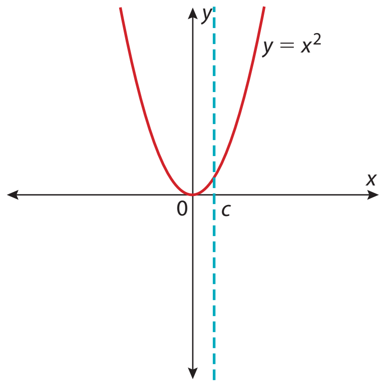
EXAMPLE 4 It is given that $f(x) = x^2$, where x is any real number ($x \in \mathbb{R}$).

(a) Show that $f(x)$ is a function.

(b) Determine the codomain of $f(x)$.

(c) Find the range of $f(x)$.

SOLUTION This is the graph of $f(x) = x^2$.



(a) Any line drawn parallel to the y-axis will cut the graph at most once. Therefore, $f(x) = x^2$ is a function.

(b) The codomain of $f(x)$ is all real numbers.

(c) The range of $f(x)$ is all non-negative numbers: $f(x) \geq 0$. The range is the set of values of $f(x)$ that are the images of x.

EXAMPLE 5 Identify the domain and range of $y = 4x - 5$.

SOLUTION Domain: since y is a linear function, x can assume any real value.

Therefore, the domain is $x \in \mathbb{R}$.

Range: for every x, we can find a corresponding y.

Hence, the range of y is $y \in \mathbb{R}$.

EXAMPLE 6 Find the domain and range of $y = x^2 - 3x + 2$.

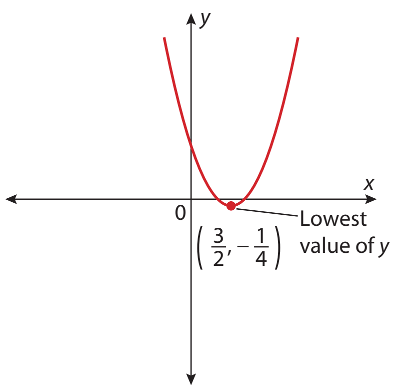
SOLUTION Domain: since y is a polynomial, the domain is $x \in \mathbb{R}$.

Range: y is a quadratic function with a minimum turning point.

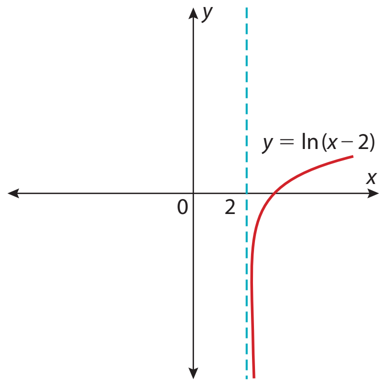
Complete the square: $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$

The minimum value of y is $-\frac{1}{4}$.

Therefore, the range is $y \geq -\frac{1}{4}$.



EXAMPLE 7 Identify the domain and range of $y = \ln(x - 2)$.



SOLUTION Since we are dealing with a log function, $x - 2$ must be greater than 0.

$x - 2 > 0$

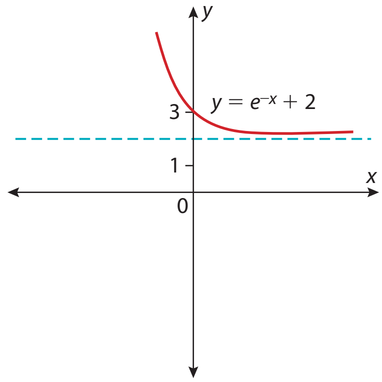
Therefore, $x > 2$.

Hence, the domain is $x: x \in \mathbb{R}, x > 2$.

The value of y can be negative or positive ranging from $-\infty$ to ∞.

Therefore, the range of y is $y \in \mathbb{R}$.

EXAMPLE 8 Identify the domain and range of $y = e^{-x} + 2$.



SOLUTION The domain is $x \in \mathbb{R}$.

Range: as $x \to \infty, e^{-x} \to 0 \Rightarrow y \to 2$

as $x \to -\infty, e^{-x} \to \infty \Rightarrow y \to \infty$

The range is $y > 2$.

Try these 6.2

Identify the domain and range of each of these.

(a) $y = \sqrt{x - 4}$

(b) $y = \ln(x + 1)$

(c) $y = 2e^x + 1$

> **Note**
>
> When identifying the domain, start with the assumption that x can take any real value. Look at the function and decide if there are any restrictions. For the range, use the domain to identify all values that y can attain.

One-to-one function (injective function)

A function $f: X \to Y$ is called a **one-to-one function** if and only if every element of Y is mapped onto one and only one element of X. This means that no two elements of X can have the same image in Y. A one-to-one function is also called an **injective function**. We can prove that functions are one-to-one in either of the following ways.

(i) Show that $f(x)$ is a one-to-one function if $f(a) = f(b) \Rightarrow a = b$.

(ii) Sketch the graph of $y = f(x)$ and draw a line parallel to the x-axis. $f(x)$ is one-to-one if this line cuts the graph at most once.

EXAMPLE 9 Show that $f(x) = 2x + 1$ is one-to-one.

SOLUTION

Method 1

Since $f(x) = 2x + 1$

$f(a) = 2a + 1$

$f(b) = 2b + 1$

For $f(x)$ to be one-to-one, $f(a) = f(b)$

$\Rightarrow 2a + 1 = 2b + 1$

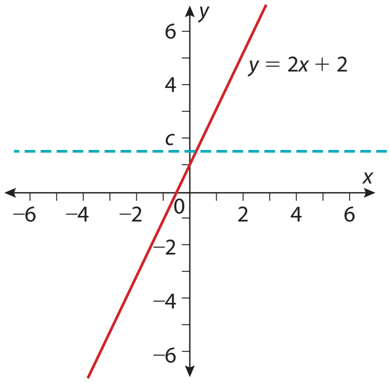
$\Rightarrow 2a = 2b$

$\Rightarrow a = b$

Since $f(a) = f(b) \Rightarrow a = b$, $f(x)$ is one-to-one.

Method 2

Sketch the graph of $f(x) = 2x + 1$.



Any line drawn parallel to the x-axis will cut the graph of $f(x) = 2x + 1$ at most once. Hence, $f(x)$ is one-to-one.

EXAMPLE 10 Show that $f(x) = \dfrac{2x + 1}{3x - 2}$ is one-to-one.

SOLUTION

If $f(x)$ is one-to-one, then $f(a) = f(b) \Rightarrow a = b$.

Since $f(x) = \dfrac{2x + 1}{3x - 2}$

$f(a) = \dfrac{2a + 1}{3a - 2}$

$f(b) = \dfrac{2b + 1}{3b - 2}$

$f(a) = f(b)$

$\Rightarrow \dfrac{2a + 1}{3a - 2} = \dfrac{2b + 1}{3b - 2}$

Cross-multiplying gives:

$(2a + 1)(3b - 2) = (2b + 1)(3a - 2)$

Expanding gives:

$6ab - 4a + 3b - 2 = 6ab - 4b + 3a - 2$

$\therefore\ 6ab - 6ab + 3b + 4b - 2 + 2 = 3a + 4a$

$\Rightarrow 7b = 7a$

$\Rightarrow a = b$

Since $f(b) = f(a) \Rightarrow a = b$, $f(x)$ is one-to-one.

Note

We could also sketch the graph of $f(x) = \dfrac{2x + 1}{3x - 2}$ and use the graph to show that $f(x)$ is one-to-one. In this case, drawing the graph would take longer than using algebra.

EXAMPLE 11 Is $f(x) = x^2 + 2x + 3$ one-to-one?

SOLUTION ***Method 1***

If $f(x)$ is one-to-one, then $f(a) = f(b) \Rightarrow a = b$.

Since $f(x) = x^2 + 2x + 3$

$$f(a) = a^2 + 2a + 3$$

$$f(b) = b^2 + 2b + 3$$

$$f(a) = f(b) \Rightarrow a^2 + 2a + 3 = b^2 + 2b + 3$$

$\therefore\ a^2 - b^2 + 2a - 2b = 0$

Factorising gives:

$(a - b)(a + b) + 2(a - b) = 0$

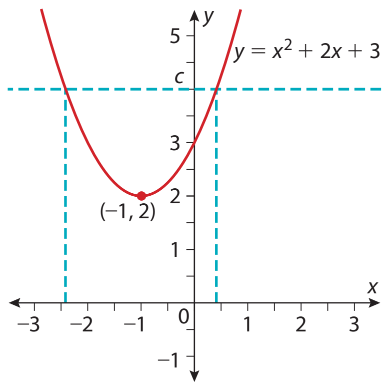
$\Rightarrow (a - b)(a + b + 2) = 0$

$\therefore\ a - b = 0, a + b + 2 = 0$

$\Rightarrow a = b, a = -2 - b$

Since $f(a) = f(b) \Rightarrow a = b$ or $a = -2 - b$, $f(x)$ is not a one-to-one function.

Method 2



The graph of $f(x) = x^2 + 2x + 3$ is shown.

When $x = 0, f(0) = 3$.

Since the coefficient of x^2 is positive, the curve has a minimum point. The minimum point exists at $x = -\frac{2}{2} = -1$, when $x = -1$, $f(-1) = (-1)^2 + 2(-1) + 3 = 2$.

The minimum point is at $(-1, 2)$.

A line drawn parallel to the x-axis cuts the graph twice. Hence, there are two values of y for one value of x. The function is not one-to-one.

Onto function (surjective function)

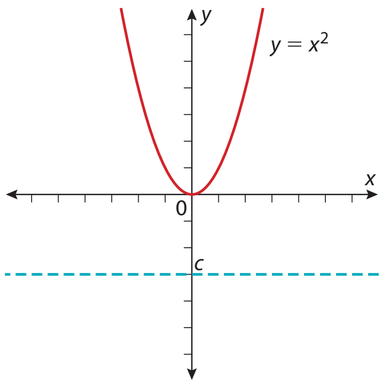
A function $f: X \rightarrow Y$ is called an **onto function** or **surjective function** if and only if every element of Y is mapped onto by at least one element of X.

For a surjective function the range and codomain are the same.

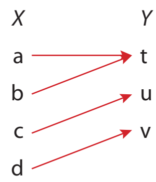
To determine whether a function $f(x)$ is onto, we can use a graphical method as follows. Sketch the graph of $y = f(x)$, draw a line parallel to the x-axis and if this line cuts the graph at least once, then the function $f(x)$ is onto.

EXAMPLE 12 Which of the following functions are onto?

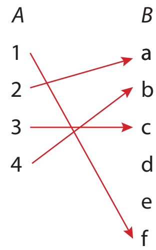
(a)



(b)

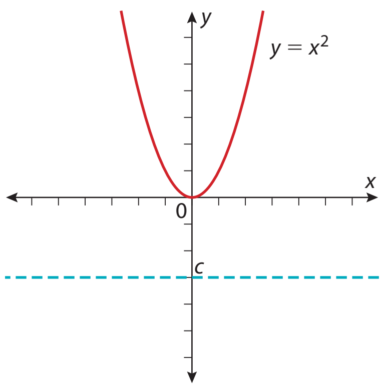


(c)



SOLUTION

(a) This function is not onto. Any line drawn parallel to the x-axis must cut the graph at *least* once. The line drawn below the x-axis does not cut the graph. Note that we can restrict the codomain to non-negative real numbers and the function will be onto.



(b) This function is onto since all values in the codomain have a value in the domain mapped onto it.

(c) This function is not onto since the elements d and e in the codomain have no values in the domain mapped onto them.

Showing that a function is onto

To show that a function is onto, we must show that for every element in the codomain there exists an element in the domain which maps to it. We can prove that a function is onto either algebraically or graphically.

Algebraic proof

Let y be any element in the codomain and x an element of the domain. We solve the equation $y = f(x)$ for x.

EXAMPLE 13 Prove that the function f defined by $f: \mathbb{R} \to \mathbb{R}$ for $f(x) = 7x - 2$ is onto.

SOLUTION Let y be an element of the codomain.

$y = 7x - 2$

Making x the subject of the formula gives:

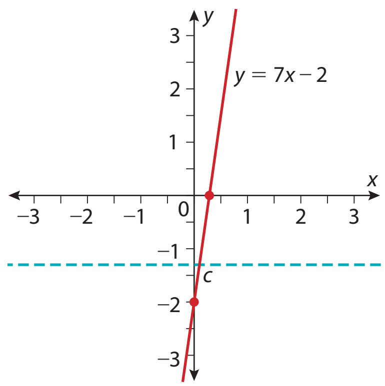
$y + 2 = 7x$

$x = \dfrac{y + 2}{7}$

Since y is a real number, then $\frac{y+2}{7}$ is a real number and $f\left(\frac{y+2}{7}\right) = 7\left(\frac{y+2}{7}\right) - 2 = y + 2 - 2 = y$.

Hence, for every x in the domain there is a corresponding y in the codomain. Therefore, $f(x)$ is an onto function.

Graphical proof



Any line drawn parallel to the x-axis cuts the graph at least once (in this case once).

Hence, the graph shows that $f(x) = 7x + 2$ is an onto function.

EXAMPLE 14 Is the function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x$ onto?

SOLUTION Let $y = x^2 + 4x$.

We rearrange the equation:

$x^2 + 4x - y = 0$

$x = \dfrac{-4 \pm \sqrt{16 + 4y}}{2}$ (Using the quadratic formula: $a = 1, b = 4, c = -y$)

$x = \dfrac{-4 \pm 2\sqrt{4 + y}}{2} = -2 \pm \sqrt{4 + y}$

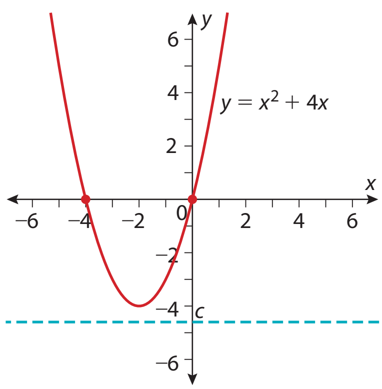
Hence, $x = -2 + \sqrt{4 + y}$ or $x = -2 - \sqrt{4 + y}$.

We can easily show that $f(-2 + \sqrt{4 + y}) = y$.

Let $y = -5, x = -2 + \sqrt{4 - 5}$.

Hence, x is not a real number when $y = -5$ and therefore $f(x)$ is not onto.

We can also show this graphically.



$f(x) = x^2 + 4x$

When $f(x) = 0, x^2 + 4x = 0$.

$\Rightarrow x(x + 4) = 0$

$\Rightarrow x = 0, x = -4$

Since the coefficient of x^2 is positive, the curve has a minimum point. The minimum point exists at $x = -\frac{4}{2} = -2, f(-2) = 4 - 8 = -4$.

The minimum point is at $(-2, -4)$

The graph shows that there are values of y that do not correspond to values of x.

Bijective functions

A **bijective function** is a function that is both surjective and injective.

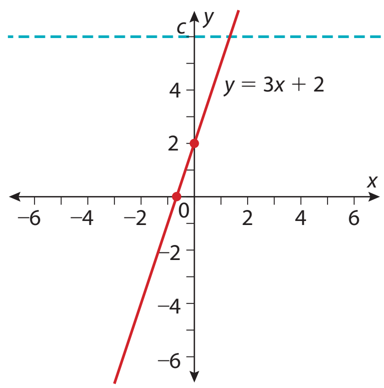
Note

The inverse of a function exists if and only if the function is bijective.

EXAMPLE 15 Given that $f(x) = 7x + 2$, show that $f(x)$ is bijective.

SOLUTION To show that $f(x)$ is bijective, we need to show that $f(x)$ is both one-to-one and onto.

We draw the graph of $f(x) = 3x + 2$.



Any line drawn parallel to the x-axis cuts the graph exactly once. Hence, the function $f(x)$ is one-to-one. Since any line drawn parallel to the x-axis cuts the graph once, for every y-value there is a corresponding x-value. Therefore, the function is onto.

Since $f(x) = 3x + 2$ is both one-to-one and onto, $f(x)$ is bijective.

EXERCISE 6A

1 Which of the following relations in the set of ordered pairs are functions? State the rule of each function. Illustrate your answers using arrow diagrams.

(a) $\{(5, 6), (6, 7), (7, 8), (7, 9), (8, 10)\}$

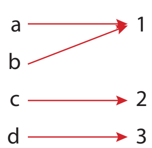
(b) $\{(-1, 2), (1, 2), (-3, 8), (3, 8)\}$

(c) $\{(a, b), (c, b), (c, d), (e, f)\}$

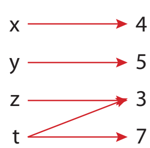
(d) $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$

2 Which of the following diagrams below represent a function or a mapping from A to B? For those that are functions, write the functions as set(s) of ordered pairs.

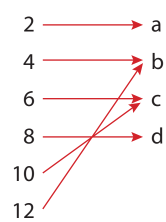
(a)



(b)



(c)



3 The function g is defined by $g(x) = 4x - 5$ and the domain of g is $\{0, 1, 2, 3, 4\}$.

(a) Find the range of g.

(b) List the set of ordered pairs of the function g.

4 The function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined by the following, where $\mathbb{R}$ is the set of real numbers.

$$f(x) = \begin{cases} x + 2 & \text{if } x < 2 \\ x - 1 & \text{if } x > 8 \\ (x + 1)^2 & \text{if } 2 \le x \le 8 \end{cases}$$

Find the following.

(a) $f(-4)$ (b) $f(9)$

(c) $f(2)$ (d) $f(8)$

5 Given that $f(x) = 5x - 2$, $x \in \mathbb{R}$, show that the following are true.

(a) $f(x)$ is a one-to-one function

(b) $f(x)$ is an onto function

6 Show that the function $f(x) = \dfrac{1}{x - 4}$, $x \in \mathbb{R}$, $x \neq 4$ is injective.

7 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the mapping defined by

$$g(x) = \begin{cases} x + 2 & \text{for } x \ge 0 \\ x & \text{when } x < 0 \end{cases}$$

(a) Is $g(x)$ injective?

(b) Is $g(x)$ surjective? Show all working clearly.

8 Show that the function $g(x) = (x - 2)^2$, $x \in \mathbb{R}$ is surjective.

9 (a) Show that the function defined by the following is a bijective function.

$$f(x) = \begin{cases} x^2 + 2 & \text{for } x \ge 0 \\ x + 2 & \text{for } x < 0 \end{cases}$$

(b) Find the inverse of the function.

10 The function $g(x)$ is defined by $g(x) = x^2 + 1$, $x \in \mathbb{R}^+$.

Decide whether each of these is true.

(a) $g(x)$ is injective

(b) $g(x)$ is surjective

11 Is the function $f(x)$ bijective?

$$f(x) = \begin{cases} 2x + 1 & \text{when } x \ge 0 \\ x - 1 & \text{when } x < 0 \end{cases}$$

12 The mapping $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is defined by $f(x) = x^2 + 4$

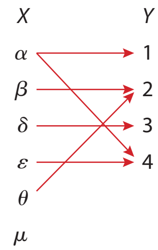
(a) Sketch the graph and state the range of the function.

(b) Show that f is neither injective nor surjective.

(c) Find a restriction g of f which is bijective and has the same image as f.

(d) Find the inverse of g.

13 A mathematics teacher is training six students α, β, δ, ε, θ, μ for a mathematics competition. The teacher makes assignments, g, of students to different areas of mathematics: geometry (1), calculus (2), algebra (3) and statistics (4). This is shown in the diagram at right in which $X = \{\alpha, \beta, \delta, \varepsilon, \theta, \mu\}$ and $Y = \{1, 2, 3, 4\}$ and the areas of mathematics and students are expressed as ordered pairs.



(a) State two reasons why g is not a function.

(b) Hence, with a minimum change to g, construct a function $f: a \rightarrow b$ as a set of ordered pairs.

Inverse functions

A function f has an inverse f^{-1} if and only if f is bijective. Since a function can be represented as a mapping, a set of ordered pairs, a graph, and a formula, you must be able to find the inverse of a function given in any form.

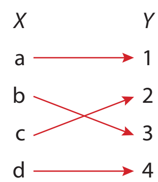
Suppose that f is a one-to-one function. The correspondence from the range of f back to the domain of f is called the inverse of f. The relationship between a function and its inverse are as follows:

(i) The domain of f is the range of f^{-1}.

(ii) The range of f is the domain of f^{-1}.

(iii) The graph of f^{-1} can be derived from the graph of f by reflecting f in the line $y = x$.

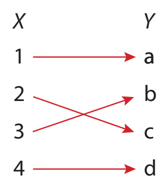
EXAMPLE 16 Find the inverse of the function shown in the diagram.



SOLUTION Reversing the domain and range, the inverse of the function is as shown.

The domain of the inverse is {1, 2, 3, 4} and the range of the inverse is {a, b, c, d}.

Inverse function



EXAMPLE 17 Find the inverse of the function {(1, 5), (2, 10), (3, 15), (4, 20), (5, 25)}.

SOLUTION Reversing the values in each pair gives the inverse as {(5, 1), (10, 2), (15, 3), (20, 4), (25, 5)}.

EXAMPLE 18 A function f is defined by $f: x \to 4x + 2, x \in \mathbb{R}$. Find in a similar form f^{-1}.

SOLUTION Let $y = 4x + 2$.

First, we interchange x and y. By exchanging x and y, we are exchanging the domain and range so that $y = f^{-1}(x)$ and $f(y) = x$.

$x = 4y + 2$

Make y the subject of the formula.

$4y = x - 2$

$y = \dfrac{x-2}{4}$

$\therefore f^{-1}(x) = \dfrac{x-2}{4}$

Hence, $f^{-1}: x \to \dfrac{x-2}{4}, x \in \mathbb{R}$.

EXAMPLE 19 A function f is defined by $f: x \to \dfrac{2x+1}{x-3}, x \neq 3$.

Find in similar form f^{-1}, and state the value of x for which f^{-1} is not defined.

SOLUTION Let $y = \dfrac{2x+1}{x-3}$.

Interchange x and y:

$x = \dfrac{2y+1}{y-3}$

Make y the subject of the formula:

$x(y-3) = 2y + 1$

$xy - 3x = 2y + 1$

$xy - 2y = 1 + 3x$

$y(x-2) = 1 + 3x$

$y = \dfrac{1+3x}{x-2}$

$\therefore f^{-1}(x) = \dfrac{1+3x}{x-2}, x \in \mathbb{R}, x \neq 2$.

Hence, $f^{-1}: x \to \dfrac{1+3x}{x-2}, x \in \mathbb{R}, x \neq 2$.

To find the inverse of a function $f(x)$:

(i) Let $y = f(x)$.
(ii) Interchange x and y.
(iii) Make y the subject of the formula, y is now the inverse of $f(x)$.

EXAMPLE 20 A function g is defined by $g(x) = \dfrac{3x-1}{x+2}, x \neq -2$.

Find:

(a) $g^{-1}(x)$

(b) $g^{-1}(-3)$.

SOLUTION **(a)** Let $y = \dfrac{3x-1}{x+2}$.

Interchange x and y:

$x = \dfrac{3y-1}{y+2}$

$$\Rightarrow x(y + 2) = 3y - 1 \qquad \text{(We make } y \text{ the subject of the formula.)}$$

$$xy + 2x = 3y - 1$$

$$xy - 3y = -1 - 2x$$

$$y(x - 3) = -1 - 2x$$

$$y = \frac{-1 - 2x}{x - 3}$$

$$= \frac{-(2x + 1)}{-(3 - x)}$$

$$= \frac{2x + 1}{3 - x}$$

Therefore, $f^{-1}: x \to \frac{2x + 1}{3 - x}, x \in \mathbb{R}, x \neq 3$

(b) Substitute $x = -3$ into:

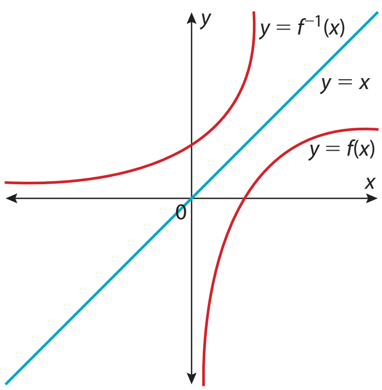
$$f^{-1}(x) = \frac{2x + 1}{3 - x}$$

$$\Rightarrow f^{-1}(-3) = \frac{2(-3) + 1}{3 - (-3)}$$

$$= \frac{-5}{6}$$

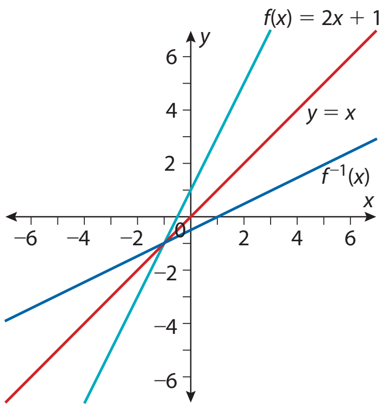
Graphs of inverse functions

Given the graph of $y = f(x)$, we can sketch the graph of $f^{-1}(x)$ by reflecting $y = f(x)$ in the line $y = x$.



EXAMPLE 21 Sketch the graph of $f(x) = 2x + 1$, and hence, sketch $f^{-1}(x)$.

SOLUTION



EXAMPLE 22 If $y = 2x^2 + 3x - 1$, identify the domain and range of y. Does the inverse of y exist?

SOLUTION Since y is a polynomial, the domain of y is $x \in \mathbb{R}$.

To find the range of y we need to identify the minimum point.

$y = 2x^2 + 3x - 1$

The minimum point is at $x = \frac{-b}{2a}$ where $a = 2$, $b = 3$.

$\therefore x = -\frac{3}{2(2)} = -\frac{3}{4}$

When $x = -\frac{3}{4}$, $y = 2\left(-\frac{3}{4}\right)^2 + 3\left(-\frac{3}{4}\right) - 1$

$$= 2\left(\frac{9}{16}\right) - \frac{9}{4} - 1$$

$$= \frac{9}{8} - \frac{9}{4} - 1$$

$$= -\frac{17}{8}$$

We can also complete the square:

$y = 2\left(x + \frac{3}{4}\right)^2 - \frac{17}{8}$

The coordinates of the minimum points are $\left(-\frac{3}{4}, -\frac{17}{8}\right)$

Since the lowest point on the curve is $\left(-\frac{3}{4}, -\frac{17}{8}\right)$, the range of the function is:

$y \geq -\frac{17}{8}$

Hence, $x \in \mathbb{R}$, $y \geq -\frac{17}{8}$.

To decide whether y has an inverse, we decide whether y is bijective.

When $x = 0$, $y = -1$

When $x = -\frac{3}{2}$, $y = 2\left(-\frac{3}{2}\right)^2 + 3\left(-\frac{3}{2}\right) - 1 = -1$

Since $y = -1$ when $x = 0$ and $x = -\frac{3}{2}$, y is not one-to-one.

Hence, y is not bijective.

Therefore, the inverse does not exist.

EXAMPLE 23 A function f is defined by $f: x \to x^2 - 2x + 3$, $x \in \mathbb{R}$, $x \geq 1$.

Write $f(x)$ in the form $a(x + b)^2 + c$.

Hence, find:

(a) the range of $f(x)$

(b) the inverse of $f(x)$

(c) the domain and range of $f^{-1}(x)$.

SOLUTION Complete the square of $f(x)$:

$$x^2 - 2x + 3 = x^2 - 2x + (-1)^2 + 3 - (-1)^2$$
$$= (x-1)^2 + 2$$

$\therefore f(x) = (x-1)^2 + 2$

(a) Since $f(x)$ is a quadratic with the coefficient of x^2 positive, $f(x)$ has a minimum point at $(1, 2)$.

$\therefore$ minimum value of $f(x) = 2$

$\Rightarrow$ range of $f(x)$ is $y \geq 2$.

(b) Let $y = x^2 - 2x + 3$.

Since $x^2 - 2x + 3 = (x-1)^2 + 2$

$y = (x-1)^2 + 2$

Interchange x and y:

$x = (y-1)^2 + 2$

Make y the subject of the formula:

$(y-1)^2 = x - 2$

$y - 1 = \pm\sqrt{x-2}$

$y = 1 \pm \sqrt{x-2}$

Since the domain of $f(x)$ = range of $f^{-1}(x)$, the range of $f^{-1}(x)$ is $y \geq 1$.

Hence, $f^{-1}(x) = 1 + \sqrt{x-2}$.

(c) The domain of $f(x)$ is $x \geq 1$.

The range of $f(x)$ is $y \geq 2$.

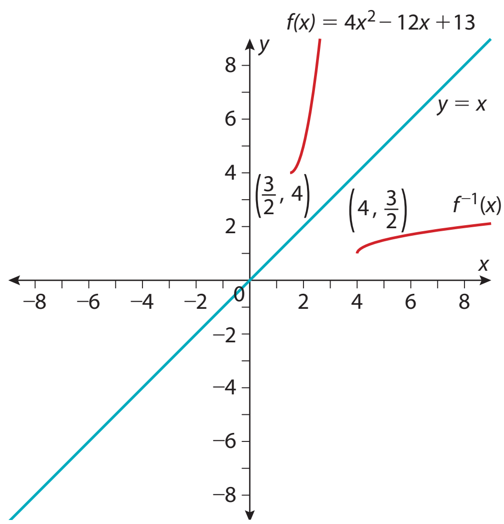
Interchanging the domain and range gives these.

The domain of $f^{-1}(x)$ is $x \geq 2$.

The range of $f^{-1}(x)$ is $y \geq 1$.

> **Note**
>
> To find the inverse of a quadratic, we complete the square first and use this form of the function.

EXAMPLE 24 The function f is defined by $f: x \rightarrow 4x^2 - 12x + 13, x \in \mathbb{R}, x \geq \frac{3}{2}$ sketch the graph of $f(x)$ and hence, sketch $f^{-1}(x)$.



SOLUTION Since $f(x) = 4x^2 - 12x + 13$

Completing the square of $f(x)$:

$$4x^2 - 12x + 13 = 4(x^2 - 3x) + 13$$
$$= 4\left(x^2 - 3x + \left(-\frac{3}{2}\right)^2\right) + 13 - 4\left(-\frac{3}{2}\right)^2$$
$$= 4\left(x - \frac{3}{2}\right)^2 + 13 - \left(4 \times \frac{9}{4}\right)$$
$$= 4\left(x - \frac{3}{2}\right)^2 + 4$$

The turning point is at $\left(\frac{3}{2}, 4\right)$.

$f^{-1}(x)$ is a reflection of $f(x)$ in the line $y = x$.

Odd and even functions

Odd functions

DEFINITION

A function $f(x)$ is said to be an odd function if and only if $f(-x) = -f(x)$.

EXAMPLE 25 Prove that $f(x) = x^3$ is an odd function.

SOLUTION

Since $f(x) = x^3$

$f(-x) = (-x)^3 = -(x^3) = -f(x)$

Since $f(-x) = -f(x) \Rightarrow f(x) = x^3$ is an odd function.

Note

The graph of an odd function has rotational symmetry about the origin. This means that if the graph of $f(x)$ is rotated through 180° about the origin, then the graph remains unchanged.

Even functions

DEFINITION

A function $f(x)$ is an even function if and only if $f(x) = f(-x)$.

EXAMPLE 26 Prove that $f(x) = x^2$ is an even function.

SOLUTION

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

Since $f(x) = f(-x) \Rightarrow f(x) = x^2$ is an even function.

Note

The graph of an even function is symmetric with respect to the line $x = 0$. The graph of $f(x)$ remains unchanged after reflection in the line $x = 0$.

Periodic functions

A periodic function is a function that repeats its values in regular intervals.

DEFINITION

A function $f(x)$ is said to be periodic with period k if and only if $f(x) = f(x + k)$. The period k is the x-distance between any point and the next point at which the same pattern of y values repeats itself.

The modulus function

DEFINITION

The absolute value of x or modulus of x denoted by $|x|$ is defined as:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

By definition, the absolute value function or modulus function is a positive function. This means that it returns positive values only.

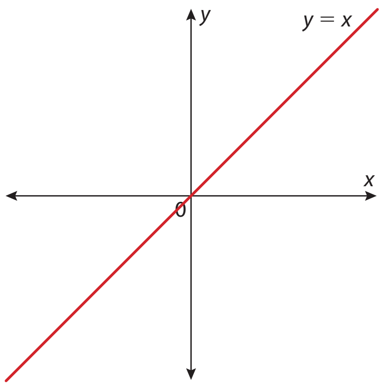
Graph of the modulus function

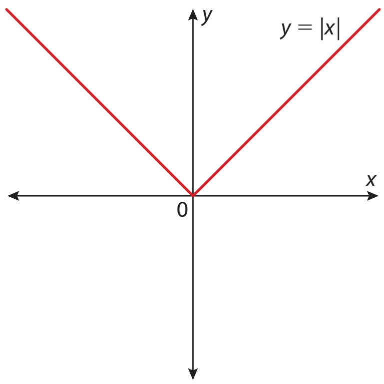
The graph of the function $|f(x)|$ can be drawn from the graph of $y = f(x)$ by reflecting the section of $y = f(x)$ that is below the x-axis to above the x-axis, with the x-axis being the line of reflection.

The part of the curve $y = f(x)$ that is above the x-axis remains unchanged.

EXAMPLE 27 Sketch the graph of $y = x$. Hence, sketch $y = |x|$.

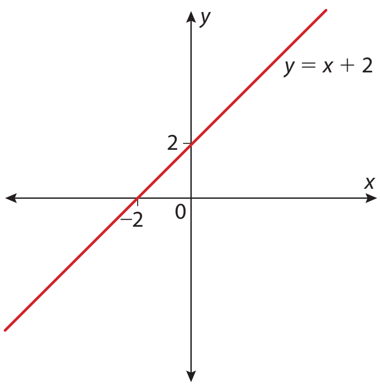
SOLUTION

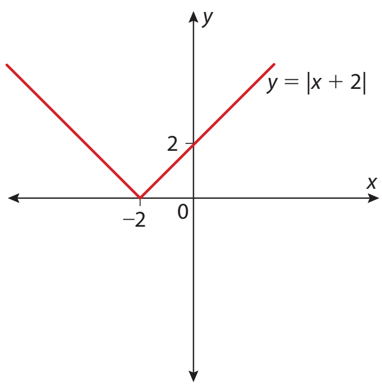




EXAMPLE 28 Sketch the graph of $y = x + 2$. Hence, sketch $y = |x + 2|$.

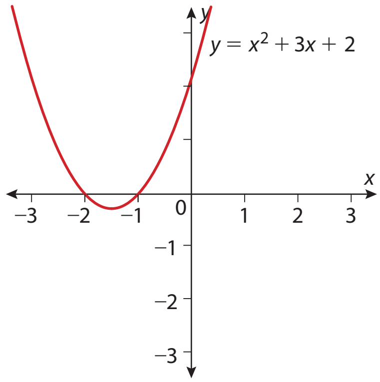
SOLUTION

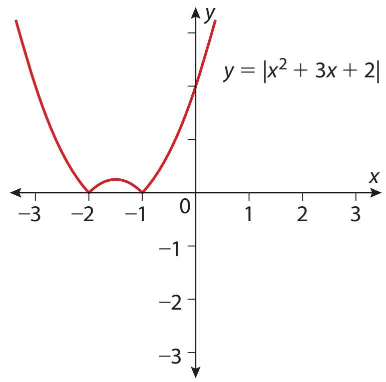




EXAMPLE 29 Sketch the graph of $y = x^2 + 3x + 2$. Hence, sketch $y = |x^2 + 3x + 2|$.

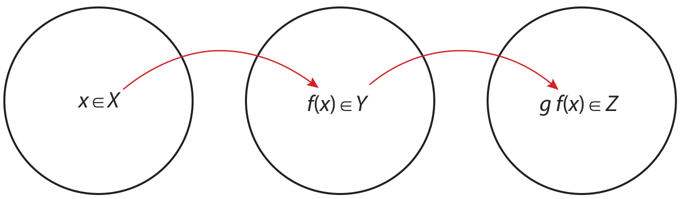
SOLUTION





Composite functions

Let f be a function of X into Y and let g be a function of Y into Z. Let $x \in X$. Then its image $f(x)$ is the domain of g and we can find the image of $f(x)$ under g, which is $gf(x)$. There exists a rule which assigns each element $x \in X$ a corresponding element $gf(x) \in Z$.



The range of f is the domain of g.

The function $gf(x)$ is called a composite function of g and f and is denoted by gf or $g \circ f$.

EXAMPLE 30 Let $f(x) = 4x + 2$ and $g(x) = 3x - 4$. Find

(a) fg

(b) gf.

SOLUTION

(a) $fg(x) = f(g(x))$

We now substitute $g(x) = 3x - 4$ into f.

$f(g(x)) = f(3x - 4)$

We can now find $f(3x - 4)$ by replacing x by $3x - 4$ into $f(x)$.

$$\begin{aligned} fg(x) &= 4(3x - 4) + 2 \\ &= 12x - 16 + 2 \\ &= 12x - 14 \end{aligned}$$

(b) $gf = gf(x)$

Substituting $f(x) = 4x + 2$ into g, we have:

$gf(x) = g(4x + 2)$

Replacing x by $4x + 2$ into $g(x)$ we have:

$$\begin{aligned} gf &= 3(4x + 2) - 4 \\ &= 12x + 6 - 4 \\ &= 12x + 2 \end{aligned}$$

The composite functions fg and gf are two different functions.

EXAMPLE 31 Functions f and g are defined by $f : x \rightarrow 2x + 1$ and $g : x \rightarrow \frac{2}{3x + 2}$, $x \neq -\frac{2}{3}$.

Express in a similar form the functions fg and gf. Hence, find $fg(2)$ and $gf(3)$.

SOLUTION

$$f(x) = 2x + 1$$

$$g(x) = \frac{2}{3x + 2}$$

$$fg = fg(x)$$

Substituting $g(x) = \frac{2}{3x + 2}$ into f:

$$\begin{aligned} fg &= f\left(\frac{2}{3x + 2}\right) \\ &= 2\left(\frac{2}{3x + 2}\right) + 1 && \left(\text{Replacing } x \text{ by } \frac{2}{3x + 2} \text{ in } f\right) \\ &= \frac{4}{3x + 2} + 1 \\ &= \frac{4 + 3x + 2}{3x + 2} \\ &= \frac{6 + 3x}{3x + 2}, x \neq -\frac{2}{3} \end{aligned}$$

$$\begin{aligned} gf &= gf(x) \\ &= g(2x + 1) && (\text{Using } f(x) = 2x + 1 \text{ into } g) \\ &= \frac{2}{3(2x + 1) + 2} && (\text{Replacing } x \text{ by } 2x + 1 \text{ in } g) \\ &= \frac{2}{6x + 3 + 2} \\ &= \frac{2}{6x + 5}, x \neq \frac{-5}{6} \end{aligned}$$

Since $fg(x) = \frac{6 + 3x}{3x + 2}$, substituting $x = 2$, we have:

$$\begin{aligned} fg(2) &= \frac{6 + 3(2)}{3(2) + 2} \\ &= \frac{12}{8} \\ &= \frac{3}{2} \end{aligned}$$

Since $gf(x) = \frac{2}{6x + 5}$

Substituting $x = 3$ we have:

$$\begin{aligned} gf(3) &= \frac{2}{6(3) + 5} \\ &= \frac{2}{23} \end{aligned}$$

EXAMPLE 32 A function g is defined by $g : x \to \frac{x}{2x+1}, x \neq -\frac{1}{2}$.

(a) Obtain expressions for g^2 and g^3.

(b) State the values of x for which the functions g^2 and g^3 are not defined.

SOLUTION

> **Note**
>
> Remember that g^2 does not mean 'the square of g'.

(a) Since $g(x) = \frac{x}{2x+1}$

$$g^2 = g^2(x)$$

$$= gg(x)$$

$$= g\left(\frac{x}{2x+1}\right) \qquad \left(\text{Replacing } g(x) = \frac{x}{2x+1} \text{ in } gg(x)\right)$$

$$= \frac{\frac{x}{2x+1}}{2\left(\frac{x}{2x+1}\right)+1} \qquad \left(\text{Substituting } x \text{ by } \frac{x}{2x+1} \text{ into } g(x)\right)$$

$$= \frac{\frac{x}{2x+1}}{\frac{2x}{2x+1}+1}$$

$$= \frac{\frac{x}{2x+1}}{\frac{2x+2x+1}{2x+1}} \qquad (\text{Finding the LCM of the denominator})$$

$$= \frac{\frac{x}{2x+1}}{\frac{4x+1}{2x+1}}$$

$$= \frac{x}{2x+1} \times \frac{2x+1}{4x+1}$$

$$= \frac{x}{4x+1}$$

$$\therefore g^2 = \frac{x}{4x+1}, x \neq -\frac{1}{4}, x \neq -\frac{1}{2}$$

(b) $g^3 = gg^2(x)$

$$= g\left(\frac{x}{4x+1}\right) \qquad \left(\text{Using } g^2(x) = \frac{x}{4x+1}\right)$$

$$= \frac{\frac{x}{4x+1}}{2\left(\frac{x}{4x+1}\right)+1}$$

$$= \frac{\frac{x}{4x+1}}{\frac{2x+4x+1}{4x+1}}$$

$$= \frac{\frac{x}{4x+1}}{\frac{6x+1}{4x+1}}$$

$$= \frac{x}{\cancel{4x+1}} \times \frac{\cancel{4x+1}}{6x+1}$$

$$= \frac{x}{6x+1}, x \neq -\frac{1}{6}$$

$$\therefore g^3 = \frac{x}{6x+1}, x \neq -\frac{1}{6}, x \neq -\frac{1}{4}, x \neq -\frac{1}{2}$$

g^2 is not defined at $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$.

g^3 is not defined at $x = -\frac{1}{2}$, $x = -\frac{1}{4}$ and $x = -\frac{1}{6}$.

EXAMPLE 33 Functions f and g are defined by $f : x \to 3 - 2x$ and $g : x \to x^2 + 2x + 1$.

Find the solutions of the equations.

(a) $fg(x) = 0$ **(b)** $gf(x) = 0$

SOLUTION

(a) First we find $fg(x)$ by replacing $g(x)$ by $x^2 + 2x + 1$:

$fg(x) = f(x^2 + 2x + 1)$

Replacing x by $x^2 + 2x + 1$ into $f(x)$, we have

$$\begin{aligned} fg(x) &= 3 - 2(x^2 + 2x + 1) \\ &= 3 - 2x^2 - 4x - 2 \\ &= -2x^2 - 4x + 1 \end{aligned}$$

$fg(x) = 0$

$\Rightarrow -2x^2 - 4x + 1 = 0$

$\Rightarrow 2x^2 + 4x - 1 = 0$

$$\begin{aligned} \Rightarrow x &= \frac{-4 \pm \sqrt{16 - (4)(2)(-1)}}{2(2)} \\ &= \frac{-4 \pm 2\sqrt{6}}{4} \\ &= -1 \pm \frac{1}{2}\sqrt{6} \end{aligned}$$

(b) $gf(x) = g(3 - 2x)$

Replacing x by $3 - 2x$ into g, we have

$$\begin{aligned} gf(x) &= (3 - 2x)^2 + 2(3 - 2x) + 1 \\ &= 9 - 12x + 4x^2 + 6 - 4x + 1 \\ &= 4x^2 - 16x + 16 \end{aligned}$$

$gf(x) = 0$

$\Rightarrow 4x^2 - 16x + 16 = 0$

$x^2 - 4x + 4 = 0$

$\Rightarrow (x - 2)^2 = 0$

$\Rightarrow x - 2 = 0, x = 2$

Relationship between inverse functions

We can show that if g is the inverse function of $f(x)$, then $fg(x) = x$. Similarly, if f is the inverse of g, $gf(x) = x$.

EXAMPLE 34 Let $f(x) = 2x + 1$. Find $f^{-1}(x)$ and hence show that $ff^{-1}(x) = x$.

SOLUTION Let $y = 2x + 1$.

Interchange x and y:

$x = 2y + 1$

Make y the subject:

$y = \frac{x-1}{2}$

Therefore, $f^{-1}(x) = \frac{x-1}{2}$

$$\begin{aligned}\text{Now } ff^{-1}(x) &= f\left(\frac{x-1}{2}\right)\\ &= 2\left(\frac{x-1}{2}\right) + 1\\ &= x - 1 + 1\\ &= x\end{aligned}$$

Hence, $ff^{-1}(x) = x$.

EXAMPLE 35 Find the inverse of $f(x) = \frac{x+1}{x-2}$. Hence, show that $ff^{-1}(x) = x$, where $f^{-1}(x)$ is the inverse of x.

SOLUTION Let $y = \frac{x+1}{x-2}$.

Interchange x and y:

$x = \frac{y+1}{y-2}$.

Make y the subject of the formula:

$x(y - 2) = y + 1$

$xy - 2x = y + 1$

$xy - y = 2x + 1$

$y(x - 1) = 2x + 1$

Hence, $y = \frac{2x+1}{x-1}$.

Hence, $f^{-1}(x) = \frac{2x+1}{x-1}$.

$$\begin{aligned}\text{Now, } ff^{-1}(x) &= f\left(\frac{2x+1}{x-1}\right)\\ &= \frac{\frac{2x+1}{x-1} + 1}{\frac{2x+1}{x-1} - 2}\end{aligned}$$

Note

If $f^{-1}(x)$ is the inverse of any function $f(x)$, then $ff^{-1}(x) = x$ and $f^{-1}f(x) = x$.

$$= \frac{\frac{2x+1+x-1}{x-1}}{\frac{2x+1-2x+2}{x-1}} \quad \text{(Simplifying the numerator and denominator)}$$

$$= \frac{\frac{3x}{x-1}}{\frac{3}{x-1}}$$

$$= \frac{3x}{x-1} \times \frac{x-1}{3}$$

$$= \frac{3x}{3}$$

$$= x$$

EXERCISE 6B

In questions **1** to **4**, obtain expressions in the same form for gf and fg.

1 $f: x \rightarrow 4x - 2$ $\quad g: x \rightarrow 6x + 1$

2 $f: x \rightarrow 3x + 5$ $\quad g: x \rightarrow 2x^2 + x + 1$

3 $f: x \rightarrow x + 4$ $\quad g: x \rightarrow \frac{2}{x}, x \neq 0$

4 $f: x \rightarrow 1 + 5x$ $\quad g: x \rightarrow \frac{x+1}{x-1}, x \neq 1$

In questions **5** to **8**, obtain expressions in the same form for g^2 and g^3.

5 $g: x \rightarrow 2x - 1$

6 $g: x \rightarrow \frac{x}{2x+1}, x \neq -\frac{1}{2}$

7 $g: x \rightarrow \frac{2x+1}{x+1}, x \neq 1$

8 $g: x \rightarrow \frac{3}{4x-2}, x \neq \frac{1}{2}$

9 Functions g and h are defined by $g: x \rightarrow x^2 + 2x + 3$, $h: x \rightarrow x - 2$. Obtain expressions for gh and hg. Find the value of x satisfying the equation $gh = hg$.

10 Two functions are defined by $f: x \rightarrow 3x - 4$ and $g: x \rightarrow \frac{x}{x+2}, x \neq 2$. Find the value(s) of x for which $fg = gf$.

11 Find f^{-1} in similar form for each of the following functions.

(a) $f: x \rightarrow 4x - 3$

(b) $f: x \rightarrow \frac{5}{x-2}, x \neq 2$

(c) $f: x \rightarrow \frac{3x-1}{x+2}, x \neq -2$

12 Express $4x^2 + 12x + 3$ in the form $a(x + b)^2 + c$ where a, b and c are integers. The function f is defined by $f: x \rightarrow 4x^2 + 12x + 3$ for $x \in \mathbb{R}$.

(a) Find the range of f.

(b) Explain why f does not have an inverse.

13 A function f is defined by $f : x \to \frac{4x - 1}{3x + 2}, x \neq -\frac{2}{3}$.

(a) Find $f^{-1}(1)$ and $f^{-1}(-1)$.

(b) Show that there are no values of x for which $f^{-1}(x) = x$.

14 Functions f and g are defined by $f : x \to \frac{4x}{x + 1}, x \neq k$ and $g : x \to x + 4$.

(a) State the value of k.

(b) Express fg in similar form and state the value of x for which fg is not defined.

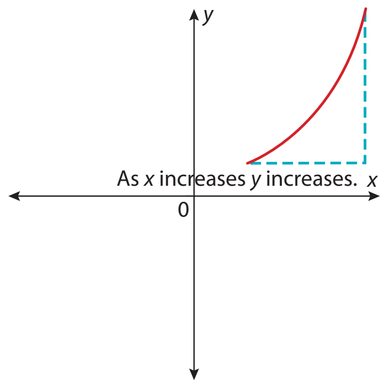
(c) Find $f^{-1}(x)$.

(d) Find the value of a for which $f^{-1}(a) = g(-1)$.

Increasing and decreasing functions

Increasing functions

A function is increasing if its graph moves upwards as x moves to the right.



Let $x = a$ and $x = b$ be two points on the graph where $b > a$. The function $f(x)$ is increasing in the interval [a, b] if and only if $\frac{f(b) - f(a)}{b - a} > 0$.

EXAMPLE 36 Show that $f(x) = 4x + 5$ is an increasing function.

SOLUTION Let $x = a$, then $f(a) = 4a + 5$

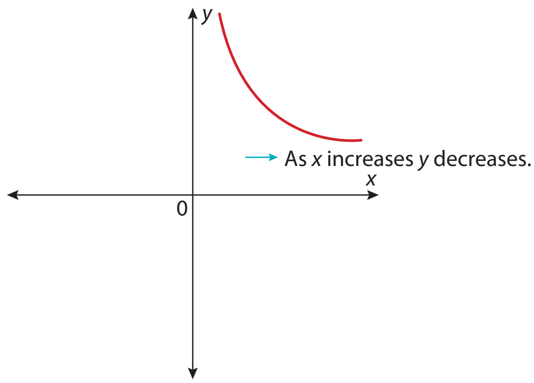
Let $x = b$, where $b > a$, then $f(b) = 4b + 5$.

Substituting these into $\frac{f(b) - f(a)}{b - a} = \frac{4b + 5 - (4a - 5)}{b - a} = \frac{4b - 4a}{b - a} = \frac{4(b - a)}{b - a} = 4$

Since $\frac{f(b) - f(a)}{b - a} > 0$, the function $f(x) = 4x + 5$ is an increasing function.

Decreasing functions

A function is decreasing if its graph moves downwards as x moves to the right.



Let $x = a$ and $x = b$ be two points on the graph where $b > a$. The function $f(x)$ is increasing in the interval $[a, b]$ if and only if $\frac{f(b) - f(a)}{b - a} < 0$.

EXAMPLE 37 Show that the function $f(x) = -2x + 3$ is a decreasing function.

SOLUTION We need to show that for $b > a$, $\frac{f(b) - f(a)}{b - a} < 0$.

Let $x = a$, then $f(a) = -2a + 3$,

Let $x = b$, where $b > a$, then $f(b) = -2b + 3$.

$$\frac{f(b) - f(a)}{b - a} = \frac{-2b + 3 - (-2a + 3)}{b - a} = \frac{2a - 2b}{b - a} = \frac{-2(b - a)}{b - a} = -2$$

Since $\frac{f(b) - f(a)}{b - a} < 0$, $f(x) = -2x + 3$, is a decreasing function.

Transformations of graphs

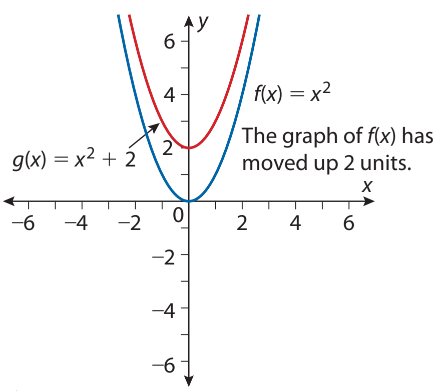
Vertical translation

EXAMPLE 38 Transform the graph of $f(x) = x^2$ to graph the function $g(x) = x^2 + 2$.

SOLUTION The graph of $g(x)$ can be obtained by adding 2 to the graph of $f(x)$. All points on the graph are shifted up by 2 units.

For $f(x) = x^2$:
when $x = 0$, $f(0) = 0^2 = 0$
when $x = 2$, $f(2) = 2^2 = 4$

For $g(x) = x^2 + 2$:
when $x = 0$, $g(0) = 0^2 + 2 = 2$
when $x = 2$, $g(2) = 2^2 + 2 = 6$



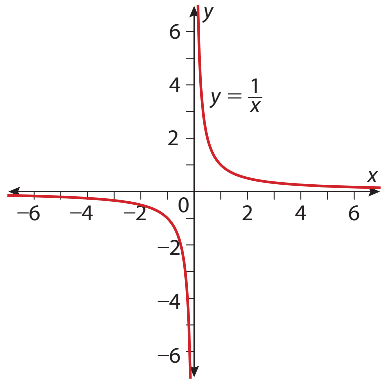
$\therefore$ the point (0, 0) on $f(x)$ maps onto (0, 2) on $g(x)$

The point (2, 4) on $f(x)$ maps onto (2, 6) on $g(x)$.

If g is a function and $h = g(x) + a$, then any point (p, q) on g maps onto $(p, q + a)$ onto h. h is the graph of $g(x)$ translated a units upwards (if $a > 0$) and a units downwards if $a < 0$.

EXAMPLE 39 Sketch the graph of $y = \frac{1}{x}$ and hence, sketch the graph of $y = \frac{1}{x} - 2$

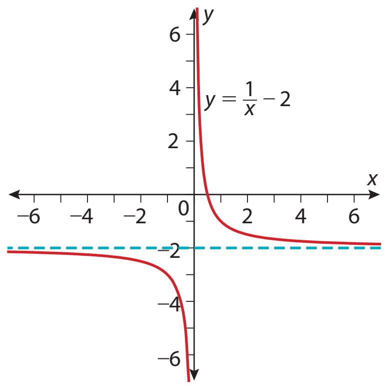
SOLUTION The graph of $\frac{1}{x}$ is as shown.



The graph of $\frac{1}{x} - 2$ is shifted downwards by 2 units.

Note

The original graph has an asymptote that is the x-axis ($y = 0$). The new graph has an asymptote that is the line $y = -2$.

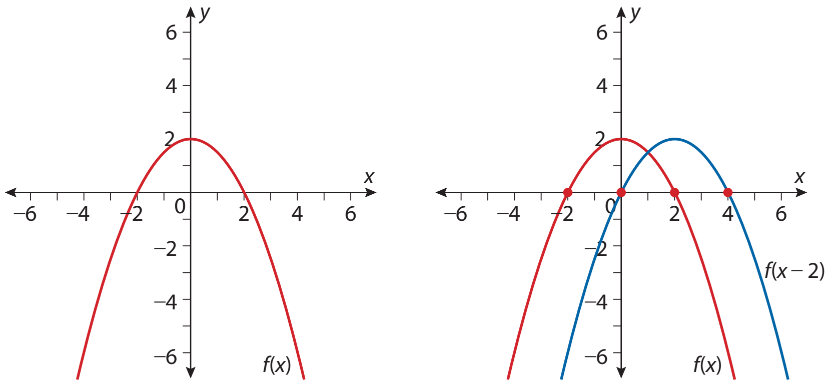


Horizontal translation

The graph of $f(x + a)$ can be obtained from the graph of $f(x)$ by shifting $f(x)$ to the left by a units if $a > 0$ and shifting $f(x)$ to the right by a units if $a < 0$.

When shifting to the right, a is added to all x-values. When shifting to the left, a is subtracted from all x-values.

EXAMPLE 40 The graph of $f(x)$ is shown below. Sketch the graph of $f(x-2)$, showing each point on the new graph.



SOLUTION $f(x-2)$ is the graph of $f(x)$ shifted to the right by 2 units.

Points on $f(x)$	Corresponding points on $g(x)$
$(-2, 0)$	$(0, 0)$
$(0, 2)$	$(2, 2)$
$(2, 0)$	$(4, 0)$
$(3, -3)$	$(5, -3)$

We add 2 units to each x-value and the y-values remain the same.

Horizontal stretch

If g is a function and $h(x) = g(ax)$ where $a \neq 0$, the graph of h is that of g stretched parallel to the x-axis by factor $\frac{1}{a}$. Any point (p, q) on $h(x)$ maps onto a point $\left(\frac{1}{a}p, q\right)$ on $g(x)$.

EXAMPLE 41 Sketch the graph of $f(x) = x^2 + 2x - 3$. Hence, sketch the graph of $f(2x)$.

SOLUTION $f(x) = x^2 + 2x - 3$

When $x = 0$, $f(0) = 0^2 + 2(0) - 3 = -3$

Since the coefficient of x^2 is positive, the graph has a minimum point.

When $x = -\dfrac{b}{2a} = -\dfrac{2}{2(1)} = -1$

When $x = -1$, $f(-1) = (-1)^2 + 2(-1) - 3 = -4$

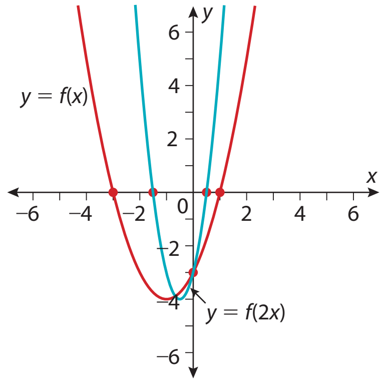
Minimum point is at $(-1, -4)$.

When $f(x) = 0$, $x^2 + 2x - 3 = 0$

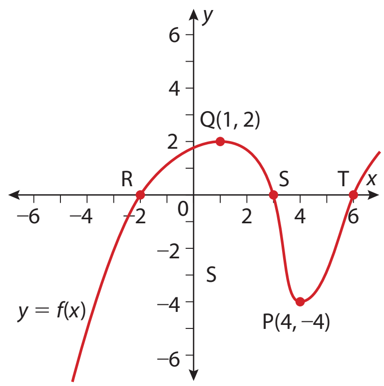
$\Rightarrow (x + 3)(x - 1) = 0$

$\Rightarrow x = -3, 1$

$f(x)$	$f(2x)$
$(-3, 0)$	$\left(-\frac{3}{2}, 0\right)$
$(1, 0)$	$\left(\frac{1}{2}, 0\right)$
$(0, -3)$	$(0, -3)$
$(-1, -4)$	$\left(-\frac{1}{2}, -4\right)$

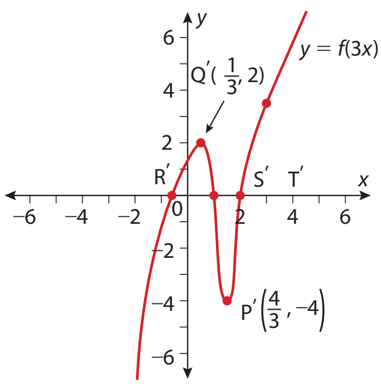


EXAMPLE 42 The curve shown in the diagram has equation $y = f(x)$. There is a minimum point at P(4, −4), a maximum point at Q(1, 2) and the curve cuts the x-axis at R(−2, 0), S(3, 0) and T(6, 0). Sketch the graph of $f(3x)$, showing the coordinates of the points corresponding to P, Q, R, S and T.



SOLUTION $f(3x)$ is a sketch along the x-axis by factor $\frac{1}{3}$. The points corresponding to P, Q, R, S and T are shown in the table. Remember that $(a, b) \rightarrow \left(\frac{1}{3}a, b\right)$.

$f(x)$	$f(3x)$
P(4, −4)	$\left(\frac{4}{3}, -4\right)$
Q(1, 2)	$\left(\frac{1}{3}, 2\right)$
R(−2, 0)	$\left(-\frac{2}{3}, 0\right)$
S(3, 0)	$(1, 0)$
T(6, 0)	$(2, 0)$



Each x-value is divided by 3 and the y-values remain the same.

Vertical stretch

If g is a function and $h(x) = ag(x)$, where $a \neq 0$, the graph of h is that of g stretched parallel to the y-axis by factor a. A point (p, q) on g maps onto (p, aq) on h.

EXAMPLE 43 Sketch the graph of $f(x) = x^2$ and hence, sketch $2f(x)$.

SOLUTION $f(x) = x^2$ is a quadratic curve with a minimum point at $(0, 0)$.

When $x = 1, f(1) = 1^2 = 1$

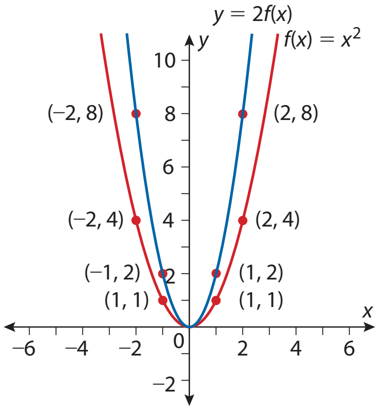
When $x = -1, f(-1) = (-1)^2 = 1$

When $x = 2, f(2) = 2^2 = 4$

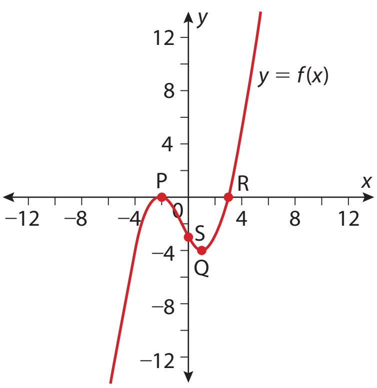
When $x = -2, f(-2) = (-2)^2 = 4$

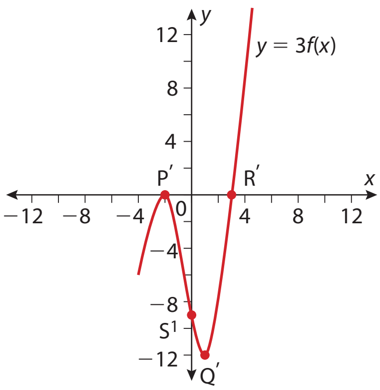
$\therefore$ $(0, 0), (-1, 1), (1, 1), (-2, 4)$ and $(2, 4)$ are on $f(x)$.

$f(x)$	$2f(x)$
(0, 0)	(0, 0)
(−1, 1)	(−1, 2)
(1, 1)	(1, 2)
(−2, 4)	(−2, 8)
(2, 4)	(2, 8)



EXAMPLE 44 The curve shown in the diagram has equation $y = f(x)$. There is a maximum point at $P(-2, 0)$, a minimum point at $Q(1, -4)$ and the curve cuts the x-axis at $R(3, 0)$ and the y-axis at $S(0, -3)$. Sketch the graph of $3f(x)$ showing the coordinates of the points corresponding to P, Q, R and S.





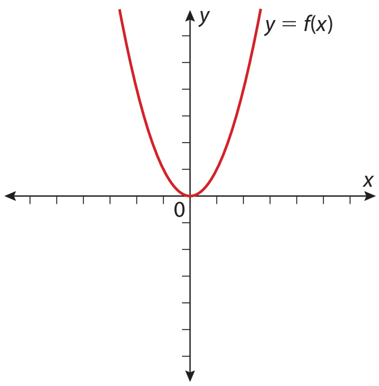
SOLUTION $3f(x)$ is a sketch of $f(x)$ parallel to the y-axis by factor 3. The y-coordinates of corresponding points on $f(x)$ are multiplied by 3.

$f(x)$	$3f(x)$
$(-2, 0)$	$(-2, 0)$
$(0, -3)$	$(0, -9)$
$(1, -4)$	$(1, -12)$
$(3, 0)$	$(3, 0)$

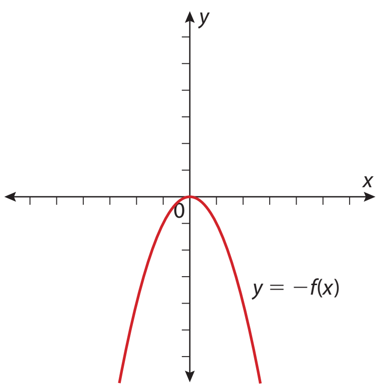
Reflection in the x-axis

The graph of $y = -f(x)$ can be obtained from the graph of $y = f(x)$ by reflecting the graph of $f(x)$ in the x-axis.

EXAMPLE 45 The curve shown in the diagram has equation $y = f(x)$. Sketch the graph of $g(x) = -f(x)$.



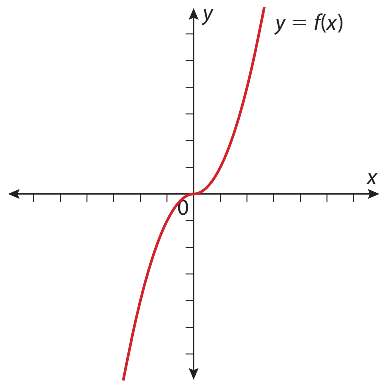
SOLUTION $g(x) = -f(x)$ can be obtained by reflecting $f(x)$ along the x-axis.



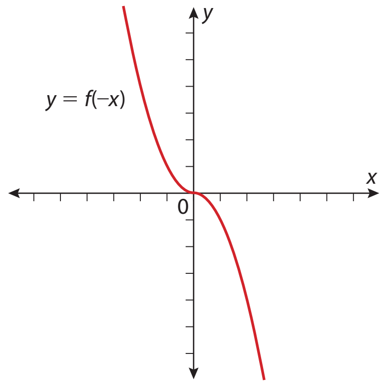
Reflection in the y-axis

The graph of $y = f(-x)$ can be obtained from the graph of $y = f(x)$ by reflecting the graph of $f(x)$ in the y-axis.

EXAMPLE 46 The curve shown in the diagram has equation $y = f(x)$. Sketch the graph of $y = f(-x)$.



SOLUTION Reflecting the graph along the y-axis we get this



We can also combine the transformations above to sketch graphs.

EXAMPLE 47 The equation of a curve is given by $g(x) = 2(x + 1)^2 - 3$. Starting with the graph of $f(x) = x^2$, describe clearly the transformations that will give the graph of $g(x)$. Hence, sketch the graph of $g(x)$, showing all the movements of the graph of $f(x) = x^2$.

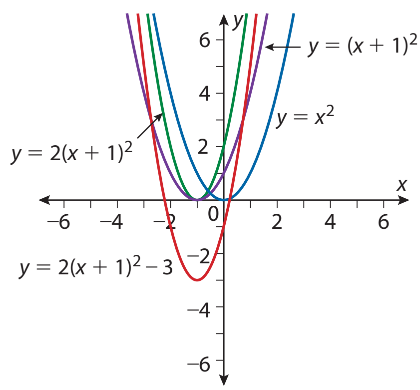
SOLUTION

$f(x) = x^2$

$f(x + 1) = (x + 1)^2$ This shifts the graph of $f(x)$ by 1 unit to the left.

$2f(x + 1) = 2(x + 1)^2$ This stretches the graph of $f(x + 1)$ along the y-axis by factor 2.

$g(x) = 2f(x + 1) - 3$
$\quad\;\; = 2(x + 1)^2 - 3$ This moves the graph of $2f(x + 1)$ downwards by 3 units.



Graphs of simple rational functions

EXAMPLE 48 Sketch the graph of $f(x) = \frac{2x+3}{x-1}$.

SOLUTION Since this function is a linear function divided by a linear function, we can write $f(x)$ as a mixed fraction first.

$$\begin{array}{r|l} & \quad 2 \\ x-1 & 2x+3 \\ & -(2x-2) \\ & \quad 5 \end{array}$$

Therefore, $\frac{2x+3}{x-1} = 2 + \frac{5}{x-1}$

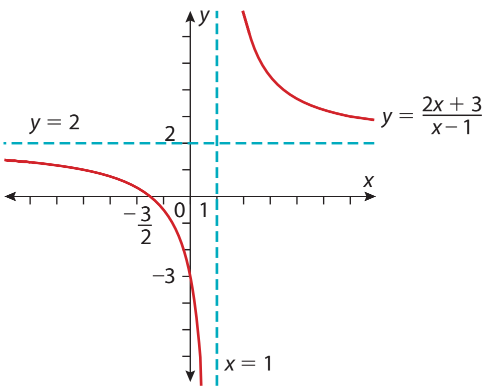
Let $g(x) = \frac{1}{x}$

$g(x-1) = \frac{1}{x-1}$

$5g(x-1) = \frac{5}{x-1}$

$2 + 5g(x-1) = 2 + \frac{5}{x-1}$, which is $f(x)$.

The graph of $f(x)$ can be obtained from the graph of $g(x)$ by shifting $g(x)$ by 1 unit to the right, then stretching along the y-axis by scale factor 5, and finally by moving this graph upwards by 2 units.



EXAMPLE 49 Use the graph of $f(x) = \frac{1}{x^2}$ to sketch the graph of $g(x) = \frac{2}{x^2+2x+1}$.

SOLUTION We need to look at the relationship between $f(x)$ and $g(x)$.

Now, $x^2 + 2x + 1 = (x+1)^2$

Therefore, $g(x) = \frac{1}{(x+1)^2}$

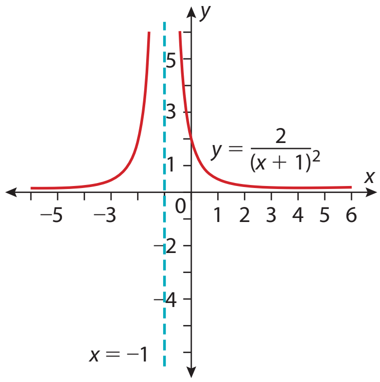
Since, $f(x) = \frac{1}{x^2}$, then $f(x+1) = \frac{1}{(x+1)^2}$

$2f(x+1) = \frac{2}{(x+1)^2}$

Therefore, $g(x) = 2f(x+1)$

> **Note**
>
> You will be doing more work on curve sketching in the section on applications of differentiation.

The graph of $g(x)$ can be obtained from $f(x)$ by moving $f(x)$ to the left by 1 unit and stretching the new graph by scale factor 2 along the y-axis.



EXAMPLE 50 Given that $g(x) = \dfrac{x+1}{x+2}$. Describe the transformation that moves the graph of $f(x) = \dfrac{1}{x}$ onto the graph of $g(x)$. Hence, sketch $g(x)$.

SOLUTION

$\dfrac{x+1}{x+2}$.

By long division

$$\begin{array}{r}
1 \\
x+2\,\overline{)\,x+1} \\
-\;\underline{x+2} \\
-\,1
\end{array}$$

$$\frac{x+1}{x+2} = 1 - \frac{1}{x+2}$$

Starting with $f(x) = \dfrac{1}{x}$:

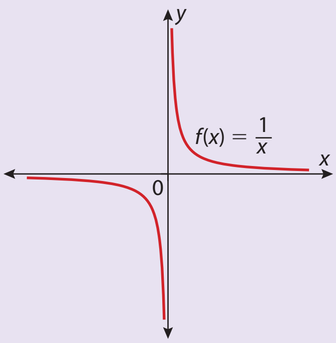
$$f(x+2) = \frac{1}{x+2}$$

$f(x+2)$ This is a shift of $f(x)$ to the left by 1 unit.

$$-f(x+2) = -\frac{1}{x+2}$$

$-f(x+2)$ This is a reflection of $f(x+2)$ in the x-axis.

$g(x) = 1 - f(x+2) = 1 - \dfrac{1}{x+2}$ This is a translation of 1 unit upwards of the graph of $-f(x+2)$.

> **Note**
>
> The graph of $f(x) = \frac{1}{x}$ is
>
> 
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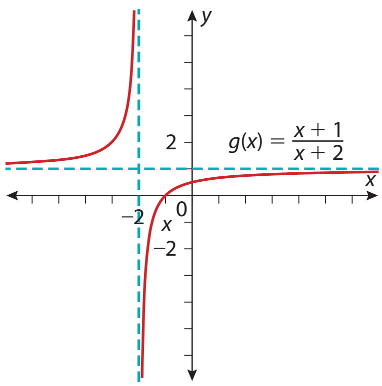


Any graph of the form $y = \frac{ax + b}{cx + d}$ can be obtained from the graph of $y = \frac{1}{x}$.
We first use division to write the fraction as a mixed fraction. Then we use the transformations described previously.

Try these 6.3

Describing all transformations clearly, sketch the graphs of:

(a) $y = \frac{x + 1}{x - 2}$

(b) $y = \frac{2x + 1}{x + 1}$, starting with the graph of $y = \frac{1}{x}$.

Piecewise defined functions

When functions are defined differently on different parts of its domain, they are called piecewise functions. Recall that the modulus function is defined by three different equations as follows:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

When evaluating $|2|$ we use the part of the function for $x > 0$, that is $|2| = 2$. When evaluating $|-2|$, we use the function defined for $x > 0$, that is $-(-2) = 2$.

EXAMPLE 51

The function f is defined by

$$f(x) = \begin{cases} -x + 2 & \text{if } -2 \leq x < 2 \\ 4 & \text{if } x = 2 \\ \frac{1}{2}x^2 & \text{if } x > 2 \end{cases}$$

(a) Find $f(1)$, $f(2)$ and $f(3)$.

(b) Determine the domain of $f(x)$.

(c) Sketch the graph of $f(x)$.

SOLUTION

(a) The equation for $f(x)$ when $x = 1$ is $x + 2$.

Therefore, $f(1) = -1 + 2 = 1$

When $x = 2$, the equation is 4.

Therefore, $f(2) = 4$

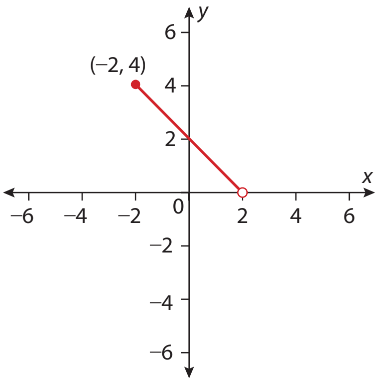
When $x = 3$, the equation for $f(x)$ is $\frac{1}{2}x^2$.

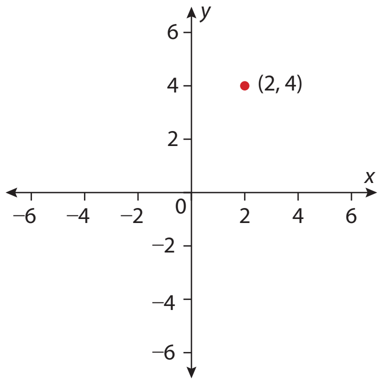
Therefore, $f(3) = \frac{1}{2}(3)^2 = \frac{9}{2}$

(b) From the definition of $f(x)$, the domain is $x \geq -2$.

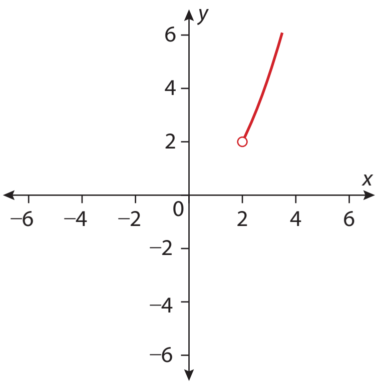
(c) When sketching $f(x)$, we draw the graph piece by piece.

When $-2 \le x < 2$, $f(x) = -x + 2$ When $x = 2$, $f(x) = 4$

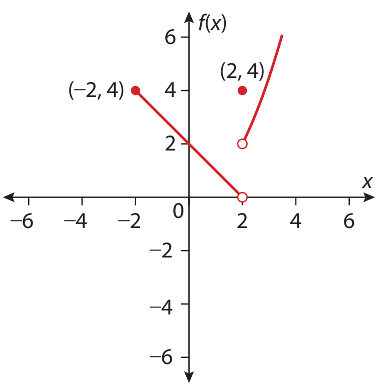




When $x > 2$, $f(x) = \dfrac{x^2}{2}$



We put all of the above on one set of axes. This is graph of $f(x)$:



EXAMPLE 52 If $f(x) = \begin{cases} 2x + 2 & \text{if } -1 \le x \le 3 \\ x^2 - x + 1 & \text{if } 3 < x \le 5 \end{cases}$ find

(a) $f(0)$

(b) $f(3)$

(c) $ff(1)$

SOLUTION

(a) Since $f(x) = 2x + 2$, when $-1 \le x \le 3$

$f(0) = 2(0) + 2 = 2$

(b) Since $f(x) = 2x + 2$, when $-1 \leq x \leq 3$

$f(3) = 2(3) + 2 = 8$

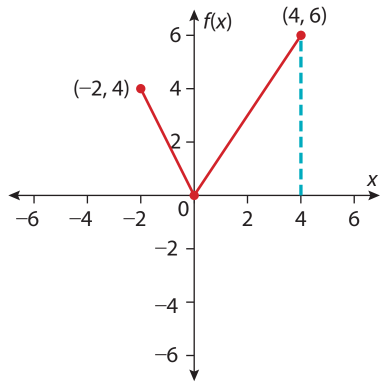
(c) Since $f(x) = 2x + 2$, when $-1 \leq x \leq 3$

$f(1) = 2(1) + 2 = 4$

Since $f(x) = x^2 - x + 1$, when $3 < x \leq 5$

$f(4) = (4)^2 - 4 + 1 = 13$

EXAMPLE 53 The graph of a piecewise function is given below. Write a definition of a function.



SOLUTION For $-2 \leq x \leq 0$, the gradient of the line $= \dfrac{4-0}{-2-0} = -2$.

Since the line passes through (0, 0), the equation of the line is $y = -2x$.

For $0 \leq x \leq 6$, the gradient of the line $= \dfrac{6-0}{4-0} = \dfrac{3}{2}$.

Since the line passes through (0, 0), the equation of the line is $y = \dfrac{3}{2}x$.

Hence, $f(x) = \begin{cases} -2x & \text{if } -2 \leq x \leq 0 \\ \dfrac{3}{2}x & \text{if } 0 \leq x \leq 6 \end{cases}$

EXERCISE 6C

1 On a single clear diagram, sketch the graphs of these.

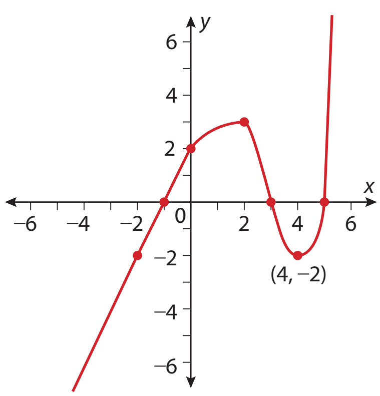
(a) $y = x^2$

(b) $y = (x + 2)^2$

(c) $y = 2(x + 2)^2$

(d) $y = 2(x + 2)^2 + 3$

2 The diagram shows the graph of $y = f(x)$. On separate diagrams, sketch the graphs of $y = 2f(x)$ and $y = f(x) - 4$. Your sketch should show clearly the coordinates of the intersections with the axes.



3 On a single clear diagram, sketch the graphs of these.

(a) $y = |x|$

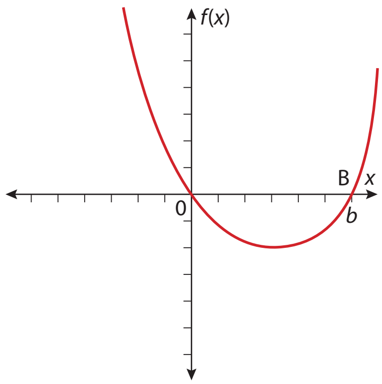
(b) $y = |x + 2|$

(c) $y = |x + 2| + 2$

4 Given that $y = 2x^2 - 4x + 6$. Express y in the form $a(x + b)^2 + c$. Describe a series of transformations which map the graph of $y = x^2$ onto the graph of $y = 2x^2 + 8x + 5$. Draw each graph on the same diagram.

5 Show that $y = \frac{x+1}{x+2}$ can be written in the form $A + \frac{B}{x+2}$ where A and B are constants to be found. Describe the series of transformation which map the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{x+1}{x+2}$. Hence, sketch the curve $y = \frac{x+1}{x+2}$.

6 The diagram shows the graph of $y = f(x)$. The curve passes through the origin and the point B with coordinates $(b, 0)$, where $b > 0$. Sketch, on separate diagrams, the graph of these.



(a) $y = f(x + b)$

(b) $y = |f(x)|$

7 Sketch the graph of $f(x) = \frac{1}{x^2}$. Hence, sketch these.

(a) $f(x - 2)$

(b) $f(x) - 2$

(c) $2f(x) - 3$

8 On a single diagram, sketch the graphs of these.

(a) $y = |x - 1|$

(b) $y = 2|x - 1|$

(c) $y = 2|x - 1| + 3$

9 The equation of a curve C is given by $y = \frac{3x+1}{x+2}$. Write the equation in the form $y = A + \frac{B}{x+2}$. Describe the series of transformations which map the graph of $y = \frac{3x+1}{x+2}$ onto the graph $y = \frac{1}{x}$.

10 Sketch the graph of $y = \frac{2x^2 - 8x + 9}{x^2 - 4x + 4}$. (Hint: Look at the relationship with $y = \frac{1}{x^2}$.)

11 Sketch the following graphs.

(a) $y = \frac{4x - 2}{x - 4}$

(b) $y = \frac{4x^2 + 4x + 2}{4x^2 + 4x + 1}$

SUMMARY

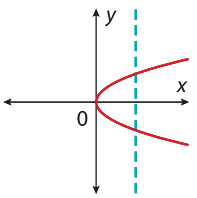
Functions

A function f from a set A to a set B assigns to each $a \in A$ a single element $f(a)$ in set B. The element in set B is called the image of a under f. The set A is called the **domain** of the function, the set B is called the **codomain** of the function. The **range** of the function is the set of elements that are the **images of $a \in A$**.

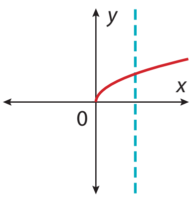
A function can be described:

(i) using an arrow diagram
(ii) as a set of ordered pairs
(iii) using a graph
(iv) by a formula
(v) by listing values.

Vertical line test: If a line drawn parallel to the y-axis cuts the graph at most once, the relation is a function.



Not a function



Function

One-to-one (injective): A function $f: X \to Y$ is injective iff every element of Y is mapped onto by one and only one element of X. **(No two x-values can have the same y-image).**

To show that $f(x)$ is one-to-one:
***Method* 1**: If $f(a) = f(b) \Rightarrow a = b$, then $f(x)$ is one-to-one.

***Method* 2**: A line drawn parallel to the x-axis on the graph of $f(x)$ must cut $f(x)$ at most once.

Onto (surjective): A function $f: X \to Y$ is surjective iff every y is mapped onto by at least one x.

For a surjective function the codomain and the range must be the same.

To show that $f(x)$ is onto: a line drawn parallel to the x-axis must cut $f(x)$ at least once if $f(x)$ is onto.

Bijective: A function $f: X \to Y$ is bijective iff it is one-to-one and onto.

Inverse functions: The inverse of $f(x)$ is denoted by $f^{-1}(x)$. $f^{-1}(x)$ exists iff $f(x)$ is one-to-one.

Domain of $f(x) \equiv$ range of $f^{-1}(x)$

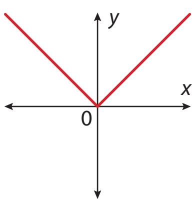
Range of $f(x) \equiv$ domain of $f^{-1}(x)$

To sketch $f^{-1}(x)$: reflect $f(x)$ in the line $y = x$.

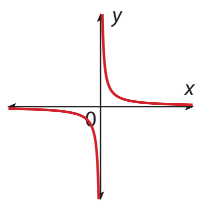
$$ff^{-1}(x) = f^{-1}f(x) = x$$

For $gf(x)$, the range of f is the domain of g.

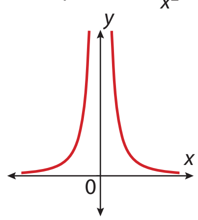
Graph of $y = |x|$



Graph of $y = \frac{1}{x}$



Graph of $y = \frac{1}{x^2}$



If $\dfrac{f(b) - f(a)}{b - a} > 0$ for $b > a$, then $f(x)$ is increasing in $[a, b]$.

If $\dfrac{f(b) - f(a)}{b - a} < 0$ for $b > a$, then $f(x)$ is decreasing in $[a, b]$.

Sketching graphs from $y = f(x)$:

(i) $f(ax)$: stretch along the x-axis by factor $\frac{1}{a}$.

(ii) $af(x)$: stretch along the y-axis by factor a.

(iii) $f(x + a)$: shift to the left by a units if a is positive and a shift to the right by a units if a is negative.

(iv) $f(x) + a$: shift upwards by a units if a is positive and a shift downwards by a units if a is negative.

(v) $f(-x)$: reflect $f(x)$ in the y-axis.

(vi) $-f(x)$: reflect $f(x)$ in the x-axis.

(vii) $|f(x)|$: reflect anything below the x-axis to above the x-axis, the section of the curve above the x-axis remains as it is.

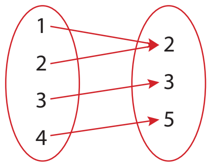
Checklist

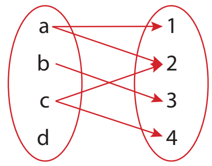
Can you do these?

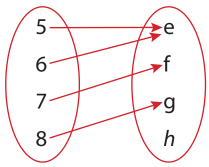
- Decide whether a relation is a function.
- Identify the domain of a function.
- Identify the range of a function.
- Show that a function is one-to-one (injective).
- Show that a function is onto (surjective).
- Show that a function is bijective (both one-to-one and onto).
- Find the inverse of a function.
- Understand the relationship between a function and its inverse.
- Find a composite function.
- Define functions as a set of ordered pairs.
- Define functions as a formula.
- Plot and sketch functions and their inverses (if they exist).
- State the geometrical relationship between a function and its inverse.
- Perform calculations using functions.
- Identify increasing and decreasing functions, using the sign of $\dfrac{f(a) - f(b)}{a - b}$ where $a \neq b$.
- Understand the relationship between the graph of $y = f(x)$ and $y = af(x)$ and sketch these graphs.
- Understand the relationship between the graph of $y = f(x)$ and $y = f(x)$.
- Understand the relationship between the graph of $y = f(x)$ and $y = f(x + a)$.
- Understand the relationship between the graph of $y = f(x)$ and $y = f(ax)$.
- Understand the relationship between the graph of $y = f(x)$ and $y = af(x + b)$.
- Understand the relationship between the graph of $y = f(x)$ and $y = |f(x)|$.
- Graph a rational function.
- Identify a piecewise defined function and its values.

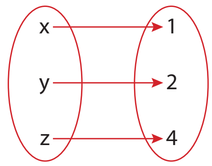
Review Exercise 6

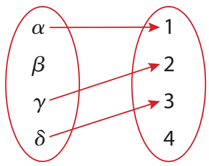
1 For each of the following relations, state which relation defines a function. If the relation is not a function, give a reason.

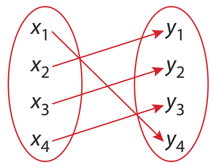
(a) 

(b) 

(c) 

(d) 

(e) 

(f) 

2 Which of the following relations are functions? If any of the relations is not a function, give a reason.

(a) $\{(1, 5), (2, 10), (3, 15), (4, 20)\}$ (b) $\{(2, 2), (4, 2), (-2, 6), (3, 8)\}$

(c) $\{(4, 5), (5, 6), (6, 7), (6, 8)\}$ (d) $\{(a, b), (c, d), (c, e), (f, g)\}$

3 A function f is defined by $f: x \to 4x - 3$. What are the images of 4, -3, $\frac{1}{2}$ and $\frac{-1}{8}$ under f?

4 Given the function $f: x \to 4 - \frac{3}{4}x$, evaluate the following:

(a) $f(1)$ (b) $f(2)$

(c) $f(0)$ (d) $f(4)$

5 Given the functions $f: x \to 3x - 7, x \in \mathbb{R}$ and $g: x \to 4x + 2, x \in \mathbb{R}$. Find the following.

(a) $f(0) + g(2)$ (b) $2f(3) + g(1)$

(c) $2f(1) - 3g(2)$ (d) $4f(-1) + 3g(-2)$

6 Given the function $f(x) = \frac{1}{2}x + \frac{3}{4}$ and $g(x) = \frac{5}{6}x + \frac{2}{3}$. Find the value of x for which

(a) $f(x) = g(x)$ (b) $f(x) = \frac{1}{4}x$

(c) $f(2x) = 3g(x)$ (d) $f\left(\frac{1}{2}x\right) = g\left(\frac{1}{4}x\right)$.

7 Sketch the graphs of the following functions on separate diagrams and state the range in each case.

(a) $f: x \to 4x + 1, -2 \le x \le 2$ (b) $f: x \to 2 - 3x, -1 \le x \le 1$

(c) $f: x \to 2 + x, x \ge 1$ (d) $f: x \to -3x + 6, x \le 0$

8 State the domain and range of the following functions.

(a) $f(x) = x^2 - 4$

(b) $f(x) = x^2 + 2x + 3$

(c) $f(x) = -4x^2 + x + 1$

(d) $f(x) = -x^2 - 3x + 5$

9 (a) State the minimum value of $f(x) = 3(x - 2)^2 + 1$ and the corresponding value of x.

(b) Sketch the graph of the function $f: x \to 3(x - 2)^2 + 1$ for the domain $-3 \leq x \leq 3$ and write down the range of the function for the corresponding domain.

10 Determine the domain and range of the following functions.

(a) $y = 1 - \frac{2}{x}$

(b) $f: x \to \dfrac{4x + 2}{x - 3}$

(c) $f: x \to \sqrt{x - 4}$

11 If $f: x \to 6x + 2$ and $g: x \to 7x - 1$, find the composite functions fg and gf. What are the values of $fg(0)$, $fg(-2)$, $gf(0)$, $gf(-2)$?

12 If $g: x \to x + 3$, find the function h such that $gh: x \to x^2 + 3x - 2$.

13 If $f: x \to 4x - 2$, find the function g such that $fg: x \to \dfrac{x + 1}{x + 2}$.

14 The functions f and g are such that $f: x \to \dfrac{3}{x + 1}$ (for $x \neq -1$) and $g: x \to 4x - 2$.

(a) Find these in similar form.

(i) fg (ii) gf

(b) Show that there are two real distinct solution to the equation $fg = gf$.

15 Functions f and g are defined on the set of real numbers by $f: x \to \dfrac{6}{x - 3}$, $x \neq k$, and $g: x \to 5x - 3$.

(a) State the value of k.

(b) Express these in a similar form.

(i) gf (ii) $f^{-1}(x)$

(c) Evaluate $fg^{-1}(4)$.

16 A function is defined by $f: x \to \dfrac{4x + 1}{3x - 2}$ for any $x \in \mathbb{R}$, $x \neq \frac{2}{3}$. Express f^{-1} in a similar form. Find the value of

(a) $f^{-1}(2)$ (b) $ff\left(\frac{1}{2}\right)$

(c) $ff^{-1}(4)$.

17 Functions f and g are defined by $f: x \to \dfrac{2}{x - 1}$, $x \neq 1$, $g: x \to \lambda x^2 - 1$, where λ is a constant.

(a) Given that $gf(3) = \frac{1}{5}$, evaluate λ.

(b) Express $f^2(x)$ in the form $\dfrac{ax + b}{cx + d}$, stating the values of a, b, c and d.

18 Functions f and g are defined as:

$f: x \rightarrow 3x - 2, x \in \mathbb{R}$

$g: x \rightarrow \frac{2}{x-2}, x \in \mathbb{R}, x \neq 2$

(a) Find f^{-1} and g^{-1}, stating the value of x for which g^{-1} is undefined.

(b) Find the values of x for which $fg(x) = x$.

(c) Sketch the graphs of f and f^{-1} on the same diagram, showing clearing the relationship between the two functions.

19 Functions f and g are defined by $f: x \rightarrow \frac{3}{x+3}, x \neq k$ and $g: x \rightarrow 2x + 1$.

(a) State the value of k.

(b) Express fg in similar form, and state the value of x for which fg is not defined.

(c) Find $f^{-1}(x)$.

(d) Find the value of a for which $f^{-1}(a) = g(4)$.

20 Function f and g are defined by $f: x \rightarrow \frac{4x}{x-1}, x \neq 1$ and $g: x \rightarrow \frac{x+\lambda}{x}, x \neq 0$.

(a) Find f^{-1} in a similar form.

(b) Given that $gf^{-1}(5) = 5$, calculate the value of λ.

21 (a) Express $3x^2 + 12x + 5$ in the form $a(x+b)^2 + c$ where a, b and c are integers. The function f is defined by $f: x \rightarrow 3x^2 + 12x + 5$ for $x \in \mathbb{R}$.

(i) Find the range of f.

(ii) Explain why f does not have an inverse.

(b) The function g is defined by $g: x \rightarrow 3x^2 + 12x + 5$ for $x \geq k$.

(i) Find the smallest value of k for which g has an inverse.

(ii) For this value of k, find an expression for g^{-1}.

22 If $f(x) = \begin{cases} 4x - 1, & -3x \leq x < 2 \\ 2x + 5, & 2 \leq x < 4 \\ x^2 + 3, & x > 4 \end{cases}$, find the following.

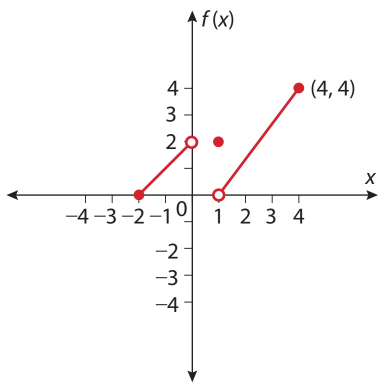
(a) $f(1)$

(b) $f(3.5)$

(c) $f(5)$

(d) $ff\left(\frac{1}{2}\right)$

23 The graph of a piecewise function is given below. Write a definition for the function.



24 Sketch the graph of the following function.

$f(x) = \begin{cases} 1 + x, & \text{if } x > 0 \\ x^2 + 2, & \text{if } x \geq 0 \end{cases}$

CHAPTER 7

Cubic Polynomials

At the end of this chapter you should be able to:

- Connect the roots of a cubic equation and the coefficients of the terms in the equations
- Find a cubic equation, given the roots of the equation
- Find $\alpha^2 + \beta^2 + \gamma^2$, $\alpha^3 + \beta^3 + \gamma^3$ etc. where α, β and γ are the roots of the equation
- Solve application problems using inequalities

KEYWORDS/TERMS

roots • cubic equation • coefficients

Review: Roots of a quadratic and the coefficient of the quadratic

(i) Let α, β be the roots of $ax^2 + bx + c = 0$.

$\alpha + \beta = \frac{-b}{a}$ and $\alpha\beta = \frac{c}{a}$

(ii) The quadratic equation is:

$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

(iii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

EXAMPLE 1 Given that α and β are the roots of $3x^2 - 6x + 12 = 0$, find a quadratic equation with roots

(a) $\frac{1}{\alpha}, \frac{1}{\beta}$

(b) α^2, β^2

SOLUTION

(a) $3x^2 - 6x + 12 = 0$

$$\alpha + \beta = \frac{-b}{a} = \frac{-(-6)}{3} = 2$$

$$\alpha\beta = \frac{c}{a} = \frac{12}{3} = 4$$

$$\text{Sum of the roots} = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$\text{Product of the roots} = \left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) = \frac{1}{\alpha\beta}$$

$$= \frac{1}{4}$$

This equation is:

$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

$x^2 - \frac{1}{2}x + \frac{1}{4} = 0$

$4x^2 - 2x + 1 = 0$

(b) $$\text{Sum of the roots} = \alpha^2 + \beta^2$$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (2)^2 - 2(4)$$

$$= 4 - 8 = -4$$

$$\text{Product of the roots} = \alpha^2\beta^2$$

$$= (\alpha\beta)^2$$

$$= (4)^2$$

$$= 16$$

The equation is:

$x^2 - (\text{sum of the roots})x + (\text{product of the roots}) = 0$

$x^2 - (-4)x + 16 = 0$

$x^2 + 4x + 16 = 0$

Cubic equations

A cubic equation with real coefficients has three roots that can be classified as:

(i) all three real or

(ii) 1 real and 2 not real.

Let α, β and γ be the three roots of the equation $ax^3 + bx^2 + cx + d = 0$.

$\Rightarrow x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$

Then $x = \alpha$, $x = \beta$ and $x = \gamma$

$\Rightarrow (x - \alpha)(x - \beta)(x - \gamma) = 0$

$\Rightarrow (x^2 - \alpha x - \beta x + \alpha\beta)(x - \gamma) = 0$

$\Rightarrow (x^3 - \gamma x^2 - \alpha x^2 + \alpha\gamma x - \beta x^2 + \beta\gamma x + \alpha\beta x - \alpha\beta\gamma = 0$

Combining terms gives:

$x^3 - \gamma x^2 - \alpha x^2 - \beta x^2 + \alpha\gamma x + \beta\gamma x + \alpha\beta x - \alpha\beta\gamma = 0$

$x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \alpha\gamma)x - \alpha\beta\gamma = 0$

Comparing coefficients with $x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ we have:

$-(\alpha + \beta + \gamma) = \frac{b}{a} \Rightarrow \alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$-(\alpha\beta\gamma) = \frac{d}{a} \Rightarrow \alpha\beta\gamma = \frac{-d}{a}$

Given α, β, γ are the roots of a cubic equation we can obtain the equation using:

$x^3 - (\text{sum of the roots})x^2 + (\text{sum of the product of two of the roots at a time})x - (\text{product of the roots}) = 0$

Given a cubic equation $ax^3 + bx^2 + cx + d = 0$ with roots α, β and γ:
(i) sum of the roots $= \alpha + \beta + \gamma = \frac{-b}{a}$
(ii) sum of the product of two of the roots at a time $= \alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
(iii) product of the roots $= \alpha\beta\gamma = \frac{-d}{a}$

EXAMPLE 2 Given that $4x^3 - 3x^2 + 2x + 1 = 0$ has roots α, β and γ, find

(a) $\alpha + \beta + \gamma$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha$

(c) $\alpha\beta\gamma$

SOLUTION $4x^3 - 3x^2 + 2x + 1 = 0$

Since the roots are α, β and γ, we compare with $ax^3 + bx^2 + cx + d = 0$ where $a = 4$, $b = -3$, $c = 2$ and $d = 1$.

(a) $\alpha + \beta + \gamma = \frac{-b}{a} = -\frac{(-3)}{4} = \frac{3}{4}$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{2}{4} = \frac{1}{2}$

(c) $\alpha\beta\gamma = \frac{d}{a} = \frac{-(1)}{4} = \frac{-1}{4}$

EXAMPLE 3 Let α, β and γ be the roots of the equation $7x^3 + 2x^2 - 14x + 4 = 0$.

Find

(a) $\alpha + \beta + \gamma$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha$

(c) $\alpha\beta\gamma$.

SOLUTION $7x^3 + 2x^2 - 14x + 4 = 0$

$a = 7, b = 2, c = -14, d = 4$

(a) $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{-2}{7}$

(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{-14}{7} = -2$

(c) $\alpha\beta\gamma = \frac{-d}{a} = \frac{-4}{7}$

EXAMPLE 4 α, β and γ are the roots of the equation $3x^2 = 4x^3 - 1$. Find

(a) $\alpha + \beta + \gamma$

(b) $\alpha\beta\gamma$.

SOLUTION Writing the equation in the form $ax^3 + bx^2 + cx + d = 0$ we have $4x^3 - 3x^2 - 1 = 0$, where $a = 4$, $b - 3$, $c = 0$ and $d = -1$.

$\therefore$ *(a)* $\alpha + \beta + \gamma = \frac{-b}{a} = \frac{-(-3)}{4} = \frac{3}{4}$

(b) $\alpha\beta\gamma = \frac{-d}{a} = \frac{-(-1)}{4} = \frac{1}{4}$

EXAMPLE 5 Given that $7x^3 - 4x^2 + 2 = 0$ has roots α, β and γ. Find

(a) $\alpha^2 + \beta^2 + \gamma^2$

(b) $\alpha^3 + \beta^3 + \gamma^3$

(c) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

SOLUTION Since the roots of $7x^3 - 4x^2 + 2 = 0$ are α, β and γ

$\Rightarrow \alpha + \beta + \gamma = \frac{-(-4)}{7} = \frac{4}{7}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{0}{7} = 0$

$\alpha\beta\gamma = \frac{-2}{7}$

(a) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= \left(\frac{4}{7}\right)^2 - 2(0)$$

$$= \frac{16}{49}$$

(b) Since $x = \alpha$, $7\alpha^3 - 4\alpha^2 + 2 = 0$ [1]

$x = \beta$, $7\beta^3 - 4\beta^2 + 2 = 0$ [2]

$x = \gamma$, $7\gamma^3 - 4\gamma^2 + 2 = 0$ [3]

$7(\alpha^3 + \beta^3 + \gamma^3) - 4(\alpha^2 + \beta^2 + \gamma^2) + 6 = 0$ [1] + [2] + [3]

Substituting $\alpha^2 + \beta^2 + \gamma^2 = \frac{16}{49}$

$\Rightarrow 7(\alpha^3 + \beta^3 + \gamma^3) - 4\left(\frac{16}{49}\right) + 6 = 0$

$\Rightarrow 7(\alpha^3 + \beta^3 + \gamma^3) = \frac{64}{49} - 6$

$= \frac{-230}{49}$

$\therefore\ \alpha^3 + \beta^3 + \gamma^3 = \frac{-230}{49 \times 7} = \frac{-230}{343}$

(c) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma}$ $\left(\text{Use } \alpha\beta + \beta\gamma + \alpha\gamma = 0 \text{ and } \alpha\beta\gamma = \frac{-2}{7}\right)$

$= \frac{0}{\frac{-2}{7}} = 0$

Notation

We can represent the relationships between the roots like this.

$\alpha + \beta + \gamma = \sum \alpha$

$\alpha\beta + \beta\gamma + \alpha\gamma = \sum \alpha\beta$

$\alpha^2 + \beta^2 + \gamma^2 = \sum \alpha^2$

$\alpha^3 + \beta^3 + \gamma^3 = \sum \alpha^3$

This leads to:

$$\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum(\alpha\beta)$$

Using this notation, if $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the roots of an equation, then:

$$\sum \frac{1}{\alpha} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$$

$$\sum \frac{1}{\alpha^2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$$

Finding $\alpha^3 + \beta^3 + \gamma^3$, using a formula

Let $\alpha + \beta + \gamma$ be the roots of a polynomial.

$(\alpha + \beta + \gamma)^3 = (\alpha + \beta + \gamma)(\alpha + \beta + \gamma)^2$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\alpha\gamma)$

$= \alpha^3 + \alpha\beta^2 + \alpha\gamma^2 + 2\alpha^2\beta + 2\alpha\beta\gamma + 2\alpha^2\gamma$

$+ \alpha^2\beta + \beta^3 + \beta\gamma^2 + 2\alpha\beta^2 + 2\beta^2\gamma + 2\alpha\beta\gamma$

$+ \alpha^2\gamma + \beta^2\gamma + \gamma^3 + 2\alpha\beta\gamma + 2\beta\gamma^2 + 2\alpha\gamma^2$

$$= \alpha^3 + \beta^3 + \gamma^3 + 3\alpha\beta^2 + 3\alpha\gamma^2 + 3\beta\gamma^2 + 3\alpha^2\beta + 3\alpha^2\gamma + 3\beta^2\gamma + 6\alpha\beta\gamma$$

$$= (\alpha^3 + \beta^3 + \gamma^3) + 3\beta\gamma(\beta + \gamma) + 3\alpha\beta(\alpha + \beta) + 3\alpha\gamma(\alpha + \gamma) + 6\alpha\beta\gamma$$

$$= (\alpha^3 + \beta^3 + \gamma^3) + 3\beta\gamma(\alpha + \beta + \gamma) - 3\alpha\beta\gamma + 3\alpha\beta(\alpha + \beta + \gamma) - 3\alpha\beta\gamma + 3\alpha\gamma(\alpha + \beta + \gamma) - 3\alpha\beta\gamma + 6\alpha\beta\gamma$$

$$= (\alpha^3 + \beta^3 + \gamma^3) + (\alpha + \beta + \gamma)(3\beta\gamma + 3\alpha\beta + 3\alpha\gamma) - 9\alpha\beta\gamma + 6\alpha\beta\gamma$$

$$= (\alpha^3 + \beta^3 + \gamma^3) + 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) - 3\alpha\beta\gamma$$

Hence, $(\alpha^3 + \beta^3 + \gamma^3) = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma$, which can be written as

$$\sum\alpha^3 = \left(\sum\alpha\right)^3 - 3\sum\alpha\sum\alpha\beta + 3\sum\alpha\beta\gamma$$

EXAMPLE 6 Given that $7x^3 - 4x^2 + 2 = 0$ has roots α, β and γ, find $\alpha^3 + \beta^3 + \gamma^3$.

SOLUTION We can use $(\alpha^3 + \beta^3 + \gamma^3) = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma$.

Since $7x^3 - 4x^2 + 2 = 0$

$\alpha + \beta + \gamma = \frac{4}{7}$

$\alpha\beta + \alpha\gamma + \beta\gamma = 0$

$\alpha\beta\gamma = \frac{-2}{7}$

Substituting into $(\alpha^3 + \beta^3 + \gamma^3) = (\alpha + \beta + \gamma)^3 - 3(\alpha + \beta + \gamma)(\alpha\beta + \alpha\gamma + \beta\gamma) + 3\alpha\beta\gamma$ gives:

$$\alpha^3 + \beta^3 + \gamma^3 = \left(\frac{4}{7}\right)^3 - 3\left(\frac{4}{7}\right)(0) + 3\left(\frac{-2}{7}\right)$$

$$= \frac{64}{343} - \frac{6}{7} = \frac{-230}{343}$$

Note

Recall that

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \alpha\gamma = \frac{c}{a}$

$\alpha\,\beta\,\gamma = -\frac{d}{a}$

Finding a cubic equation, given the roots of the equation

EXAMPLE 7 Given that the cubic equation $ax^3 + bx^2 + cx + d = 0$ has roots 2, 3, and 4, find a, b, c and d.

SOLUTION We can solve this problem by using two different methods.

Method 1

Remember that given the roots of a cubic equation, we can obtain the equation by using:

x^3 − (sum of the roots)x^2 + (sum of the products of two of the roots at a time)x − (product of the roots) = 0

(This was derived from the expansion of $(x - \alpha)(x - \beta)(x - \gamma) = 0$, where α, β and γ are the roots.

Sum of the roots $= 2 + 3 + 4 = 9$

Sum of the product of two roots at a time $= (2)(3) + 2(4) + 3(4)$

$= 6 + 8 + 12 = 26$

Product of the roots $= (2)(3)(4) = 24$

$\therefore$ the equation is

$x^3 -$ (sum of the roots)$x^2 +$ (sum of the product of two of the roots at a time)$x -$ (product of the roots) $= 0$

$\Rightarrow x^3 - 9x^2 + 26x - 24 = 0$

Hence, $a = 1$, $b = -9$, $c = 26$, $d = -24$.

Method 2

Since the roots are 2, 3 and 4, we know that:

$(x - 2)(x - 3)(x - 4) = 0$

$\Rightarrow (x - 2)(x^2 - 7x + 12) = 0$

$\Rightarrow x^3 - 7x^2 + 12x - 2x^2 + 14x - 24 = 0$

$\Rightarrow x^3 - 9x^2 + 26x - 24 = 0$

Hence, $a = 1$, $b = -9$, $c = 26$, $d = -24$.

EXAMPLE 8 Find the cubic equation where roots are $\frac{1}{2}$, -1 and 2.

SOLUTION Sum of the roots $= \frac{1}{2} - 1 + 2 = \frac{3}{2}$

Sum of the product of two of the roots at a time $= \left(\frac{1}{2}\right)(-1) + \frac{1}{2}(2) + (-1)(2)$

$= -\frac{1}{2} + 1 - 2$

$= \frac{-3}{2}$

Product of the roots $= \left(\frac{1}{2}\right)(-1)(2)$

$= -1$

We can obtain the equation by using:

$x^3 -$ (sum of the roots)$x^2 +$ (sum of the product of two of the roots at a time)$x -$ (product of the roots) $= 0$

$\therefore$ the equation is $x^3 - \left(\frac{3}{2}\right)x^2 + \left(\frac{-3}{2}\right)x - (-1) = 0$

$\Rightarrow x^3 - \frac{3}{2}x^2 - \frac{3}{2} + 1 = 0$

$\Rightarrow 2x^3 - 3x^2 - 3x + 2 = 0$

Alternative solution:

Since the roots are $\frac{1}{2}$, -1 and 2, we know that:

$\left(x - \frac{1}{2}\right)(x + 1)(x - 2) = 0$

$\Rightarrow \left(x - \frac{1}{2}\right)(x^2 - x - 2) = 0$

$$\Rightarrow x^3 - x^2 - 2x - \frac{1}{2}x^2 + \frac{1}{2}x + 1 = 0$$

$$\Rightarrow x^3 - \frac{3}{2}x^2 - \frac{3}{2}x + 1 = 0$$

$$\Rightarrow 2x^3 - 3x^2 - 3x + 2 = 0$$

EXAMPLE 9 Find the cubic equation where roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ where α, β and γ are the roots of $3x^3 - x^2 - 2x + 1 = 0$

SOLUTION Since $3x^3 - x^2 - 2x + 1 = 0$ has roots α, β and γ

$$\Rightarrow \alpha + \beta + \gamma = \frac{-(-1)}{3} = \frac{1}{3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-2}{3}$$

$$\alpha\beta\gamma = \frac{-1}{3}$$

We need to find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$ (Sum of the roots)

$$= \frac{\frac{-2}{3}}{\frac{-1}{3}} = 2$$

Also $\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right) + \left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) + \left(\frac{1}{\gamma}\right)\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$ (Sum of the products of two roots at a time)

$$= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma}$$

$$= \frac{\frac{1}{3}}{\frac{-1}{3}} = -1$$

And $\left(\frac{1}{\alpha}\right)\left(\frac{1}{\beta}\right)\left(\frac{1}{\gamma}\right) = \frac{1}{\alpha\beta\gamma}$ (Product of the roots)

$$= \frac{1}{\frac{-1}{3}} = -3$$

$\therefore$ sum of the roots $= 2$

Sum of the product of the roots $= -1$

Product of the roots $= -3$

Hence, the equation where roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ is $x^3 - (2)x^2 + (-1)(x) - (-3) = 0$.

$$\Rightarrow x^3 - 2x^2 - x + 3 = 0$$

Alternative solution:

Let $y = \frac{1}{x}$, $x = \frac{1}{y}$

Substituting $\frac{1}{y} = x$ into $3x^3 - x^2 - 2x + 1 = 0$ gives:

$$3\left(\frac{1}{y}\right) - \left(\frac{1}{y}\right) - 2\left(\frac{1}{y}\right) + 1 = 0$$

$$\Rightarrow \frac{3}{y^3} - \frac{1}{y^2} - \frac{2}{y} + 1 = 0$$

$$\Rightarrow 3 - y - 2y^2 + y^3 = 0$$

Since $x = \alpha, \beta, \gamma$, then $y = \frac{1}{x} = \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

Therefore, the equation $3 - y - 2y^2 + y^3 = 0$ has roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

EXAMPLE 10 The roots of the equation $x^3 + 5x^2 + 5x + 7 = 0$ are p, q, and r.

Find the equation where roots are $(p + 3)$, $(q + 3)$ and $(r + 3)$.

SOLUTION Given that p, q, and r are the roots of $x^3 + 5x^2 + 5x + 7 = 0$

$\Rightarrow p + q + r = -\frac{5}{1} = -5$

$pq + pr + qr = \frac{5}{1} = 5$

$pqr = \frac{-7}{1} = -7$

We need to find

$(p + 3) + (q + 3) + (r + 3)$, $(p + 3)(q + 3) + (p + 3)(r + 3) + (q + 3)(r + 3)$ and $(p + 3)(q + 3)(r + 3)$

Now $(p + 3) + (q + 3) + (r + 3) = p + q + r + 9$

$= -5 + 9$

$= 4$

$(p + 3)(q + 3) + (p + 3)(r + 3) + (q + 3)(r + 3)$

$= pq + 3p + 3q + 9 + pr + 3p + 3r + 9 + qr + 3q + 3r + 9$

$= (pq + pr + qr) + 6(p + q + r) + 27$ (Combining terms)

$= 5 + 6(-5) + 27$

$= 2$

$(p + 3)(q + 3)(r + 3) = (pq + 3p + 3q + 9)(r + 3)$

$= pqr + 3pq + 3pr + 9p + 3qr + 9q + 9r + 27$

$= pqr + 3(pq + pr + qr) + 9(p + q + r) + 27$

$= -7 + 3(5) + 9(-5) + 27$

$= -7 + 15 - 45 + 27$

$= -10$

The equation with roots $p + 3$, $q + 3$ and $r + 3$ is $x^3 - (4)x^2 + 2(x) - (-10) = 0$

$\Rightarrow x^3 - 4x^2 + 2x + 10 = 0$

Alternative solution:

Let $y = x + 3$.

$\Rightarrow x = y - 3$

Substituting $x = y - 3$ into the equation $x^3 + 5x^2 + 5x + 7 = 0$ we get:

$(y - 3)^3 + 5(y - 3)^2 + 5(y - 3) + 7 = 0$

$y^3 - 9y^2 + 27y - 27 + 5(y^2 - 6y + 9) + 5y - 15 + 7 = 0$ (Expanding brackets)

$\Rightarrow y^3 - 4y^2 + 2y + 10 = 0$

Since $x = p, q, r$ are the roots of the equation and $y = x + 3 \Rightarrow$ the roots of the equation in y are $p + 3$, $q + 3$ and $r + 3$.

EXAMPLE 11 If the roots of the equation $x^3 - 6x^2 + 3x - 30 = 0$ are α, β and γ, show that an equation whose roots are $\alpha - 3$, $\beta - 3$ and $\gamma - 3$ is $x^3 + 3x^2 - 6x - 48 = 0$.

Hence, find $\Sigma(\alpha - 3)^2$.

SOLUTION $x^3 - 6x^2 + 3x - 30 = 0$

Let $y = x - 3$, $\therefore x = y + 3$.

Substituting $x = y + 3$ into the equation above we get:

$(y + 3)^3 - 6(y + 3)^2 + 3(y + 3) - 30 = 0$

Expanding gives:

$(y + 3)(y^2 + 6y + 9) - 6(y^2 + 6y + 9) + 3y + 9 - 30 = 0$

$\Rightarrow y^3 + 6y^2 + 9y + 3y^2 + 18y + 27 - 6y^2 - 36y - 54 + 3y + 9 - 30 = 0$

$\Rightarrow y^3 + 3y^2 - 6y - 48 = 0.$

Since $y = x - 3$, and $x = \alpha$, $x = \beta$, and $x = \gamma$, then the equation in y has roots $\alpha - 3$, $\beta - 3$ and $\gamma - 3$. Hence, the equation with roots $\alpha - 3$, $\beta - 3$ and $\gamma - 3$ is $y^3 + 3y^2 - 6y - 48 = 0$

This is equivalent to $x^3 + 3x^2 - 6x - 48 = 0$.

From this equation we have:

$$\sum(\alpha - 3) = \frac{-3}{1} = -3$$

$$\sum(\alpha - 3)(\beta - 3) = \frac{-6}{1} = -6$$

$$\sum(\alpha - 3)(\beta - 3)(\gamma - 3) = -\frac{(-48)}{1} = 48$$

$$\sum(\alpha - 3)^2 = \left(\sum(\alpha - 3)\right)^2 - 2\sum(\alpha - 3)(\beta - 3) = (-3)^2 - 2(-6) = 21$$

EXAMPLE 12 The cubic equation $2x^3 - 3x^2 + 4x + 6 = 0$ has the roots α, β, and γ. Find the values of

(a) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$

(b) $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$

SOLUTION **(a)** $2x^3 - 3x^2 + 4x + 6 = 0$

$$\alpha + \beta + \gamma = -\left(\frac{-3}{2}\right) = \frac{3}{2}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{4}{2} = 2$$

$$\alpha\beta\gamma = -\frac{6}{2} = -3$$

(b) We write $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$ in terms of $\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$, $\alpha\beta\gamma$:

$$\begin{aligned}\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} &= \frac{2(\beta\gamma) + 2(\gamma\alpha) + 2(\alpha\beta)}{\alpha\beta\gamma} \\ &= \frac{2[\alpha\beta + \beta\gamma + \gamma\alpha]}{\alpha\beta\gamma} \\ &= \frac{2(2)}{-3} = -\frac{4}{3}\end{aligned}$$

(c) We write $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}$ in terms of $\alpha\beta\gamma$, $\alpha + \beta + \gamma$, and $\alpha\beta + \beta\gamma + \gamma\alpha$:

$$\begin{aligned}\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} &= \frac{\gamma + \alpha + \beta}{\alpha\beta\gamma} \\ &= \frac{\frac{3}{2}}{-3} = -\frac{1}{2}\end{aligned}$$

EXAMPLE 13 The cubic equation $x^3 + 3x^2 - 4x + 2 = 0$ has roots α, β and γ. Find the following.

(a) $\alpha^2 + \beta^2 + \gamma^2$

(b) $\alpha^3 + \beta^3 + \gamma^3$

(c) $\frac{1}{\alpha^3\beta^3} + \frac{1}{\beta^3\gamma^3} + \frac{1}{\gamma^3\alpha^3}$.

SOLUTION

(a) $x^3 + 3x^2 - 4x + 2 = 0$

$\alpha + \beta + \gamma = \frac{-(3)}{1} = -3$ $\qquad \sum\alpha = -3$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{-4}{1} = -4$ $\qquad \sum\alpha\beta = -4$

$\alpha\beta\gamma = \frac{-(2)}{1} = -2$

Now:

$$\begin{aligned}\sum\alpha^2 &= \left(\sum\alpha\right)^2 - 2\sum\alpha\beta \\ \sum\alpha^2 &= (-3)^2 - 2(-4) \\ &= 9 + 8 \\ &= 17\end{aligned}$$

Hence, $\alpha^2 + \beta^2 + \gamma^2 = 17$.

(b) Replacing $x = \alpha$, $x = \beta$, $x = \gamma$ into the equation, we get:

$\alpha^3 + 3\alpha^2 - 4\alpha + 2 = 0$

$\beta^3 + 3\beta^2 - 4\beta + 2 = 0$

$\gamma^3 + 3\gamma^2 - 4\gamma + 2 = 0$

Adding the above three equations gives:

$\alpha^3 + \beta^3 + \gamma^3 + 3\alpha^2 + 3\beta^2 + 3\gamma^2 - 4\alpha - 4\beta - 4\gamma + 2 + 2 + 2 = 0$

$$\sum \alpha^3 + 3\sum \alpha^2 - 4\sum \alpha + 6 = 0$$

$$\Rightarrow \sum \alpha^3 + 3(17) - 4(-3) + 6 = 0$$

$$\therefore \sum \alpha^3 = -69$$

(c) $$\frac{1}{\alpha^3\beta^3} + \frac{1}{\beta^3\gamma^3} + \frac{1}{\gamma^3\alpha^3} = \frac{\gamma^3 + \alpha^3 + \beta^3}{\alpha^3\beta^3\gamma^3}$$

$$= \frac{-69}{(-2)^3}$$

$$= \frac{69}{8}$$

SUMMARY

Cubic polynomials

Quadratics

Let α, β be the roots of the equation $ax^2 + bx + c = 0$

$\alpha + \beta = \frac{-b}{a}, \alpha\beta = \frac{c}{a}$

Given the sum of the roots and the product of the roots the equation is:
$x^2 -$ (sum of the roots)$x +$ (product of the roots) $= 0$

$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$\alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta)$

$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

If $\alpha > \beta$, $(\alpha - \beta)$ is positive

Cubics

Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$

$\alpha\beta + \beta\gamma + \gamma a = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

or $\sum \alpha^2 = \left(\sum \alpha\right)^2 - 2\sum \alpha\beta$

$\sum \alpha^3 = \left(\sum \alpha\right)^3 - 3\left(\sum \alpha\right)\sum(\alpha\beta) + 3\alpha\beta\gamma$

The cubic equation is:
$x^3 -$ (sum of the roots)$x^2 +$ (sum of the product of two of the roots at a time)$x -$ (product of the roots) $= 0$

Checklist

Can you do these?

- Connect the roots of a quadratic and the coefficients of the quadratic equation.
- Form quadratic equations.
- Connect the roots of a cubic and the coefficients of the cubic equation.
- Find a cubic equation, given the roots of the equation.

Review Exercise 7

1 Given that α and β are the roots of the equation $2x^2 = 3x - 5$, find

(a) the numerical value of $\dfrac{\alpha}{2\beta + 1} + \dfrac{\beta}{2\alpha + 1}$

(b) the equation whose roots are $\dfrac{\alpha}{2\beta + 1}$ and $\dfrac{\beta}{2\alpha + 1}$.

2 Given that α and β are the roots of the equation $2x^2 - 5x + 7 = 0$, find the equation whose roots are $\alpha + \frac{1}{\alpha}$ and $\beta + \frac{1}{\beta}$.

3 If α and β are the roots of the equation $3x^2 = -(2x - 1)$ and $\alpha > \beta$. Calculate

(a) $\alpha^2 + \beta^2 + 2\alpha\beta$

(b) $\alpha^4 - \beta^4 + \alpha^2 - \beta^2$.

4 If α, β and γ are the roots of the equation $x^3 - 10x + 6 = 0$, find the values of the following.

(a) $\alpha + \beta + \gamma$

(b) $\alpha^2 + \beta^2 + \gamma^2$

(c) $\alpha^3 + \beta^3 + \gamma^3$

5 If α, β and γ are the roots of the equation $2x^3 - x^2 - 10x - 6 = 0$, find the values of the following.

(a) $\alpha + \beta + \gamma$

(b) $\alpha^2 + \beta^2 + \gamma^2$

(c) $\alpha^3 + \beta^3 + \gamma^3$

6 The roots of the equation $x^3 + 4x + 1 = 0$, are p, q and r.

(a) Show that $p^2 + q^2 + r^2 = -8$.

(b) Find $p^3 + q^3 + r^3$.

7 Given that α, β and γ are the roots of the equation $2x^3 - 4x^2 + 6x - 1 = 0$, find the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

8 Given that the roots of the equation $x^3 + \alpha x^2 + \beta x + \gamma = 0$ are -2, -3 and 4, find the values of α, β and γ.

9 The roots of the equation $x^3 + 6x^2 + 10x + 14 = 0$ are α, β and γ. Find the equation whose roots are:

(a) α^2, β^2 and γ^2

(b) $\alpha + 3$, $\beta + 3$ and $\gamma + 3$

10 The roots of the equation $3x^3 - 4x^2 + 8x - 7 = 0$ are α, β and γ. Find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

11 The roots of the equation $2x^3 - x^2 + 10x - 6 = 0$ are α, β and γ.

(a) Write down $\alpha + \beta + \gamma$.

(b) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

(c) Given that $9\beta^2 + 9\gamma^2 = 32\alpha^2$, and that α, γ are both negative and β is positive, find the exact values of α, β and γ.

12 The cubic equation $x^3 + ax^2 + bx + c = 0$ has roots α, β and γ. Given that $\alpha + \beta + \gamma = 6$, $\alpha^2 + \beta^2 + \gamma^2 = 14$ and $\alpha^3 + \beta^3 + \gamma^3 = 36$, find the values of a, b and c.

13 The cubic equation $x^3 + ax^2 + bx + c = 0$ has roots α, β and γ. Given that $\alpha + \beta + \gamma = 0$, $\alpha^2 + \beta^2 + \gamma^2 = 14$ and $\alpha^3 + \beta^3 + \gamma^3 = 18$, find the values of a, b and c.

14 The roots of the equation $x^3 - 6x + 3 = 0$ are α, β and γ. Find the equation with roots $\frac{\alpha + 1}{\alpha}$, $\frac{\beta + 1}{\beta}$ and $\frac{\gamma + 1}{\gamma}$.

15 If α, β and γ are the roots of the equation $x^3 - 2x^2 + 4x + 5 = 0$, find the cubic equation for each of these sets of roots.

(a) 2α, 2β and 2γ

(b) $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$

CHAPTER 8

Inequalities and the Modulus Function

At the end of this chapter you should be able to:

- Solve linear inequalities
- Solve quadratic inequalities
- Solve inequalities of the form $\frac{ax + b}{cx + d} > 0$
- Solve modulus inequalities
- Define the modulus function
- Use the definition of the modulus function to solve equations
- Identify the properties of the modulus function
- Use the properties to solve modulus equalities and inequalities
- Use the triangle inequality
- Solve application problems involving inequalities

KEYWORDS/TERMS

modulus • absolute values • triangle inequality • inequalities • sign table • zero of the function • linear inequality • quadratic inequality • modulus inequality

An inequality is a relationship of numbers connected by less than ($<$), less than or equal to ($\leq$), greater than ($>$), or greater than or equal to ($\geq$). A solution for an inequality is any number satisfying the inequality. An inequality typically has an infinite set of solutions and the solution set is given in an interval using set brackets. For example $\{x: x < 2\}$ represents the set of values of x less than 2.

Theorems of inequalities

Theorem 1

If $a > b$, then $a + c > b + c$.

We may add the same number to both sides of an inequality without changing the direction of the inequality.

For example, if $x > 5$, then $x + 2 > 5 + 2$.

$\Rightarrow x + 2 > 7$

Theorem 2

If $a > b$ and $c > 0$, then $ac > bc$.

We can multiply two sides of an inequality by a positive number and the inequality sign remains the same.

For example, if, $x > 2$ then $2x > 4$.

Theorem 3

If $a > b$ and $c < 0$, then $ac < bc$.

When we multiply both sides of an inequality by a negative number, the inequality sign reverses.

For example, if $a > b$ then $-3a < -3b$.

Theorem 4

If $a > b$, then $\frac{1}{a} < \frac{1}{b}$.

Quadratic inequalities

To solve $ax^2 + bx + c > 0$, $a \neq 0$, we can graph the function $f(x) = ax^2 + bx + c$ and identify the values of x for which the curve is above the x-axis.

To solve $ax^2 + bx + c < 0$, $a \neq 0$, we can graph the function $f(x) = ax^2 + bx + c$ and identify the values of x for which the curve is below the x-axis.

EXAMPLE 1 Find the range of values of x for which $x - 3x + 2 < 0$.

SOLUTION Let us sketch the graph of $f(x) = x^2 - 3x + 2$.

When solving quadratic inequalities, we can sketch the graph by identifying the coordinates of the turning point, and read off the solution set.

When $x = 0, f(x) = 2$.

$\therefore$ (0, 2) is on the curve.

$f(x) = 0 \Rightarrow x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$

$\Rightarrow x = 1, x = 2$

$\therefore$ (1, 0) and (2, 0) are on the curve.

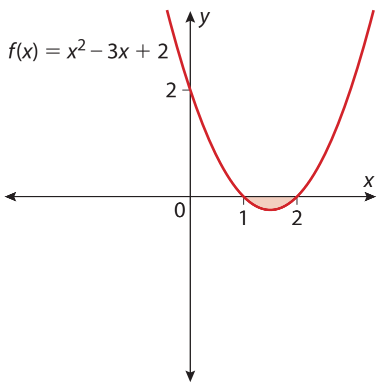
The graph has a minimum point for:

$$x = \frac{-(-3)}{2(1)} = \frac{3}{2}$$

When $x = \frac{3}{2}$, $f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 3\left(\frac{3}{2}\right) + 2 = \frac{9}{4} - \frac{9}{2} + 2$

$= -\frac{1}{4}$

Minimum point at $\left(\frac{3}{2}, -\frac{1}{4}\right)$



Since we are solving $x^2 - 3x + 2 < 0$, we look where the graph is negative.

The graph is negative for $-1 < x < 2$.

$\therefore \{x: -1 < x < 2\}$

EXAMPLE 2 Find the range of values of x for which $2x^2 < 3x + 2$.

SOLUTION Rearrange the inequality so that all the terms are on one side.

$2x^2 < 3x + 2$

$\Rightarrow 2x^2 - 3x - 2 < 0$

We sketch $f(x) = 2x^2 - 3x - 2$.

When $x = 0, f(0) = -2$.

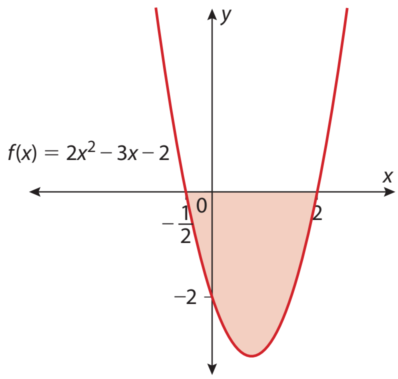
$\therefore (0, -2)$ lies on the curve.

When $f(x) = 0$, $2x^2 - 3x - 2 = 0$

$\Rightarrow (2x + 1)(x - 2) = 0$

$\Rightarrow 2x + 1 = 0$, $x - 2 = 0$

$\Rightarrow x = -\frac{1}{2}, x = 2$



$\therefore (-\frac{1}{2}, 0)$ and (2, 0) lie on the curve.

The curve has a minimum point when $x = \frac{-b}{2a} = \frac{-(-3)}{2(2)} = \frac{3}{4}$.

$f\left(\frac{3}{4}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) - 2 = \frac{-25}{8}$

Therefore, the minimum point is at $\left(\frac{3}{4}, \frac{-25}{8}\right)$.

$2x^2 - 3x - 2 < 0$

$f(x) < 0$

The curve is negative when it is below the x-axis

This is when $\frac{-1}{2} < x < 2$.

$\therefore \left\{x: \frac{-1}{2} < x < 2\right\}$

EXAMPLE 3 Solve the inequality $-x^2 + 4x - 3 > 0$.

SOLUTION

Let $f(x) = -x^2 + 4x - 3$

When $x = 0, f(0) = -3$.

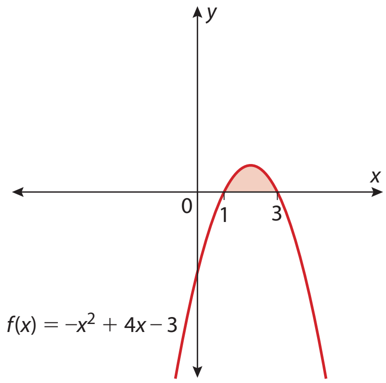
$\therefore (0, -3)$ is on the curve.

When $f(x) = 0, \quad -x^2 + 4x - 3 = 0$

$\Rightarrow -(x - 1)(x - 3) = 0$

$\Rightarrow x = 1, x = 3$

$\therefore$ (1, 0) and (3, 0) are on the graph.



Since the coefficient of x^2 is negative, there is a maximum point when

$x = \frac{-b}{2a} = \frac{-4}{2(-1)} = 2$.

$f(2) = -(2)^2 + 4(2) - 3$

$= -4 + 8 - 3$

$= 1$

Maximum point is at (2, 1)

Since we are solving $f(x) > 0$, we look for where the curve is above the x-axis.

This is $1 < x < 3$.

$\therefore \{x: 1 < x < 3\}$

Sign table

The critical values of a function are the values of x for which the function becomes zero or infinity. Critical values of a function $f(x)$ can be found by solving $f(x) = 0$. If the function is of the form $\frac{P(x)}{Q(x)}$, critical values of $f(x)$ are found by solving $P(x) = 0$ and $Q(x) = 0$. These values are also called the zeros of the function.

EXAMPLE 4 Find the critical values of $f(x) = \frac{2x + 1}{x - 1}$.

SOLUTION The critical values of $f(x)$ occur at:

$2x + 1 = 0 \Rightarrow x = \frac{-1}{2}$

and $x - 1 = 0 \Rightarrow x = 1$

We can use the critical values of $f(x)$ to solve inequalities by drawing up a sign table. A sign table consists of the factors of $f(x)$, non-overlapping intervals for x using the zeros of $f(x)$ and the sign of each factor and $f(x)$.

EXAMPLE 5 Find the set of values of x satisfying $x^2 - 5x + 4 < 0$.

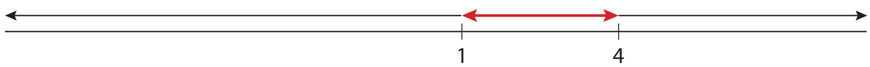
SOLUTION Let us use a sign table.

$x^2 - 5x + 4 < 0$

$\Rightarrow (x - 4)(x - 1) < 0$

$\therefore\ x = 4, x = 1$, for the critical values.

The critical values of $f(x)$ are $x = 1, x = 4$, in ascending order. Identifying the critical values on a number line means we can set up non-overlapping intervals for x.



	$x - 4$	$x - 1$	$(x - 4)(x - 1)$
$x < 1$	$-$ve	$-$ve	$+$ve
$1 < x < 4$	$-$ve	$+$ve	$-$ve
$x > 4$	$+$ve	$+$ve	$+$ve

To check the sign of $x^2 - 5x + 4$ we use one value of x in each interval and check the sign of each factor and then the sign of the product.

In the interval $x < 1$, using $x = 0$ we see that $x - 4$ is negative and $x - 1$ is negative.

$\therefore\ (x - 4)(x - 1) = -\text{ve} \times -\text{ve} = +\text{ve}$

Since we are interested in $x^2 - 5x + 4 < 0$, we can see that this occurs in the interval $1 < x < 4$.

$\therefore$ the solution set is $\{x: 1 < x < 4\}$.

EXAMPLE 6 Solve the inequality $12x^2 - 5x - 2 > 0$.

SOLUTION Factorising the function gives:

$12x^2 - 5x - 2 > 0$

$\Rightarrow (4x + 1)(3x - 2) > 0$

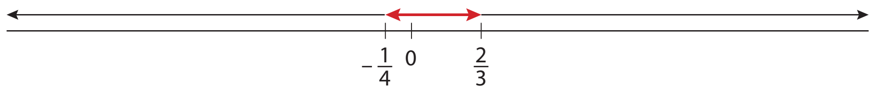
Finding the zeros we have:

$(4x + 1)(3x - 2) = 0$

$\Rightarrow 4x + 1 = 0, \quad 3x - 2 = 0$

$x = \frac{-1}{4}, x = \frac{2}{3}$

Place the zeros on a number line in ascending order:



Sign table:

	4x + 1	**3x − 2**	**(4x + 1)(3x − 2)**
$x < \frac{-1}{4}$	− ve	− ve	+ve
$\frac{-1}{4} < x < \frac{2}{3}$	+ve	− ve	− ve
$x > \frac{2}{3}$	+ve	+ve	+ve

We can use one value of x in each region to test the sign of the function.

For $x < -\frac{1}{4}$, if $x = -1$:

$4x + 1 = 4(-1) + 1 = -3$

$-3 < 0$

Therefore, $4x + 1$ is negative when $x < -\frac{1}{4}$.

Also, if $x = -1$:

$3x - 2 = 3(-1) - 2 = -5$

$-5 < 0$

When we find the product of $(4x + 1)(3x - 2)$, we get a positive value.

Since we are interested in $12x^2 - 5x - 2 > 0$, there are two regions where this occurs:

$\therefore \left\{x\text{: } x < \frac{-1}{4}\right\} \cup \left\{x\text{: } x > \frac{2}{3}\right\}$

EXAMPLE 7 Find the range of values of x for which $x^2 + 2x - 3 \geq +12$.

SOLUTION Bring all the terms to one side of the inequality:

$x^2 + 2x - 3 \geq 12$

$\Rightarrow x^2 + 2x - 15 \geq 0$

Factorising gives:

$(x + 5)(x - 3) \geq 0$

Remember

Make the function either greater than or less than zero.

Finding the zeros of the function gives:

$x + 5 = 0, \quad x - 3 = 0$

$\Rightarrow x = -5, x = 3$

Place the zeros on a number line in ascending order:

Sign table:

	$x + 5$	$x - 3$	$(x + 5)(x - 3)$
$x < -5$	$-$ve	$-$ve	$+$ve
$-5 < x < 3$	$+$ve	$-$ve	$-$ve
$x > 3$	$+$ve	$+$ve	$+$ve

Since we are interested in $x^2 + 2x - 15 \geq 0$, we must include the end points as part of our solution set.

$\therefore \{x: x \leq -5\} \cup \{x: x \geq 3\}$

Try these 8.1

(a) Find the values of x for which $3x < x^2 - 4$.

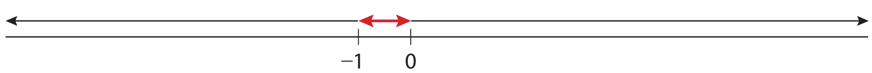
(b) Find the values of x for which $6x^2 - 11x - 7 \geq 0$.

Rational functions and inequalities

EXAMPLE 8 Find the range of values of x for which $\frac{x}{x + 1} > 0$.

SOLUTION The critical values of x are $x = 0, x + 1 = 0 \Rightarrow x = -1$.

Place the critical values of x on the number line in ascending order.



Sign table

Divide the number line into regions banded by the critical values

$\therefore x < -1, \quad -1 < x < 0, \quad x > 0$

	x	$x+1$	$\frac{x}{x+1}$
$x < -1$	$-$ ve	$-$ ve	$+$ve
$-1 < x < 0$	$-$ ve	$+$ve	$-$ ve
$x > 0$	$+$ve	$+$ve	$+$ve

Solution set is $\{x: x < -1\} \cup \{x: x > 0\}$.

EXAMPLE 9 Find the range of values of x satisfying $\frac{x+1}{x+2} > 3$.

SOLUTION ***Method 1***

Bring all terms to one side of the inequality:

$$\frac{x+1}{x+2} - 3 > 0$$

Write as one fraction:

$$\frac{x+1-3(x+2)}{x+2} > 0$$

$$\Rightarrow \frac{x+1-3x-6}{x+2} > 0$$

$$\Rightarrow \frac{-2x-5}{x+2} > 0$$

Find the critical values:

$$-2x - 5 = 0 \Rightarrow -2x = 5 \Rightarrow x = \frac{-5}{2}$$

$$x + 2 = 0 \Rightarrow x = -2$$

Place the critical values in ascending order on a number line, splitting into non-overlapping regions:

Sign table:

	$-2x-5$	$x+2$	$\frac{-2x-5}{x+2}$
$x < \frac{-5}{2}$	$+$ve	$-$ ve	$-$ ve
$\frac{-5}{2} < x < -2$	$-$ ve	$-$ ve	$+$ve
$x > -2$	$-$ ve	$+$ve	$-$ ve

$$\therefore \left\{x: \frac{-5}{2} < x < -2\right\}$$

Method 2

We can also solve these inequalities by multiplying throughout by the square of the denominator.

$\frac{x+1}{x+2} \times (x+2)^2 > 3 \times (x+2)^2$ (Multiplying both sides by $(x+2)^2$)

$\Rightarrow (x+1)(x+2) > 3(x+2)^2$

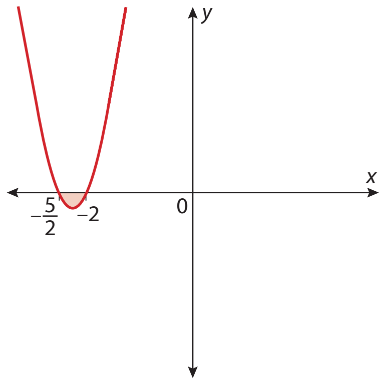
$\Rightarrow 3(x+2)^2 - (x+1)(x+2) < 0$

$\Rightarrow (x+2)(3(x+2) - (x+1)) < 0$

$\Rightarrow (x+2)(3x+6-x-1) < 0$

$\Rightarrow (x+2)(2x+5) < 0$

From the graph, $\{x: -\frac{5}{2} < x < -2\}$



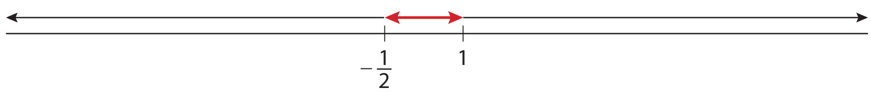
EXAMPLE 10 Solve the inequality $\frac{2x+1}{x-1} < 0$.

SOLUTION Find the critical values:

$2x + 1 = 0 \Rightarrow x = \frac{-1}{2}$

$x - 1 = 0 \Rightarrow x = 1$

Place the values on a number line in ascending order, splitting into non-overlapping regions:



Sign table:

	$2x+1$	$x-1$	$\frac{2x+1}{x-1}$
$x < \frac{-1}{2}$	$-$ve	$-$ve	$+$ve
$\frac{-1}{2} < x < 1$	$+$ve	$-$ve	$-$ve
$x > 1$	$+$ve	$+$ve	$+$ve

Since the function $\frac{2x+1}{x-1}$ is negative for $-\frac{1}{2} < x < 1$,

the solution set is $\left\{x: \frac{-1}{2} < x < 1\right\}$.

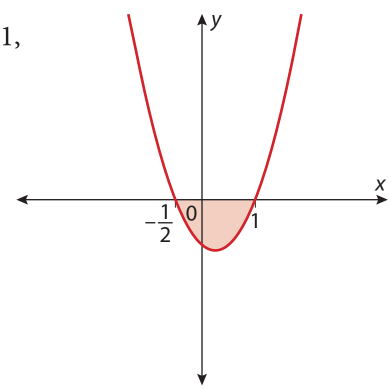
Alternative solution:

$\frac{2x+1}{x-1} \times (x-1)^2 < 0 \times (x-1)^2$

(Multiplying both sides by $(x-1)^2$)

$\Rightarrow (2x+1)(x-1) < 0$

Therefore $\left\{x : -\frac{1}{2} < x < 1\right\}$



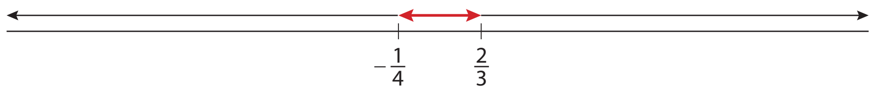
EXAMPLE 11 Find the solution set of $\frac{4x+1}{3x-2} < 0$.

SOLUTION The critical values of the function are:

$4x + 1 = 0 \Rightarrow x = -\frac{1}{4}$

$3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

Place the critical values on a number line in ascending order, splitting into non-overlapping regions:



Sign table:

	$4x + 1$	$3x - 2$	$\frac{4x+1}{3x-2}$
$x < \frac{-1}{4}$	− ve	− ve	+ve
$\frac{-1}{4} < x < \frac{2}{3}$	+ve	− ve	− ve
$x > \frac{2}{3}$	+ve	+ve	+ve

Since $\frac{4x+1}{2x-2} < 0 \Rightarrow \left\{x: \frac{-1}{4} < x < \frac{2}{3}\right\}$.

Try these 8.2

(a) Find the range of values of x for which $\frac{3x+2}{4x-1} < 0$.

(b) Solve the inequality $\frac{3x-4}{x-2} < 5$.

(c) Find the solution set of $\frac{x+1}{x+2} > 3$.

EXERCISE 8A

In questions **1** to **7**, find the solution set of the inequalities.

1 $x^2 + 8x + 15 < 0$

2 $x^2 + 3x - 4 < 0$

3 $x^2 - x < 6$

4 $3x^2 + 4x < -3x - 2$

5 $6x^2 + 7x + 2 < 0$

6 $5x^2 + 6x + 1 < 0$

7 $x^2 - 2 > 0$

8 Find the range of values of k for which the equation $kx^2 + 2kx + 2x + 7 = 0$ has real roots.

9 Find the range of values of k for which the equation $2x^2 + 5x + k + 2 = 0$ has real and distinct roots.

10 Find the range of values of p for which the equation $(p + 2)x^2 - 4x + 3 = 0$ has real and distinct roots.

11 Calculate the smallest positive integer k for which the equation $4x^2 - 2kx + 3 = 0$ has real roots.

12 Solve the following inequalities.

(a) $\dfrac{x+4}{x+5} > 2$

(b) $\dfrac{2x-1}{3x+1} > 1$

(c) $\dfrac{7x+2}{x+1} > 5$

(d) $\dfrac{3x-1}{x-2} < 1$

13 Find the range of values of x satisfying the inequality $\dfrac{x+1}{x-2} > \dfrac{x}{x+3}$.

14 Find the range of values of x satisfying the inequality $\dfrac{2x+1}{x-3} < \dfrac{1}{x+2}$.

EXAMPLE 12 Use the definition of the modulus function to solve $|x + 2| = 3$.

SOLUTION

$|x + 2| = 3$

By definition:

$x + 2 = 3 \quad \text{or} \quad -(x + 2) = 3$

$\therefore x = 1 \quad \text{or} \quad x + 2 = -3$

$x = -3 - 2$

$= -5$

$\therefore x = 1, -5$

Remember

The absolute value of x or modulus of x denoted by $|x|$ is defined as

$$|x| = \begin{cases} x \text{ if } x \geqslant 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

By definition, the absolute value function or modulus function is a positive function. This means that it returns positive values only.

EXAMPLE 13 Find the value(s) of x satisfying the equation $|2x - 1| = 5$.

SOLUTION $|2x - 1| = 5$

By definition

$2x - 1 = 5$ or $-(2x - 1) = 5$

$2x = 6$ or $2x - 1 = -5$

$x = 3$ $2x = -4$

$x = -2$

$\therefore x = 3, -2$

EXAMPLE 14 Solve the equation $|2x + 1| = |3x - 4|$.

SOLUTION $|2x + 1| = |3x - 4|$

By definition:

$2x + 1 = 3x - 4$ or $2x + 1 = -(3x - 4)$

$\Rightarrow 2x - 3x = -4 - 1$ or $2x + 1 = -3x + 4$

$-x = -5$ $5x = 3$

$x = 5$ $x = \frac{3}{5}$

$\therefore x = 5, \frac{3}{5}$

Try these 8.3 Find the value(s) of x satisfying each of the following.

(a) $|x + 1| = 3$

(b) $|4x - 3| = 7$

(c) $|2x + 5| = |4x - 7|$

General results about the absolute value function

Result 1

$|x|^2 = x^2$

EXAMPLE 15 Solve the equation $|2x - 1| = 3$.

SOLUTION $|2x - 1| = 3$

We can solve this equation by squaring both sides:

$|2x - 1|^2 = 3^2$

Using $|x|^2 = x^2$

$\Rightarrow |2x - 1|^2 = (2x - 1)^2$

Our equation becomes:

$(2x - 1)^2 = 3^2$

$\Rightarrow 4x^2 - 4x + 1 = 9$

$\Rightarrow 4x^2 - 4x - 8 = 0$

Divide both sides by 4:

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow (x - 2)(x + 1) = 0$

Hence, $x = 2$, $x = -1$.

We can also use the definition of modulus to find solutions to other types of problems.

EXAMPLE 16 Solve the equation $|2x - 3| = |4x - 1|$.

SOLUTION Squaring both sides gives:

$|2x - 3|^2 = |4x - 1|^2$

Using $|x|^2 = x^2$, gives $|2x - 3|^2 = (2x - 3)^2$ and $|4x - 1|^2 = (4x - 1)^2$.

Our equation becomes:

$(2x - 3)^2 = (4x - 1)^2$

$\Rightarrow 4x^2 - 12x + 9 = 16x^2 - 8x + 1$

$\Rightarrow 12x^2 + 4x - 8 = 0$

Divide both sides by 4:

$3x^2 + x - 2 = 0$

Factorising gives:

$(3x - 2)(x + 1) = 0$

$3x - 2 = 0 \Rightarrow x = \frac{2}{3}$

$x + 1 = 0 \Rightarrow x = -1$

Hence, $x = \frac{2}{3}, -1$.

EXAMPLE 17 Find all the values of x satisfying the equation $x^2 - 6|x| + 8 = 0$.

SOLUTION ***Method 1***

(Using the definition)

Using $|x| = x$ we have:

$x^2 - 6x + 8 = 0$

$\Rightarrow (x - 2)(x - 4) = 0$

$\Rightarrow x - 2 = 0, x - 4 = 0$

$\Rightarrow x = 2, x = 4$

Using $|x| = -x$ we have:

$x^2 - 6(-x) + 8 = 0$

$\Rightarrow x^2 + 6x + 8 = 0$

$\Rightarrow (x + 2)(x + 4) = 0$

$\Rightarrow x + 2 = 0, \quad x + 4 = 0$

$x = -2, x = -4$

$\therefore\ x = -2, 2, -4, 4.$

Method 2

(Using Result 1)

Keep the modulus on one side of the equation and carry everything else to the other side.

$x^2 - 6|x| + 8 = 0$

$\Rightarrow x^2 + 8 = 6|x|$

Square both sides:

$(x^2 + 8)^2 = (6|x|)^2$

$\Rightarrow x^4 + 16x^2 + 64 = 36|x|^2$

Using $|x|^2 = x^2$ gives:

$x^4 + 16x^2 + 64 = 36x^2$

$\therefore\ x^4 - 20x^2 + 64 = 0$

$\Rightarrow (x^2 - 16)(x^2 - 4) = 0$

$\Rightarrow x^2 - 16 = 0, \quad x^2 - 4 = 0$

$\Rightarrow x^2 = 16, \quad x^2 = 4$

$\Rightarrow x = \pm 4, x = \pm 2$

$\therefore\ x = -2, 2, -4, 4$

Here is an alternative method of solving $x^4 - 20x^2 + 64 = 0$.

$x^4 - 20x^2 + 64 = 0$ is a quadratic in x^2.

Let $y = x^2$.

$\Rightarrow y^2 - 20y + 64 = 0$

$\Rightarrow (y - 16)(y - 4) = 0$

$\Rightarrow y = 16, 4$

$\Rightarrow x^2 = 16, x^2 = 4$

$\Rightarrow x = 2, -2, 4, -4$

Result 2

$|xy| = |x||y|$

Modulus of a product = product of the modulus

EXAMPLE 18 Solve the equation $|(2x + 1)(x - 2)| = 0$.

SOLUTION Using the result $|xy| = |x||y|$, we have:

$|(2x + 1)(x - 2)| = 0$

$\Rightarrow |2x + 1| \times |x - 2| = 0$

$\Rightarrow |2x + 1| = 0, \quad |x - 2| = 0$

Using the definition:

$2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$

$-(2x + 1) = 0 \Rightarrow x = -\frac{1}{2}$

$x - 2 = 0 \Rightarrow x = 2$

$-(x - 2) = 0 \Rightarrow x = 2$

$\therefore\ x = -\frac{1}{2}, 2$

Result 3

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Modulus of a quotient = quotient of the moduli

EXAMPLE 19 Solve the equation $\left|\frac{3x + 1}{x + 2}\right| = 2$.

SOLUTION $\left|\frac{3x + 1}{x + 2}\right| = 2$

Separating, we get:

$\frac{|3x + 1|}{|x + 2|} = 2$

$\Rightarrow |3x + 1| = 2|x + 2|$

Squaring both sides gives:

$|3x + 1|^2 = 2^2|x + 2|^2$

Using $|3x + 1|^2 = (3x + 1)^2$ and $|x + 2|^2 = (x + 2)^2$

$(3x + 1)^2 = 4(x + 2)^2$

$\Rightarrow 9x^2 + 6x + 1 = 4(x^2 + 4x + 4)$

$\Rightarrow 9x^2 + 6x + 1 = 4x^2 + 16x + 16$

$\Rightarrow 5x^2 - 10x - 15 = 0$

Divide both sides by 5:

$x^2 - 2x - 3 = 0$

Factorising gives:

$(x - 3)(x + 1) = 0$

$\Rightarrow x - 3 = 0, \quad x + 1 = 0$

$\Rightarrow x = 3 \qquad x = -1$

Result 4

Given any constant c, $|x| < c$, if and only if $-c < x < c$.

This means that for any constant c, the modulus of x is less that c, if x lies between c and negative c.

EXAMPLE 20 Solve the inequality $|x + 2| < 5$.

SOLUTION

$|x + 2| < 5$

$\Rightarrow -5 < x + 2 < 5$ (Using $|x| < c \Rightarrow -c < x < c$)

$\Rightarrow -7 < x < 3$

EXAMPLE 21 Find the solution set of the inequality $|2x - 1| < 7$.

SOLUTION

$|2x - 1| < 7$

$\Rightarrow -7 < 2x - 1 < 7$

$\Rightarrow -6 < 2x < 8$

$\Rightarrow -3 < x < 4$

$\therefore \{x: -3 < x < 4\}$

Result 5

Given a constant c, $|x| > c$ if and only if $x > c$ or $x < -c$.

This means that for any constant c, the modulus of x is greater than c if and only if x is either greater than c or less than negative c.

EXAMPLE 22 Solve the inequality $|3x + 1| > 7$.

SOLUTION

$|3x + 1| > 7$

$\Rightarrow 3x + 1 > 7, \quad 3x + 1 < -7$

$\Rightarrow 3x > 6 \quad 3x < -8$

$\Rightarrow x > 2 \quad x < \frac{-8}{3}$

$\therefore \{x: x > 2\} \cup \left\{x: x < \frac{-8}{3}\right\}$

Result 6

$|x| < |y|$ if and only if $x^2 < y^2$

The modulus of x is less than the modulus of y if and only if x^2 is less than y^2.

Since $|x| < |y| \Rightarrow |x|^2 < |y|^2$

$\Rightarrow x^2 < y^2$

EXAMPLE 23 Solve the inequality $|2x + 5| < |2x - 1|$.

SOLUTION

$|2x + 5| < |2x - 1|$

$\Rightarrow (2x + 5)^2 < (2x - 1)^2$

Expanding gives:

$4x^2 + 20x + 25 < 4x^2 - 4x + 1$

$\Rightarrow 0 < -24x - 24$

$\Rightarrow 24x < -24$

$\Rightarrow x < -1$

Square root of x^2

Consider the expression $\sqrt{x^2}$. Since x^2 is always non-negative, the principal square root of x^2 is defined whether $x < 0$ or $x > 0$. The principal square root is non-negative and in order to ensure the non-negative result we have $|x| = +\sqrt{x^2}$, for any $x \in \mathbb{R}$. The table of values below shows this result for some values of x.

x	$+\sqrt{x^2}$	$\|x\|$
-3	$+\sqrt{(-3)^2} = 3$	$\|-3\| = 3$
-2	$+\sqrt{(-2)^2} = 2$	$\|-2\| = 2$
-1	$+\sqrt{(-1)^2} = 1$	$\|-1\| = 1$
0	$+\sqrt{(0)^2} = 0$	$\|0\| = 0$
1	$+\sqrt{(1)^2} = 1$	$\|1\| = 1$
2	$+\sqrt{(2)^2} = 2$	$\|2\| = 2$
3	$+\sqrt{(3)^2} = 3$	$\|3\| = 3$
4	$+\sqrt{(4)^2} = 4$	$\|4\| = 4$

The triangle inequality

Is the following statement true?

$|A + B| = |A| + |B|$

Consider this:

If $A = 10$ and $B = -4$, then the left-hand side is $|10 + (-4)| = |6| = 6$.

The right hand side is $|10| + |-4| = 10 + 4 = 14$.

So we have that the left-hand side is less than the right-hand side.

If, instead, $A = 5$ and $B = 3$ then the left-hand side is $|5 + 3| = |8| = 8$.

The right-hand side is $|5| + |3| = 5 + 3 = 8$.

Now we have that the left-hand side is equal to the right-hand side.

Clearly the statement must be false and we can therefore say that the modulus of a sum is not necessarily equal to the sum of the moduli.

Let us consider:

$|x + y| \leq |x| + |y|$

The above result is known as the triangle inequality.

PROOF

It is known that $x \leq |x|$ for all x.

$\Rightarrow 2xy \leq 2|x||y|$ for real x and y.

$\Rightarrow |x|^2 + |y|^2 + 2xy \leq |x|^2 + |y|^2 + 2|x||y|$

$\Rightarrow x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|x||y|$ (By result 1)

Factorising gives:

$(x + y)^2 \leq (|x| + |y|)^2$

$\Rightarrow |x + y|^2 \leq (|x| + |y|)^2$ (By result 1)

$\Rightarrow |x + y| \leq |x| + |y|$ (Taking square roots)

EXERCISE 8B

1 Solve the following equations.

(a) $|2x + 3| = 7$ (b) $|5x - 1| = 8$

(c) $|4x + 3| = 1$ (d) $|1 - 2x| = 6$

2 Find the values of x satisfying the following equations.

(a) $|3x + 1| = |2x - 4|$

(b) $|x - 1| = |x + 2|$

(c) $|7x + 1| = |5x + 3|$

3 Solve the equations.

(a) $|x| = 2 - |x|$ (b) $2|x| = 3 + 2x - x^2$

(c) $|x^2 - 1| - 1 = 3x - 2$ (d) $2 - |x + 1| = |4x - 3|$

4 Given that $\left|\frac{2x + 1}{3x - 4}\right| = 2$, find x.

5 Solve the equation $|4x - 1|^2 - 6|4x - 1| + 5 = 0$.

6 Find the values of x satisfying the equation $|3x + 2|^2 - 9|3x + 2| + 20 = 0$.

7 Solve the equations.

(a) $2x^2 - 5|x| + 2 = 0$

(b) $3x^2 - 19|x| + 20 = 0$

8 Find the solution set for the following inequalities, using two methods.

(a) $|4x - 1| < 3$ (b) $|2x + 4| < 5$

(c) $|3x - 1| > 6$ (d) $|5x + 2| > 9$

9 Find the values of x satisfying the inequalities.

(a) $|3x + 1| < |2x - 5|$ (b) $|7x + 1| < |3x + 5|$

(c) $4|x + 2| < |x - 1|$

Applications problems for inequalities

EXAMPLE 24 A stone is catapulted vertically upwards with a velocity of $25\,\text{ms}^{-1}$. The distance travelled by the stone at time t seconds is given by

$s = 25t - 5t^2$

How long does its height exceed 30 m?

SOLUTION The distance travelled by the stone must be:

$s > 30$

$25t - 5t^2 > 30$

$5t^2 - 25t + 30 < 0$

$t^2 - 5t + 6 < 0$

$(t - 2)(t - 3) < 0$

$\therefore 2 < t < 3$

The stone will be above a height of 30 m for 1 second $((3 - 2) = 1)$.

EXAMPLE 25 The cost in TT dollars of producing x carnival costumes is given by

$C = -2x^2 + 1400x + 16\,000$ for $0 \leq x \leq 300$

A costume designer wants to keep his cost less than TT \$166 000 for the year 2013. What is the maximum number of costumes can be produced within this investment?

SOLUTION Since his cost must be less than TT \$166 000, we need to find x for which:

$-2x^2 + 1400x + 16\,000 < 166\,000$

$\therefore -2x^2 + 1400x + 16\,000 - 166\,000 < 0$

$-2x^2 + 1400x - 150\,000 < 0$

$x^2 - 700x + 75\,000 > 0$ (Dividing by -2; switch the inequality sign since we are dividing by a negative number)

Let us find where this function becomes zero.

Using the quadratic formula:

$x^2 - 700x + 75\,000 = 0$

$$x = \frac{+700 \pm \sqrt{(-700)^2 - 4(75\,000)}}{2(1)}$$

$\dfrac{700 \pm 435.89}{2}$

$x = 567.94$ or 132.06

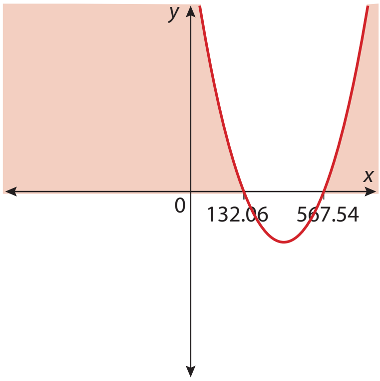
$\therefore\ (x - 567.94)(x - 132.06) > 0$

From the graph:

$\{x: x < 132.06\} \cup \{x > 567.54\}$

Since $0 \leq x \leq 300$, x must be in the interval $\{x: x < 132.06\}$.

Hence, $x = 132$.



Therefore, the maximum number of costumes the designer will produce to keep his cost less than TT $166 000 is 132.

If you are not sure whether $x = 132$ or 133, you can check the cost for each and see which is less than TT $166 000

$x = 132$, $C = -2(132)^2 + 1400(132) + 16\,000 = 165\,952$

$x = 133$, $C = -2(133)^2 + 1400(133) + 16\,000 = 166\,822$

EXAMPLE 26 A tour operator takes tourist to Blue Mountain, Jamaica, on a daily basis. The profit earned by the company for x number of tourists can be modelled by $P(x) = -x^2 + 500x - 1500$. How many tourists are needed to make a profit of at least US $3000?

SOLUTION Since the profit must be at least US $3000:

$P(x) \geq 3000$

$\therefore\ -x^2 + 500x - 1500 \geq 3000$

$\Rightarrow -x^2 + 500x - 4500 \geq 0$

$\Rightarrow x^2 - 500x + 4500 \leq 0$ (Multiplying by -1)

Let us find where the function becomes 0:

$x^2 - 500x + 4500 = 0$

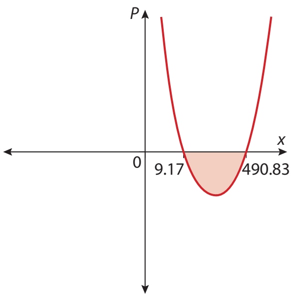
$$x = \frac{500 \pm \sqrt{(-500)^2 - 4(4500)}}{2}$$

$$= \frac{500 \pm 481.66}{2}$$

$$= 490.83,\ 9.17$$

$\therefore\ (x - 490.83)(x - 9.17) \leq 0$

$\therefore\ \{x: 9.17 \leq x \leq 490.83\}$.



To make a profit of at least $3000, the tour operator will need anywhere from 10 tourists to 490 tourists.

If we substitute $x = 9$, 10, 490 and 491 into $P(x)$ we can we can check where $P(x)$ will be greater than \$3000. When

$x = 9, P(x) = 2919$

$x = 10, P(x) = 3400$

$x = 490, P(x) = 3400$

$x = 491, P(x) = 2919$

SUMMARY

Inequalities and the modulus function

Properties of inequalities

If $a > b$, then $a + c > b + c$

If $a > b$ and $c > 0$, then $ac > bc$

If $a > b$ and $c < 0$, then $ac < bc$

If $a > b$, then $\frac{1}{a} < \frac{1}{b}$

Solving inequalities using a sign table

Make the function $f(x) > 0$ or $f(x) < 0$

Identify the critical values of $f(x)$.

Draw up a sign table using the critical values and the factors of the function.

Identify the solution set.

Modulus function

$$|x| = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -x \text{ if } x < 0 \end{cases}$$

$|x|^2 = x^2$

$|xy| = |x||y|$

$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$

$|x| < |y|$ if and only if $x^2 < y^2$

$|x| > c$ if and only if $x > c$ or $x < -c$.

$|x| < c$, where c is a constant if and only if $-c < x < c$.

Checklist

Can you do these?

- Define the modulus function.
- Use the definition of the modulus function to solve equations.
- Use the properties of the modulus function to solve equalities and inequalities.

- Use the triangle law of inequality.
- Identify different ways of solving equations and inequalities using the modulus function.
- Solve linear inequalities.
- Solve quadratic inequalities.
- Solve inequalities of the form $\frac{ax+b}{cx+d} > 0$.
- Solve applications problems involving inequalities.

Review Exercise 8

1 Solve the inequality $2x + 1 > 4x - 5$.

2 Solve $x^2 - 2\sqrt{3}x > 6$.

3 The equation $(p + 3)x^2 - 2px + p + 2 = 0$ is satisfied by two distinct real values of x. Find the range of values of p.

4 Find the range of values of λ for which the equation $4x^2 - 4\lambda x = 5\lambda - 12x - 15$ has no real roots.

5 Find the range of values of θ for which the equation $(2 - 3\theta)x^2 = (\theta - 4)x - 2$ has no real roots.

6 Show that the equation $(\lambda + 1)y^2 + (2\lambda + 3)y + \lambda + 2 = 0$ has real roots for all real values of λ.

7 For what range of values of x is $x^2 + 11x + 30$

(a) positive

(b) negative?

8 For what range of values of x is $4x^2 - 3x + 2$ greater than $3x^2$?

9 Find the values of x satisfying the equation $|3x + 2|^2 - 9|3x + 2| + 20 = 0$.

10 Find the values of x satisfying the equation $\left|\frac{2x+1}{3x-2}\right| = 1$.

11 Solve the equation $|2 - x| = 2|x + 2|$.

12 For each of the following find the range of values of x for which the inequality holds.

(a) $x^2 + 5x + 6 < 0$

(b) $x^2 + 2x - 8 > 0$

(c) $x^2 < x + 20$

(d) $(x + 3)(x - 2) > 2(x + 3)$

13 Find the range of values of p for which the line $y = 3x + p$ does not intersect the curve $x^2 + y^2 = 64$.

14 Find the range of values of p for which the line $y = 3 + px$ does not intersect the curve $x^2 + 2xy + 1 = 0$.

15 Find the solution set of the inequality $\frac{2x+1}{3x-4} < 2$.

16 Given that $f(x) = \frac{2-x}{x+3}$. Find the set of values of x for which $f(x) > 1$.

17 Solve these equations.

(a) $2x^2 - 5|x| + 2 = 0$

(b) $3x^2 - 19|x| + 20 = 0$

18 Solve the inequality $\frac{4x+2}{x-1} + 2 > 0$.

19 Find the range of values of p for which the line $y = \frac{p-x}{2}$ meets the curve $y = \frac{-8}{x} - x$.

20 The height s (in metres) of a ball thrown with an initial velocity of 80 metres per second from an initial height of 6 metres is given as:

$$s = -16t^2 + 80t + 6$$

where t is the time in seconds.

For what period of time is the ball at a height greater than 6 metres?

21 The daily profit, P (in hundreds of thousands of dollars) of a company is given by:

$$P = 8x - 0.02x^2$$

where x is the number of items produced per day. What is the range of the number of items that will have to be produced for the profit to be at least \$400 000?

22 Suppose that the manufacturer of phototcopying machines has found that, when the unit price is \$$x$, the revenue y (in dollars) is given by:

$$y = -4x^2 + 4000x$$

(a) At what prices \$$x$ is revenue zero?

(b) For what range of prices will revenue exceed \$800 000?

Module 1 Tests

Module 1 Test 1

1 (a) Without the use of tables or a calculator, simplify $\sqrt{80} + \sqrt{245} + \sqrt{320}$ in the form $k\sqrt{5}$, where k is an integer. [4]

(b) The roots of the cubic equation $x^3 + 3ax^2 - bx + 4c = 0$ are 2, -3 and 4. Find a, b and c. [8]

(c) (i) Show that $\sum_{r=1}^{r=n} r(2r-1) = \frac{n(n+1)(4n-1)}{6}$, $n \in \mathbb{N}$. [8]

(ii) Hence, or otherwise, evaluate $\sum_{r=18}^{r=40} r(2r-1)$. [5]

2 (a) Solve the inequality $\frac{2x+1}{4x-3} > 2$. [5]

(b) (i) Solve the equation $3|x| - 2x - 1 = 0$. [4]

(ii) Determine the values of the real number p for which the roots of the quadratic equation $2x^2 - px + 2p + 1 = 0$ are non-real. [4]

(c) Copy and complete the table below to show the truth values of $\sim(p \wedge \sim q)$. [6]

p	q	$\sim q$	$(p \wedge \sim q)$	$\sim(p \wedge \sim q)$

3 (a) Prove by mathematical induction that $15^r - 1$ is divisible by 7 for every non-negative integer r. [7]

(b) (i) Prove that $\log_a b = \frac{\log_2 b}{\log_2 a}$. [4]

(ii) Solve the equation $3\log_2 x + 2\log_x 2 = 7$. [4]

(iii) Find the value of x satisfying the equation $2^{2x+1} = 3^{4-x}$. [4]

(c) Solve the equation $3^{2x} - 2(3^x) = 3$, for real x. [6]

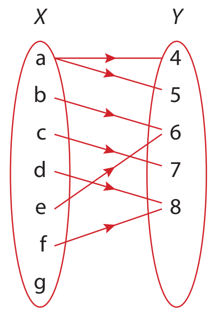
4 (a) Find the range of each of the following functions.

(i) $f: x \to x^2 + 5x + 6, x \in \mathbb{R}$ [3]

(ii) $f: x \to \frac{1}{x^2 + 5x + 6}$, where $x \neq -2, x \neq -3, x \in \mathbb{R}$ [3]

(b) A, B and C are constants such that $\frac{4x^2 + 16x + 19}{x^2 + 4x + 4} = \frac{A}{(x+B)^2} + C$, for all real values of x except $x = -2$. Find the values of A, B and C.

Hence, state precisely the sequence of transformations by which the graph of $y = \dfrac{4x^2 + 16x + 19}{x^2 + 4x + 4}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$. [9]



(c) A relation h in which $X = \{a, b, c, d, e, f, g\}$ and $Y = \{4, 5, 6, 7, 8\}$ is given by

(i) Express h as a set of ordered pairs. [4]

(ii) State two reasons why h is not a function. [2]

(iii) Construct a function $h: X \to Y$ as a set of ordered pairs. [4]

Module 1 Test 2

1 (a) The operation multiplication modulo 8 is defined on two integers a, b as follows: the remainder of $\dfrac{a \times b}{8}$, i.e. multiply the two integers and take the remainder when the product is divided by 8. The set S is $\{1, 3, 5, 7\}$.

(i) Show that S is closed under the operation multiplication modulo 8. [4]

(ii) Identify the identity in the set S. [2]

(iii) Find the inverse of each element in S. [4]

(b) Show that $\log x + \log x^2 + \log x^3 + \ldots \log x^n = \frac{1}{2}n(n + 1)\log x$. [6]

(c) The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} 4x + 1, & x \le 0 \\ 2x^2 + 1, & 0 < x < 4 \\ x^3, & x \ge 4 \end{cases}$$

Find the following.

(i) $f(-1)$

(ii) $f(2)$

(iii) $f(5)$

(iv) $ff(3)$ [9]

2 (a) Prove by mathematical induction that $\displaystyle\sum_{r=1}^{r=n} \frac{r^2 + r - 1}{r(r + 1)} = \frac{n^2}{n + 1}$. [7]

(b) Prove, using the principle of mathematical induction, that for any integer $n \ge 1$, $n^4 + 3n^2$ is divisible by 4. [8]

(c) The propositions p, q and r are defined as follows:

p: the examination is difficult

q: the pass mark is 50

r: I will pass

Write each of the following as a sentence in words.

(i) $q \Rightarrow r$

(ii) $p \wedge \sim r$ [5]

(d) Let t be 'Tobago is beautiful' and z be 'Zico likes Tobago'. Write the negation of each of the following as a sentence in words.

(i) $t \wedge \sim z$

(ii) $\sim t \wedge \sim z$ [5]

3 (a) Given the function f: $x \to x^2 - 4x + 7, x \in \mathbb{R}$.

(i) Write the function in the form $(x - a)^2 + b$, where a and b are integers. [3]

(ii) Is the function one-to-one? Give a reason for your answer. [2]

(iii) Is the function onto? Give a reason for your answer. [2]

(iv) If the function is not bijective, restrict the domain and range to form a bijective function g: $x \to x^2 - 4x + 7, x \in \mathbb{R}, x > k$, identifying k. [3]

(v) Find g^{-1}. [4]

(b) The roots of the cubic equation $2x^3 - 5x^2 + 6x + 2 = 0$ are α, β and γ.

(i) Find $\alpha^2 + \beta^2 + \gamma^2$. [2]

(ii) Find $\alpha^3 + \beta^3 + \gamma^3$. [4]

(iii) Find a cubic equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. [5]

4 (a) Solve the inequality $-1 < \frac{4x + 1}{x - 2} < 1$. [6]

(b) Solve the equation $|2x - 1|^2 - 6|2x - 1| + 8 = 0$. [8]

(c) Solve the equation $3^{3x} - 9(3^{2x}) + 26(3^x) - 24 = 0$. [11]

2

Trigonometry and Plane Geometry

CHAPTER 9

Trigonometry

At the end of this chapter you should be able to:

- Prove and use the identity $\sin^2\theta + \cos^2\theta = 1$
- Prove and use the identity $1 + \cot^2\theta = \text{cosec}^2\theta$
- Prove and use the identity $\tan^2\theta + 1 = \sec^2\theta$
- Solve trigonometric equations involving quadratics
- Prove and use $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$
- Use $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$
- Use $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$
- Prove and use the double angle results
- Convert $a\cos + b\sin\theta$ to $R\cos(\theta - \alpha)$, $R > 0$, $0° < \alpha < 90°$
- Identify the maximum and minimum of $a\cos\theta + b\sin\theta$
- Identify the angle at which $a\cos\theta + b\sin\theta$ is a maximum or minimum
- Solve equations of the form $a\cos\theta + b\sin\theta = c$, $c \neq 0$
- Solve equations using the double angle results
- Convert products to sums and differences
- Convert sums and differences to products

KEYWORDS/TERMS

trigonometric identity • double angle • maximum • minimum • periodicity • symmetry • amplitude

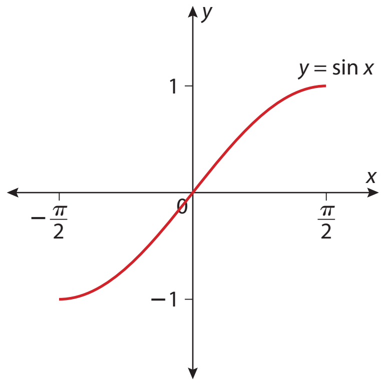
Inverse trigonometric functions and graphs

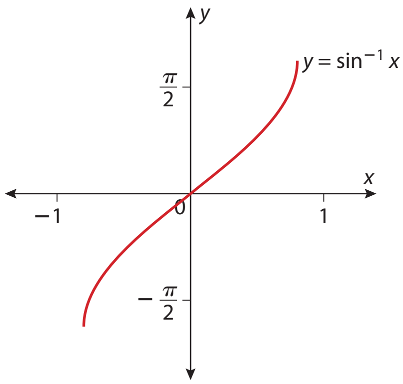
Inverse sine function

The inverse sine trigonometric function is represented by $\sin^{-1} x$ or $\arcsin x$.

Let $f(x) = \sin x$, $f^{-1}(x) = \arcsin x$. Since the function must be one-to-one and onto, $\arcsin x$ is defined over $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

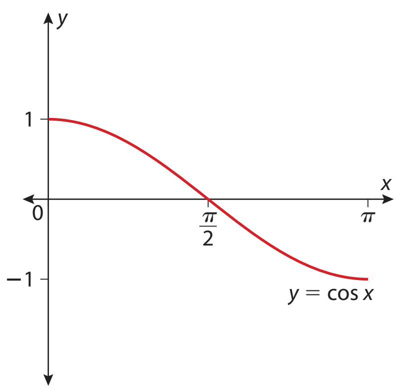
Reflecting $y = \sin x$ in the line $y = x$, the graph of $y = \arcsin x$ is this.

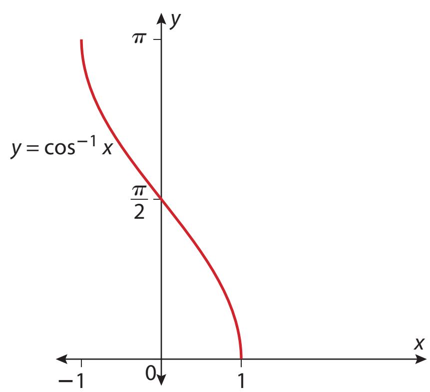




Inverse cosine function

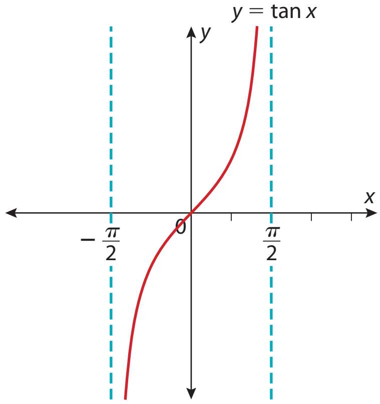
The inverse cosine function is denoted by $\cos^{-1} x$ or $\arccos x$. The principle values of $y = \cos x$ are $0 \leq x \leq \pi$, we can derive the graph of $y = \arccos x$ by reflecting $y = \cos x$ in the line $y = x$.

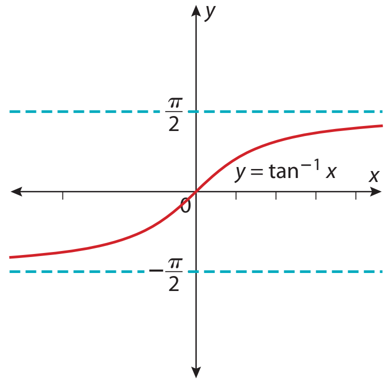




Inverse tangent function

By reflecting the graph of $y = \tan x$ in the line $y = x$ we arrive at the graph of $\arctan x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.





Solving simple trigonometric equations

Graphical solution of $\sin x = k$

EXAMPLE 1 Solve graphically the equation $\sin x = k$ where $-1 \le k \le 1$.

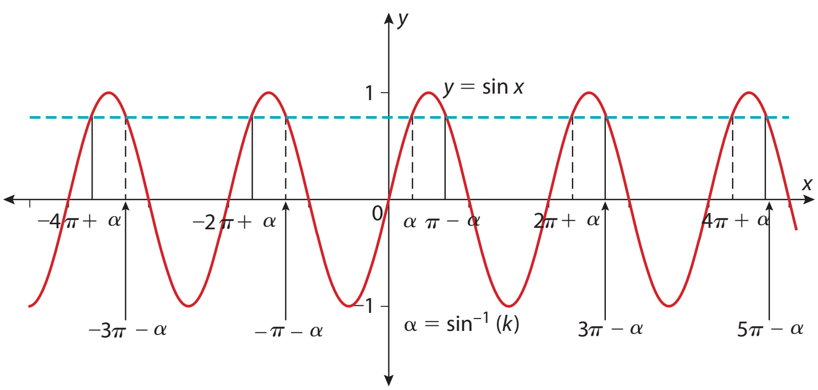
SOLUTION The solution occurs at the point of intersections of the graphs of $y = \sin x$ and $y = k$.

Draw the graph of $y = \sin x$

On the same axes draw the graph of $y = k$.

$\sin x = k \Rightarrow x = \sin^{-1} k$

Let $\sin^{-1} k = \alpha$.



Therefore, $x = \alpha$ is a solution to the equation.

Reading off the solutions from the graph we have:

$\ldots -4\pi + \alpha, -3\pi - \alpha, -2\pi + \alpha, -\pi - \alpha, \alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha \ldots$

We can write this as $x = n\pi + (-1)^n\alpha$, where $n \in \mathbb{Z}$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$.

This is called the general solution of the equation $\sin x = k$.

EXAMPLE 2 Find the general solution of $\sin x = \frac{1}{2}$.

SOLUTION Using $x = n\pi + (-1)^n\alpha$, where $n \in \mathbb{Z}$, and where $\alpha = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, we get:

$x = n\pi + (-1)^n \frac{\pi}{6}$ where $n \in \mathbb{Z}$

EXAMPLE 3 Find the general solution of $\sin 2x = \frac{1}{\sqrt{2}}$.

SOLUTION The general solution is $x = n\pi + (-1)^n\alpha$, where $n \in \mathbb{Z}$.

Since $\sin 2x = \frac{1}{\sqrt{2}}$.

$2x = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

Therefore, $\alpha = \frac{\pi}{4}$.

Hence, $2x = n\pi + (-1)^n\frac{\pi}{4}$ where $n \in \mathbb{Z}$.

$\therefore\ x = \frac{n\pi}{2} + (-1)^n\frac{\pi}{8}$ where $n \in \mathbb{Z}$.

EXAMPLE 4 Find the general solution of the equation $\sin\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$.

SOLUTION Use $x = n\pi + (-1)^n\alpha$, where $n \in \mathbb{Z}$.

$\sin\left(3x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$\Rightarrow 3x - \frac{\pi}{6} = \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}$

Therefore, $3x - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{3}, n \in \mathbb{Z}$.

Let us separate the result and find x, when $n = 2p, p \in \mathbb{Z}$.

> **Note**
>
> $(-1)^n$ is positive when n is even and negative when n is odd. We can split the solution by writing n is odd and even.

We get:

$3x - \frac{\pi}{6} = 2p\pi + (-1)^{2p}\frac{\pi}{3}, p \in \mathbb{Z}, (-1)^{2p} = [(-1^{\,2})]^p = 1$

$3x - \frac{\pi}{6} = 2p\pi + \frac{\pi}{3}, p \in \mathbb{Z}$

$3x = 2p\pi + \frac{\pi}{3} + \frac{\pi}{6}, p \in \mathbb{Z}$

$3x = 2p\pi + \frac{\pi}{2}, p \in \mathbb{Z}$

$x = \frac{2p\pi}{3} + \frac{\pi}{6}, p \in \mathbb{Z}$

When $n = 2p + 1, p \in \mathbb{Z}$:

$3x - \frac{\pi}{6} = (2p + 1)\pi + (-1)^{2p+1}\frac{\pi}{3}, p \in \mathbb{Z}, (-1)^{2p+1} = (-1)^1 \times (-1)^{2p}$

$= (-1) \times (1) = -1$

$3x - \frac{\pi}{6} = (2p + 1)\pi - \frac{\pi}{3}, p \in \mathbb{Z}$

$3x = (2p + 1)\pi - \frac{\pi}{3} + \frac{\pi}{6}, p \in \mathbb{Z}$

$3x = (2p + 1)\pi - \frac{\pi}{6}, p \in \mathbb{Z}$

$x = \frac{(2p + 1)\pi}{3} - \frac{\pi}{18}, p \in \mathbb{Z}$

Hence, the general solution is:

$x = \frac{2p\pi}{3} + \frac{\pi}{6}, p \in \mathbb{Z}$

$x = \frac{(2p + 1)\pi}{3} - \frac{\pi}{18}, p \in \mathbb{Z}$

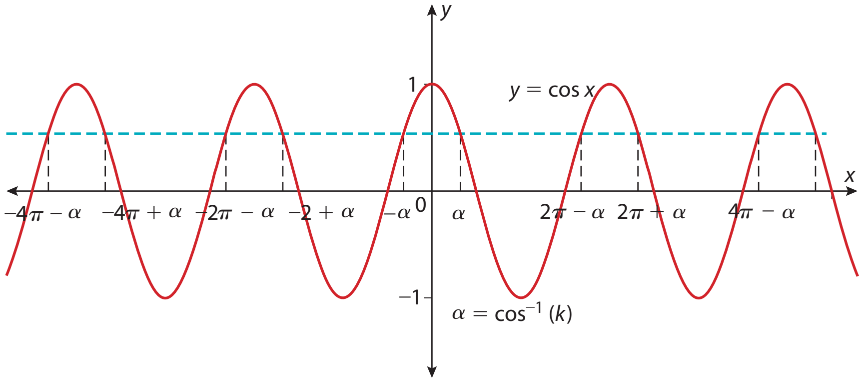
Graphical solution of $\cos x = k$

EXAMPLE 5 Solve graphically the equation $\cos x = k$ where $-1 \le k \le 1$.

SOLUTION The solution occurs at the point of intersections of the graphs of $y = \cos x$ and $y = k$.

Draw the graph of $y = \cos x$.

On the same axes, draw the graph of $y = k$.



$\cos x = k \Rightarrow x = \cos^{-1} k$

Let $\cos^{-1}(k) = \alpha$.

Therefore, $x = \alpha$ is a solution to the equation.

Reading off the solutions from the graph we have:

$\ldots -4\pi - \alpha, -4\pi + \alpha, -2\pi + \alpha, -2\pi - \alpha, \alpha, 2\pi - \alpha, 2\pi + \alpha, 4\pi - \alpha, 4\pi + \alpha \ldots$

We can write these solutions as $x = 2n\pi \pm \alpha$, where $n \in \mathbb{Z}$.

EXAMPLE 6 Find the general solution of $\cos x = \frac{1}{3}$.

SOLUTION First we find α.

Since $\cos x = \frac{1}{3}$, $\alpha = \cos^{-1}\left(\frac{1}{3}\right) = 70.5°$

General solution: $x = 360°n \pm \alpha$

Hence, the general solution is $x = 360°n \pm 70.5°$, where $n \in \mathbb{Z}$.

EXAMPLE 7 Solve the equation $\cos 3x = 0.5$.

SOLUTION First we find α.

Since $\cos 3x = 0.5$, $\alpha = \cos^{-1}(0.5) = 60°$

Hence, the general solution is $3x = 360°n \pm 60°$, where $n \in \mathbb{Z}$.

$x = \frac{360°}{3}n \pm \frac{60°}{3}, n \in \mathbb{Z}$

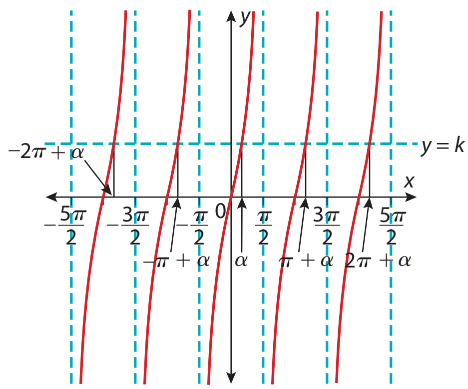
$\therefore x = 120°n \pm 20°, n \in \mathbb{Z}$.

Graphical solution of $\tan x = k$

EXAMPLE 8 Solve graphically the equation $\tan x = k$ where $-1 \leq k \leq 1$.

SOLUTION The solution occurs at the point of intersections of the graphs of $y = \tan x$ and $y = k$.

Draw the graph of $y = \tan x$



On the same axes draw the graph of $y = k$.

Since $\tan x = k \Rightarrow x = \tan^{-1} k$.

Let $\tan^{-1}(k) = \alpha$, therefore $x = \alpha$ is a solution to the equation.

Reading off the points of intersections of the two graphs, the solutions are:

$\ldots -3\pi + \alpha - 2\pi + \alpha, -\pi + \alpha, \alpha, \pi + \alpha, 2\pi + \alpha, 3\pi + \alpha, 4\pi + \alpha \ldots$

We can write these solutions as $x = n\pi + \alpha$, where $n \in \mathbb{Z}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$.

EXAMPLE 9 Find the general solution of $\tan x = 1$.

SOLUTION The general solution of $\tan x = k$ is $x = n\pi + \alpha$, where $n \in \mathbb{Z}$.

Since $\tan x = 1$, $\alpha = \tan^{-1} 1 = \frac{\pi}{4}$.

$x = n\pi + \frac{\pi}{4}$, where $n \in \mathbb{Z}$.

EXAMPLE 10 Find the general solution of $2 \sin 2x = \cos 2x$.

SOLUTION

$2 \sin 2x = \cos 2x$

$\frac{\sin 2x}{\cos 2x} = \frac{1}{2}$ (Dividing by $2 \cos 2x$)

$\tan 2x = \frac{1}{2}$

$2x = \tan^{-1}\left(\frac{1}{2}\right) = 26.6°$

General solution:

$2x = 180°n + 26.6°$, $n \in \mathbb{Z}$.

Therefore, $x = \frac{180°n}{2} + \frac{26.6°}{2}$, $n \in \mathbb{Z}$

Hence, $x = 90°n + 13.3°$, $n \in \mathbb{Z}$.

Try these 9.1 Find the general solution of

(a) $\sin 4x = 0.28$

(b) $\cos(x + 30°) = 0.6$

(c) $\tan(2x + 45°) = 0.7$.

EXERCISE 9A

In questions 1 to 6, find the general solution of the following equations, giving your answer in radians.

1 $\sin 2\theta = -\frac{1}{2}$

2 $\cos 3\theta = 0$

3 $\tan\left(2\theta + \frac{\pi}{3}\right) = 1$

4 $\cos\left(2\theta - \frac{\pi}{4}\right) = \frac{1}{2}$

5 $\sin\left(3\theta - \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

6 $\tan\left(\frac{1}{2}\theta - \frac{\pi}{2}\right) = -1$

7 Find the general solution of the equation $\sin 3x = 3\cos 3x$.

Trigonometrical identities

Reciprocal identities

Expressions involving cosecant, secant and cotangent can be changed to expressions involving sine, cosine or tangent respectively. When these expressions are changed, we are using the reciprocal identities. The reciprocal identities can be used to simplify other expressions and to solve trigonometric equations by changing the expressions to sines and/or cosines only. The list below gives these:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Every trigonometric ratio can be changed to a combination of sines and/or cosines.

EXAMPLE 11 Show that $\cot\theta\sec\theta = \operatorname{cosec}\theta$.

SOLUTION

$$\cot\theta\sec\theta = \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos\theta} \qquad \left(\text{Since } \cot\theta = \frac{\cos\theta}{\sin\theta} \text{ and } \sec\theta = \frac{1}{\cos\theta}\right)$$

$$= \frac{1}{\sin\theta}$$

$$= \operatorname{cosec}\theta \qquad \text{Q.E.D.}$$

EXAMPLE 12 Prove the identity $\sec\theta\operatorname{cosec}\theta\cot\theta = \operatorname{cosec}^2\theta$.

SOLUTION

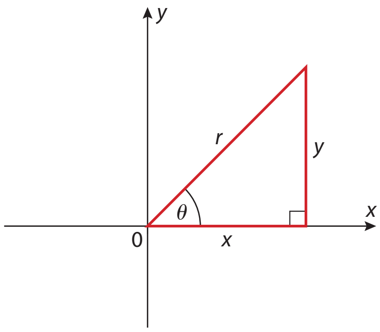
PROOF

Since $\sec\theta = \frac{1}{\cos\theta}$, $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ and $\cot\theta = \frac{\cos\theta}{\sin\theta}$:

$$\begin{aligned}\sec\theta\operatorname{cosec}\theta\cot\theta &= \frac{1}{\cos\theta}\times\frac{1}{\sin\theta}\times\frac{\cos\theta}{\sin\theta}\\ &= \frac{1}{\sin^2\theta}\\ &= \operatorname{cosec}^2\theta \qquad \text{Q.E.D}\end{aligned}$$

Pythagorean identities

The Pythagorean identities are some of the most useful identities in trigonometry because they can be used to simplify more complicated expressions. The identities are derived from the right-angled triangle and the reciprocal identities. Three basic properties of the trigonometric ratios are derived from Pythagoras' theorem as follows.



For all values of θ: $r^2 = x^2 + y^2$

$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

Since $\sin\theta = \frac{y}{r}$

$\sin^2\theta = \frac{y^2}{r^2}$ [1] (Squaring both sides)

$\cos^2\theta = \frac{x^2}{r^2}$ [2] (Squaring both sides)

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \qquad [1] + [2]\\ &= \frac{x^2 + y^2}{r^2}\end{aligned}$$

Since $r^2 = x^2 + y^2$:

$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \frac{r^2}{r^2}\\ &= 1\end{aligned}$$

Therefore, $\sin^2\theta + \cos^2\theta = 1$ holds true for all values of θ and is called a **trigonometric identity**.

We can use this to derive other trigonometric identities.

Since $\sin^2\theta + \cos^2\theta = 1$,

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$ (Dividing by $\cos^2\theta$)

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$

Dividing $\sin^2\theta + \cos^2\theta = 1$ by $\sin^2\theta$ gives:

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\Rightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

The fundamental trigonometric identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

The fundamental trigonometric identities can be used to prove other identities and solve some trigonometric equations.

Proving identities

When proving trigonometric identities, follow these steps.

(i) Start with the side containing the most information.

(ii) Keep an eye on what you are proving. Replace any identities to reach what you want.

(iii) Combine the sum or difference of fractions to form one fraction whenever possible.

(iv) Keep your eye on the identity you are proving. Keep looking back to the question.

(v) Know the fundamental trigonometric identities.

EXAMPLE 13 Prove the identity $\dfrac{1+\cos x}{1+\sec x} = \cos x$.

SOLUTION

PROOF

The left-hand side has more information. Let us start on this side: $\dfrac{1+\cos x}{1+\sec x}$

We can replace $\sec x$ with $\dfrac{1}{\cos x}$:

$$\Rightarrow \frac{1+\cos x}{1+\sec x} = \frac{1+\cos x}{1+\dfrac{1}{\cos x}}$$

$$= \frac{1+\cos x}{\dfrac{\cos x + 1}{\cos x}} \qquad \left(\text{Since } 1 + \frac{1}{\cos x} = \frac{\cos x + 1}{\cos x}\right)$$

$$= (1+\cos x) \times \frac{\cos x}{1+\cos x}$$

$$= \cos x = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 14 Prove the identity $\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = 2\,\text{cosec}^2\,\theta$.

SOLUTION

PROOF

Starting on the left-hand side, since there is more information to work with:

$\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta}$.

Find the lowest common multiple (LCM) by multiplying the two denominators $(1 + \cos\theta)(1 - \cos\theta)$. Bring together as one fraction:

$$\frac{1}{1 + \cos\theta} + \frac{1}{1 - \cos\theta} = \frac{1 - \cos\theta + 1 + \cos\theta}{(1 + \cos\theta)(1 - \cos\theta)}$$

Expanding the denominator:

$$\begin{aligned}\frac{1 - \cos\theta + 1 + \cos\theta}{(1 + \cos\theta)(1 - \cos\theta)} &= \frac{2}{1 - \cos^2\theta} \\ &= \frac{2}{\sin^2\theta} && (\text{Since } 1 - \cos^2\theta = \sin^2\theta) \\ &= 2\,\text{cosec}^2\,\theta && \left(\text{Since } \frac{1}{\sin^2\theta} = \text{cosec}^2\,\theta\right) \\ &= \text{right-hand side} && \text{Q.E.D.}\end{aligned}$$

EXAMPLE 15 Prove the identity $\cot x + \tan x = \text{cosec}\,x \sec x$.

SOLUTION

PROOF

It is normally easier to convert a sum or difference to a product. We start with the left-hand side.

Convert $\cot x$ and $\tan x$ to $\frac{\cos x}{\sin x}$ and $\frac{\sin x}{\cos x}$ respectively.

$$\cot x + \tan x = \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x}$$

Find the LCM.

$$\frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x} \qquad \left(\text{Since } \frac{\cos x}{\sin x} + \frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\sin x \cos x}\right)$$

Replace with $\cos^2 x + \sin^2 x = 1$.

$$\begin{aligned}\frac{\cos^2 x + \sin^2 x}{\sin x \cos x} &= \frac{1}{\sin x \cos x} \\ &= \frac{1}{\sin x} \times \frac{1}{\cos x} \\ &= \text{cosec}\,x \sec x = \text{right-hand side} && \text{Q.E.D.}\end{aligned}$$

EXAMPLE 16 Prove the identity $\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{2}{\sin x}$.

SOLUTION

PROOF

Start with the left-hand side, and bring together to form one fraction:

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{(\sin x)(\sin x) + (1 + \cos x)(1 + \cos x)}{(1 + \cos x)(\sin x)}$$

$$= \frac{\sin^2 x + 1 + 2\cos x + \cos^2 x}{(1 + \cos x)(\sin x)}$$

$$= \frac{2 + 2\cos x}{(1 + \cos x)\sin x} \qquad (\text{Since } \sin^2 x + \cos^2 x = 1)$$

$$= \frac{2(1 + \cos x)}{(1 + \cos x)\sin x}$$

$$= \frac{2}{\sin x} = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 17 Show that $\cos^4\theta - \sin^4\theta = 1 - 2\sin^2\theta$.

SOLUTION We know that $x^4 - y^4 = (x^2 - y^2)(x^2 + y^2)$.

$\therefore \cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= \cos^2\theta - \sin^2\theta$ (Use $\cos^2\theta + \sin^2\theta = 1$)

$= 1 - \sin^2\theta - \sin^2\theta$ (Substitute $\cos^2\theta = 1 - \sin^2\theta$)

$= 1 - 2\sin^2\theta =$ right-hand side Q.E.D.

EXAMPLE 18 Prove the identity $\dfrac{\cos^2\theta}{1 - \sin\theta} = 1 + \sin\theta$.

SOLUTION

PROOF

Starting with the left-hand side and using $\cos^2\theta = 1 - \sin^2\theta$:

$$\frac{\cos^2\theta}{1 - \sin\theta} = \frac{1 - \sin^2\theta}{1 - \sin\theta}$$

Factorising using $1 - \sin^2\theta = (1 - \sin\theta)(1 + \sin\theta)$ gives:

$$\frac{1 - \sin^2\theta}{1 - \sin\theta} = \frac{\cancel{(1 - \sin\theta)}(1 + \sin\theta)}{\cancel{1 - \sin\theta}}$$

$$= 1 + \sin\theta = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 19 Prove the identity $\dfrac{1 - \cot^2 x}{1 + \cot^2 x} = 1 - 2\cos^2 x$.

SOLUTION

PROOF

Starting on the left-hand side and using $1 + \cot^2 x = \operatorname{cosec}^2 x$:

$$\frac{1 - \cot^2 x}{1 + \cot^2 x} = \frac{1 - \cot^2 x}{\operatorname{cosec}^2 x}$$

Since $\cot^2 x = \dfrac{\cos^2 x}{\sin^2 x}$ and $\operatorname{cosec}^2 x = \dfrac{1}{\sin^2 x}$:

$$\Rightarrow \frac{1 - \cot^2 x}{\operatorname{cosec}^2 x} = \frac{1 - \dfrac{\cos^2 x}{\sin^2 x}}{\dfrac{1}{\sin^2 x}}$$

$$= \frac{\frac{\sin^2 x - \cos^2 x}{\sin^2 x}}{\frac{1}{\sin^2 x}} \qquad \left(\text{Since } 1 - \frac{\cos^2 x}{\sin^2 x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x}\right)$$

$$= \sin^2 x - \cos^2 x$$

$$= 1 - \cos^2 x - \cos^2 x$$

$$= 1 - 2\cos^2 x = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 20 Prove the identity $\frac{1 - \sin\theta}{1 + \sin\theta} = (\sec\theta - \tan\theta)^2$.

SOLUTION

PROOF

Starting with the left-hand side and multiplying by $\frac{1 - \sin\theta}{1 - \sin\theta}$:

$$\frac{1 - \sin\theta}{1 + \sin\theta} = \frac{1 - \sin\theta}{1 + \sin\theta} \times \frac{1 - \sin\theta}{1 - \sin\theta}$$

$$= \frac{1 - 2\sin\theta + \sin^2\theta}{1 - \sin^2\theta}$$

$$= \frac{1 - 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} - \frac{2\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sec^2\theta - \frac{2\sin\theta}{\cos\theta \times \cos\theta} + \tan^2\theta$$

$$= \sec^2\theta - \left(\frac{2\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}\right) + \tan^2\theta$$

$$= \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta$$

$$= (\sec\theta - \tan\theta)(\sec\theta - \tan\theta) \qquad \text{(Factorising)}$$

$$= (\sec\theta - \tan\theta)^2 = \text{right-hand side} \qquad \text{Q.E.D.}$$

Let us see what happens when we start on the right-hand side.

$$(\sec\theta - \tan\theta)^2 = (\sec\theta - \tan\theta)(\sec\theta - \tan\theta)$$

$$= \sec^2\theta - 2\sec\theta\tan\theta + \tan^2\theta \qquad \text{(Expanding the brackets)}$$

$$= \frac{1}{\cos^2\theta} - \left(\frac{2}{\cos\theta} \times \frac{\sin\theta}{\cos\theta}\right) + \frac{\sin^2\theta}{\cos^2\theta} \qquad \text{(Changing to } \sin\theta \text{ and } \cos\theta\text{)}$$

$$= \frac{1 - 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

$$= \frac{(1 - \sin\theta)(1 - \sin\theta)}{1 - \sin^2\theta} \qquad \text{(Replacing } \cos^2\theta = 1 - \sin^2\theta\text{)}$$

$$= \frac{(1 - \sin\theta)(1 - \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)} \qquad \text{(Since } 1 - \sin^2\theta = (1 - \sin\theta)(1 + \sin\theta)\text{)}$$

$$= \frac{1 - \sin\theta}{1 + \sin\theta} = \text{left-hand side} \qquad \text{Q.E.D.}$$

Notice that we can trace our steps back and forth when using the left-hand side or right-hand side.

Try these 9.2 Prove the following identities.

(a) $\dfrac{\sec\theta - \cos\theta}{\sec\theta + \cos\theta} = \dfrac{\sin^2\theta}{1 + \cos^2\theta}$

(b) $\dfrac{1 - \cos^2\theta}{\sin\theta} = \sin\theta$

(c) $\dfrac{\sec\theta + \tan\theta}{\cot\theta + \cos\theta} = \dfrac{\sin\theta}{\cos^2\theta}$

Solving trigonometric equations

We can use the trigonometric identities we know to solve equations.

EXAMPLE 21 Solve the equation $2\cos^2 x - 3\sin x - 3 = 0$ for $0° \leq x \leq 360°$.

SOLUTION $2\cos^2 x - 3\sin x - 3 = 0$

We can form a quadratic equation in $\sin x$ by replacing $\cos^2 x = 1 - \sin^2 x$.

$\Rightarrow 2(1 - \sin^2 x) - 3\sin x - 3 = 0$

$\Rightarrow 2 - 2\sin^2 x - 3\sin x - 3 = 0$

$\Rightarrow -2\sin^2 x - 3\sin x - 1 = 0$

$\Rightarrow 2\sin^2 x + 3\sin x + 1 = 0$

Let $y = \sin x$

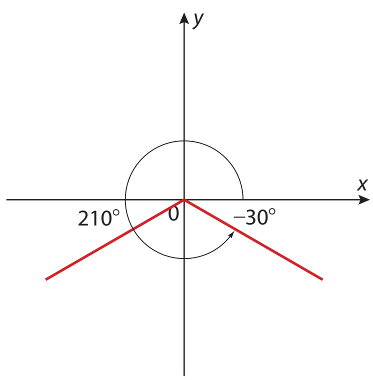
$\therefore 2y^2 + 3y + 1 = 0$

Factorising gives:

$(2y + 1)(y + 1) = 0$

$\therefore 2y + 1 = 0, y + 1 = 0$

$\Rightarrow y = -\frac{1}{2}, y = -1$



For $y = \sin x = -\frac{1}{2}$:

$\sin x = -\frac{1}{2}$

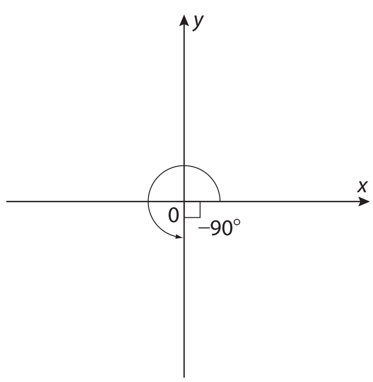
$\Rightarrow x = \sin^{-1}\left(-\frac{1}{2}\right)$

$\Rightarrow x = -30°, 330°, 210°$

For $y = \sin x = -1$:

$\Rightarrow x = \sin^{-1}(-1)$

$\Rightarrow x = -90°, 270°$



Since $0° \leq x \leq 360°$

$x = 210°, 270°, 330°$

Omit all solutions outside of the range 0° to 360°.

EXAMPLE 22 Solve the equation $3 - 3\cos\theta = 2\sin^2\theta$, giving values of θ from 0° to 360° inclusive.

SOLUTION

$3 - 3\cos\theta = 2\sin^2\theta$

We can form a quadratic in $\cos\theta$ by replacing $\sin^2\theta = 1 - \cos^2\theta$.

$\Rightarrow 3 - 3\cos\theta = 2(1 - \cos^2\theta)$

$\Rightarrow 3 - 3\cos\theta = 2 - 2\cos^2\theta$

$\Rightarrow 2\cos^2\theta - 3\cos\theta + 1 = 0$

Let $y = \cos\theta$.

The equation becomes $2y^2 - 3y + 1 = 0$.

Factorising gives:

$(2y - 1)(y - 1) = 0$

$\Rightarrow 2y - 1 = 0, y - 1 = 0$

$\Rightarrow y = \frac{1}{2}, y = 1$

For $y = \cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

Since $\cos x$ is positive in the first and fourth quadrants:

$\theta = 60°, 300°$

For $y = \cos\theta = 1$:

$\theta = \cos^{-1}(1)$

$\Rightarrow \theta = 0°, 360°$

$\therefore\ \theta = 0°, 60°, 300°, 360°$ for $0° \leq \theta \leq 360°$

EXAMPLE 23 Solve the equation $2\sin^2\theta + 1 = 3\sin\theta$.

SOLUTION

$2\sin^2\theta + 1 = 3\sin\theta$

$\Rightarrow 2\sin^2\theta - 3\sin\theta + 1 = 0$

Let $y = \sin\theta$.

$\Rightarrow 2y^2 - 3y + 1 = 0$

Factorising gives:

$(2y - 1)(y - 1) = 0$

$\Rightarrow 2y - 1 = 0, y - 1 = 0$

$\Rightarrow y = \frac{1}{2}, y = 1$

For $y = \sin\theta = \frac{1}{2}$:

$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$

$\sin\theta = \frac{1}{2} \Rightarrow \theta = n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in \mathbb{Z}$

$\sin\theta = 1 \Rightarrow \theta = n\pi + (-1)^n\left(\frac{\pi}{2}\right), n \in \mathbb{Z}$

EXAMPLE 24 Find the general solution of the equation $\operatorname{cosec}^2\theta = 3 + \cot\theta$.

SOLUTION

$\operatorname{cosec}^2\theta = 3 + \cot\theta$

We can form a quadratic in $\cot\theta$ by replacing $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

$\Rightarrow 1 + \cot^2\theta = 3 + \cot\theta$

$\therefore \cot^2\theta - \cot\theta - 2 = 0$

Let $y = \cot\theta$:

$\Rightarrow y^2 - y - 2 = 0$

Factorising gives: $(y - 2)(y + 1) = 0$

$\Rightarrow y - 2 = 0, y + 1 = 0$

$\Rightarrow y = 2, y = -1$

For $y = \cot\theta = 2$:

$$\frac{1}{\tan\theta} = 2$$

$\Rightarrow \tan\theta = \frac{1}{2}$

$\Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right)$

$\Rightarrow \theta = 26.6°$

General solution is:

$\theta = 180°n + 26.6°, n \in \mathbb{Z}$

For $y = \cot\theta = -1$:

$$\frac{1}{\tan\theta} = -1$$

$\Rightarrow \theta = -45°$

General solution is:

$\theta = 180°n + (-45°), n \in \mathbb{Z}$

$\quad = 180°n - 45°$

Hence, $\theta = 180°n + 26.6°$ and $180°n - 45°$, where $n \in \mathbb{Z}$.

EXAMPLE 25 Solve the equation $3\sec^2 x - 2\tan x - 8 = 0$, giving values of x from 0° to 360° inclusive.

SOLUTION

$3\sec^2 x - 2\tan x - 8 = 0$

We can form a quadratic equation in $\tan x$ by replacing $\sec^2 x = 1 + \tan^2 x$.

$\Rightarrow 3(1 + \tan^2 x) - 2\tan x - 8 = 0$

$\qquad 3\tan^2 x - 2\tan x - 5 = 0$

Let $y = \tan x$,

$\therefore 3y^2 - 2y - 5 = 0$

Factorising gives:

$(3y - 5)(y + 1) = 0$

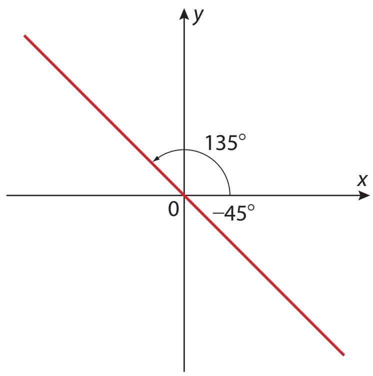
$\Rightarrow 3y - 5 = 0, y + 1 = 0$

$\Rightarrow y = \frac{5}{3}, y = -1$

For $y = \tan x = \frac{5}{3}$

$x = \tan^{-1}\left(\frac{5}{3}\right)$

Since $\tan x$ is positive in the first and third quadrants, $x = 59°, 239°$.



$\tan x = -1$

$x = \tan^{-1}(-1)$

Since $\tan x$ is negative in the second and fourth quadrants, $x = 135°, 315°$.

Hence, $x = 59°, 135°, 239°, 315°$.

EXAMPLE 26 Solve the equation $2\tan^2\theta + \sec\theta = 1$ for $0° \leq \theta \leq 180°$.

SOLUTION

$2\tan^2\theta + \sec\theta = 1$

We can write this equation as a quadratic equation in $\sec^2\theta$ by replacing $\tan^2\theta = \sec^2\theta - 1$.

$\Rightarrow 2(\sec^2\theta - 1) + \sec\theta = 1$

$\Rightarrow 2\sec^2\theta + \sec\theta - 3 = 0$

Let $y = \sec\theta$.

$\Rightarrow 2y^2 + y - 3 = 0$

Factorising gives:

$(2y + 3)(y - 1) = 0$

$\Rightarrow 2y + 3 = 0, y - 1 = 0$

$\Rightarrow y = -\frac{3}{2}, y = 1$

For $y = \sec\theta = -\frac{3}{2}$:

$\frac{1}{\cos\theta} = -\frac{3}{2}$

$\Rightarrow \cos\theta = -\frac{2}{3}$

$\Rightarrow \theta = \cos^{-1}\left(-\frac{2}{3}\right)$

$= 131.8°$

For $y = \sec\theta = 1$:

$\frac{1}{\cos\theta} = 1$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 0°, 360°$

Hence, $\theta = 131.8°, 360°$.

Note

In order to solve any trigonometric equation, first break it down and rewrite it in terms of the basic trigonometric functions $\sin x$, $\cos x$ and $\tan x$.

Try these 9.3

(a) Solve for $0° \leq \theta \leq 360°$:

(i) $3\sin^2\theta = 1 + \cos\theta$

(ii) $4\operatorname{cosec}^2\theta - 4\cot\theta - 7 = 0$

(b) Find the general solution of $20\sec^2\theta - 3\tan\theta - 22 = 0$.

EXERCISE 9B

In questions **1** to **6**, simplify the expressions.

1 $(\sin\theta + \cos\theta)^2 - 1$

2 $\sin x(\sin x - \cot x \operatorname{cosec} x)$

3 $\sin^4\theta - \cos^4\theta$

4 $\sin^2\theta(\cot^2\theta + \operatorname{cosec}^2\theta)$

5 $\dfrac{\sin^2\theta - 1}{\cos^2\theta}$

6 $\dfrac{\sec^2\theta - \tan^2\theta}{\sin\theta}$

In questions **7** to **16**, prove each identity.

7 $1 - \dfrac{\sin^2\theta}{1 - \cos\theta} = -\cos\theta$

8 $\dfrac{\cos^2\theta}{1 - \sin\theta} - 1 = \sin\theta$

9 $\dfrac{\operatorname{cosec}\theta}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta} = \tan\theta$

10 $\dfrac{1 - \cos^2\theta}{1 - \sec^2\theta} = -\cos^2\theta$

11 $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

12 $\dfrac{\cos x}{1 - \tan x} + \dfrac{\sin x}{1 - \cot x} = \sin x + \cos x$

13 $\dfrac{1 - \cos x}{1 + \cos x} = (\operatorname{cosec} x - \cot x)^2$

14 $\dfrac{1}{\sin x + 1} + \dfrac{1}{1 - \sin x} = 2\sec^2 x$

15 $\sin^4\theta - \sin^2\theta = \cos^4\theta - \cos^2\theta$

16 $\dfrac{\cot^2 x - 1}{\tan^2 x + 1} = -\cot^2 x$

In questions **17** to **24**, find all the angles between 0 and 360° such that the following are true.

17 $4\sec x - \tan x = 6\cos x$

18 $3\tan^2 x - \sec x - 1 = 0$

19 $2\cot^2 x + \operatorname{cosec} x = 1$

20 $3\cos^2 x = 4\sin x - 1$

21 $2\cot x = 3\sin x$

22 $\sin^2 x = 3\cos^2 x + 4\sin x$

23 $2\cos x = \tan x$

24 $2\cot x = 1 + \tan x$

In questions **25** to **28**, find the general solution of each equation.

25 $2 + 3\sin Z = 2\cos^2 Z$

26 $2\cot^2 x + \operatorname{cosec} x = 4$

27 $2\sec x + 3\cos x = 7$

28 $5\cos x = 6\sin^2 x$

Further trigonometrical identities

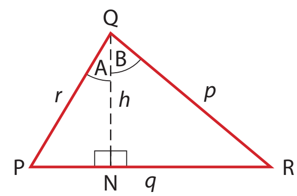
Expansion of sin (A ± B)

EXAMPLE 27 Prove the identity $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

SOLUTION

PROOF

We can split triangle PQR into two right-angled triangles PNQ and QNR.



$\therefore$ area of $\triangle PQR$ = area of $\triangle PNQ$ + area of $\triangle QNR$

Using formula for the area of triangles gives:

area of $\triangle PQR = \frac{1}{2}pr\sin(A + B)$

area of $\triangle PNQ = \frac{1}{2}hr\sin A$

area of $\triangle QNR = \frac{1}{2}ph\sin B$

Hence, $\frac{1}{2}pr\sin(A + B) = \frac{1}{2}hr\sin A + \frac{1}{2}ph\sin B$

$\left[\div\frac{1}{2}pr\right] \Rightarrow \sin(A + B) = \frac{h}{p}\sin A + \frac{h}{r}\sin B$ [1]

From $\triangle PNQ$, $\cos A = \frac{h}{r}$

From $\triangle QNR$, $\cos B = \frac{h}{p}$

Substitute these into (1):

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

> $\sin(A + B)$
> $= \sin A \cos B$
> $+ \cos A \sin B$
>
> $\sin(A - B)$
> $= \sin A \cos B$
> $- \cos A \sin B$

We can use this result to find an identity for $\sin(A - B)$.

Using $\sin(A + B) = \sin A \cos B + \cos A \sin B$ and replacing $B = -B$, we have:

$\sin(A - B) = \sin A \cos(-B) + \cos A \sin(-B)$

Recall that $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$.

$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B$

EXAMPLE 28 Show that $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

SOLUTION Using $\sin(A + B) = \sin A \cos B + \cos A \sin B$, where $A = x$ and $B = \frac{\pi}{2}$, gives:

$\sin\left(x + \frac{\pi}{2}\right) = \sin x \cos\frac{\pi}{2} + \cos x \sin\frac{\pi}{2}$

Since $\sin\frac{\pi}{2} = 1$ and $\cos\frac{\pi}{2} = 0$

$$\begin{aligned}\sin\left(x + \frac{\pi}{2}\right) &= (\sin x) \times 0 + (\cos x) \times 1\\ &= \cos x \qquad \text{Q.E.D.}\end{aligned}$$

EXAMPLE 29 Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$.

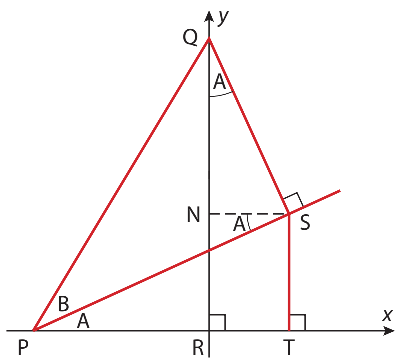
SOLUTION Using $\sin(A - B) = \sin A \cos B - \cos A \sin B$, where $A = x$, $B = \frac{\pi}{2}$, we get:

$$\begin{aligned}\sin\left(x - \frac{\pi}{2}\right) &= \sin x \cos\frac{\pi}{2} - \cos x \sin\frac{\pi}{2}\\ &= (\sin x) \times 0 - (\cos x) \times 1\\ &= -\cos x \qquad \text{Q.E.D.}\end{aligned}$$

Expansion of $\cos(A \pm B)$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

We can prove this geometrically:



Look at the diagram.

From $\triangle PQR$, $\cos(A + B) = \frac{PR}{PQ}$

Now, $PR = PT - RT$

And $RT = NS$

Therefore, $PR = PT - NS$

$$\text{Therefore, } \cos(A+B) = \frac{PT - NS}{PQ}$$
$$= \frac{PT}{PQ} - \frac{NS}{PQ}$$

We can write: $\frac{PT}{PQ} = \frac{PT}{PS} \times \frac{PS}{PQ}$

$$\frac{NS}{PQ} = \frac{NS}{QS} \times \frac{QS}{PQ}$$

$$\text{Therefore, } \cos(A+B) = \frac{PT}{PS} \times \frac{PS}{PQ} - \frac{NS}{QS} \times \frac{QS}{PQ} \qquad [1]$$

From $\triangle PST$, $\cos A = \frac{PT}{PS}$

From $\triangle PQS$, $\cos B = \frac{PS}{PQ}$

From $\triangle QNS$, $\sin A = \frac{NS}{QS}$

From $\triangle PQS$, $\sin B = \frac{QS}{PQ}$

Substituting into [1], we get:

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A+B)$
$= \cos A \cos B$
$- \sin A \sin B$

$\cos(A-B)$
$= \cos A \cos B$
$+ \sin A \sin B$

We can use this result to find an identity for $\cos(A-B)$.

Using $B = -B$, we have:

$\cos(A-B) = \cos A \cos(-B) - \sin A \sin(-B)$

Since $\cos(-B) = \cos B$ and $\sin(-B) = -\sin B$:

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

EXAMPLE 30 Show that $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$.

SOLUTION Using $\cos(A+B) = \cos A \cos B - \sin A \sin B$ where $A = \frac{\pi}{2}$, $B = x$, gives:

$$\cos\left(\frac{\pi}{2} + x\right) = \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x = 0 \times \cos x - 1 \times \sin x$$
$$= -\sin x = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 31 Prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

SOLUTION

PROOF

Using $\cos(A-B) = \cos A \cos B + \sin A \sin B$ where $A = \frac{\pi}{2}$ and $B = x$, gives:

$$\cos\left(\frac{\pi}{2} - x\right) = \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= 0 \cos x \times +1 \times \sin x$$
$$= \sin x = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 32 Find the exact value of $\sin\frac{\pi}{12}$.

SOLUTION

Since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$:

$\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$

Expanding using $\sin(A - B) = \sin A\cos B - \cos A\sin B$, where $A = \frac{\pi}{3}$, $B = \frac{\pi}{4}$, gives:

$\sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$

Recall these: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

So we have:

$$\sin\frac{\pi}{12} = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

EXAMPLE 33 Find the exact value of $\cos\frac{5\pi}{12}$.

SOLUTION

Since $\frac{5\pi}{12} = \frac{\pi}{4} + \frac{\pi}{6}$,

$\cos\left(\frac{5\pi}{12}\right) = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

Expanding gives:

$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$

Since $\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ and $\sin\frac{\pi}{6} = \frac{1}{2}$, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, we have:

$$\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$$

$$= \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$$

To check, recall that $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

Since $\cos\left(\frac{5\pi}{12}\right) = \left(\frac{\pi}{2} - \frac{\pi}{12}\right)$, $\cos\left(\frac{5\pi}{12}\right) = \sin\left(\frac{\pi}{12}\right)$

From Example 22, we know $\sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$.

Therefore, $\cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{2}}{4}(\sqrt{3} - 1)$.

Try these 9.4

Find the exact value of each of the following. Do not use tables or a calculator.

(a) $\cos\frac{7\pi}{12}$

(b) $\sin\frac{5\pi}{12}$

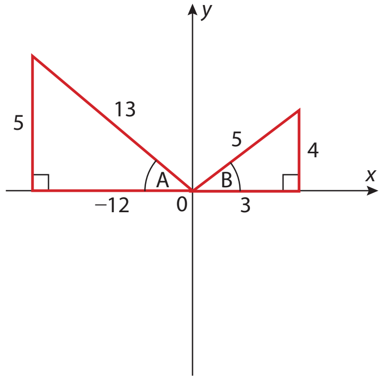
(c) $\sin\frac{7\pi}{12}$

EXAMPLE 34 If $\sin A = \frac{5}{13}$ and $\cos B = \frac{3}{5}$, where A is obtuse and B is acute, find the values of these.

(a) $\sin(A + B)$

(b) $\cos(A - B)$

SOLUTION Since A is obtuse, A will be in the second quadrant. Since B is acute, B lies in the first quadrant.



From the two triangles:

$\sin A = \frac{5}{13}$ $\qquad$ $\sin B = \frac{4}{5}$

$\cos A = -\frac{12}{13}$ $\qquad$ $\cos B = \frac{3}{5}$

(a)
$$\begin{aligned}\sin(A + B) &= \sin A\cos B + \cos A\sin B\\ &= \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) + \left(-\frac{12}{13}\right)\left(\frac{4}{5}\right)\\ &= \frac{15}{65} - \frac{48}{65}\\ &= -\frac{33}{65}\end{aligned}$$

(b)
$$\begin{aligned}\cos(A - B) &= \cos A\cos B + \sin A\sin B\\ &= \left(-\frac{12}{13}\right)\left(\frac{3}{5}\right) + \left(\frac{5}{13}\right)\left(\frac{4}{5}\right)\\ &= -\frac{36}{65} + \frac{20}{65}\\ &= -\frac{16}{65}\end{aligned}$$

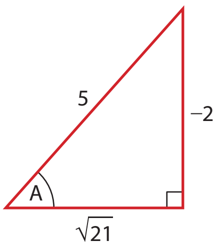
EXAMPLE 35 Given that $\sin A = -\frac{2}{5}$ and $\cos B = \frac{1}{4}$ and that A and B are in the same quadrant, find these.

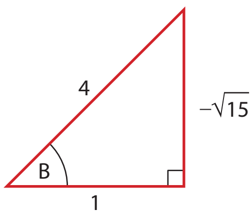
(a) $\sin(A + B)$

(b) $\cos(A + B)$

SOLUTION Since sin A is negative in the third and fourth quadrants and cos B is positive in the first and fourth quadrant, then A and B must be in the fourth quadrant.

> **Remember**
>
> Identify the quadrants that each angle is in and decide on the signs of the ratios involved.





$\sin A = -\frac{2}{5}$ $\qquad$ $\sin B = -\frac{\sqrt{15}}{4}$

$\cos A = \frac{\sqrt{21}}{5}$ $\qquad$ $\cos B = \frac{1}{4}$

(a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(-\frac{2}{5}\right)\left(\frac{1}{4}\right) + \left(\frac{\sqrt{21}}{5}\right)\left(-\frac{\sqrt{15}}{4}\right)$$

$$= -\frac{2}{20} - \frac{\sqrt{21 \times 15}}{20}$$

$$= -\frac{1}{20}(2 + 3\sqrt{35}) \qquad \left(\text{Since } \sqrt{21 \times 15} = \sqrt{3 \times 7 \times 3 \times 5} = \sqrt{9}\,\sqrt{35} = 3\sqrt{35}\right)$$

(b) $\cos(A + B) = \cos A \cos B - \sin A \sin B$

$$= \left(\frac{\sqrt{21}}{5}\right)\left(\frac{1}{4}\right) - \left(-\frac{2}{5}\right)\left(-\frac{\sqrt{15}}{4}\right)$$

$$= \frac{\sqrt{21}}{20} - \frac{\sqrt{15}}{10}$$

$$= \frac{1}{20}(\sqrt{21} - 2\sqrt{15})$$

Expansion of tan (A + B)

We can expand $\tan(A + B)$ and show that $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

PROOF

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)}$$

$$= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \qquad \text{(Expanding } (\sin(A + B) \text{ and } \cos(A + B))\text{)}$$

$$= \frac{\frac{\sin A \cos B}{\cos A \cos B} + \frac{\cos A \sin B}{\cos A \cos B}}{\frac{\cos A \cos B}{\cos A \cos B} - \frac{\sin A \sin B}{\cos A \cos B}} \qquad \text{(Divide the numerator and denominator by } \cos A \cos B\text{)}$$

$$= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}}$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B} = \text{right-hand side}$$

Similarly, we can show that: $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

EXAMPLE 36 Show that $\tan(\pi - \theta) = -\tan \theta$.

SOLUTION Using $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$, with $A = \pi$, $B = \theta$, gives:

$$\tan(\pi - \theta) = \frac{\tan \pi - \tan \theta}{1 + \tan \pi \tan \theta}$$

$$= \frac{0 - \tan \theta}{1 + (0 \times \tan \theta)}$$

$$= -\tan \theta = \text{right-hand side} \qquad \text{Q.E.D.}$$

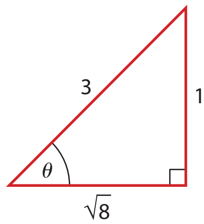
EXAMPLE 37 Given that $\sin\theta = \frac{1}{3}$, $0° < \theta < 90°$, show that $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1}{7}(9 + 4\sqrt{2})$.

SOLUTION Using $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$, where $A = \theta$ and $B = \frac{\pi}{4}$:

$$\tan\left(\theta + \frac{\pi}{4}\right) = \frac{\tan\theta + \tan\frac{\pi}{4}}{1 - \tan\theta\tan\frac{\pi}{4}}$$

$$= \frac{\tan\theta + 1}{1 - \tan\theta} \qquad \left(\text{Since } \tan\frac{\pi}{4} = 1\right)$$

Use the right-angled triangle for $\sin\theta = \frac{1}{3}$.



$$\tan\theta = \frac{1}{\sqrt{8}}$$

$$\therefore \tan\left(\theta + \frac{\pi}{4}\right) = \frac{\frac{1}{\sqrt{8}} + 1}{1 - \frac{1}{\sqrt{8}} \times 1} = \frac{\frac{1 + \sqrt{8}}{\sqrt{8}}}{\frac{\sqrt{8} - 1}{\sqrt{8}}}$$

$$= \frac{1 + \sqrt{8}}{\sqrt{8} - 1}$$

$$= \frac{1 + \sqrt{8}}{\sqrt{8} - 1} \times \frac{\sqrt{8} + 1}{\sqrt{8} + 1}$$

(Rationalise the denominator, i.e., multiply the numerator and denominator by $\sqrt{8} + 1$, which is the conjugate of $\sqrt{8} - 1$)

$$= \frac{1 + 2\sqrt{8} + 8}{8 - 1}$$

$$= \frac{9 + 2\sqrt{8}}{7}$$

$$= \frac{9 + 2\sqrt{4 \times 2}}{7}$$

$$= \frac{9 + 4\sqrt{2}}{7}$$

$$= \frac{1}{7}(9 + 4\sqrt{2})$$

Key points

$\sin(A + B) = \sin A\cos B + \cos A\sin B$

$\sin(A - B) = \sin A\cos B - \cos A\sin B$

$\cos(A + B) = \cos A\cos B - \sin A\sin B$

$\cos(A - B) = \cos A\cos B + \sin A\sin B$

$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

EXERCISE 9C

1 Find the value of $\sin 75°$ in surd form, simplifying your answer.

2 Given $\frac{\sin(A - B)}{\sin(A + B)} = \frac{5}{13}$, that show that $4\tan A = 9\tan B$.

3 Given that $\tan\alpha = \frac{5}{12}$, $\cos\beta = -\frac{3}{5}$ and that α is in the third quadrant and β in the second quadrant, calculate, without the use of a calculator, the values of the following.

(a) $\sin(\alpha + \beta)$ (b) $\tan(\alpha - \beta)$ (c) $\cos(\alpha - \beta)$

4 Prove that the identity $\sin(\theta + 30°) + \sqrt{3}\cos(\theta + 30°) = 2\cos\theta$ is correct, where θ is measured in degrees.

5 The angle θ, measured in degrees, satisfies the equation $\sin(\theta + 30°) = 2\cos(\theta + 60°)$. Show that this equation may be simplified to $\cos\theta = 3\sqrt{3}\sin\theta$.

6 If $\sin(\theta + \alpha) = k\sin(\theta - \alpha)$, where k is a constant $(k \neq 1)$, find an expression for $\tan\theta$ in terms of k and $\tan\alpha$.

7 If α is an obtuse angle such that $\cos\alpha = -\frac{12}{13}$, find the following.

(a) $\sin\alpha$ (b) $\tan\alpha$ (c) $\cos(\alpha + 30°)$

8 If $\sin\alpha = \frac{12}{13}$ and $\sin\beta = \frac{4}{5}$, where α and β are obtuse angles, find the values of the following.

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha + \beta)$ (c) $\tan(\alpha + \beta)$

9 If $\cos\alpha = \frac{1}{4}$, and α is in the fourth quadrant, find the exact value of the following.

(a) $\sin\alpha$ (b) $\sin\left(\alpha - \frac{\pi}{6}\right)$ (c) $\cos\left(\alpha + \frac{\pi}{3}\right)$

10 Show that the following are true.

(a) $\tan(2\pi - \theta) = -\tan\theta$ (b) $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

11 If $\tan A = y + 1$ and $\tan B = y - 1$, show that $2\cot(A - B) = y^2$.

12 Express each the following as a single trigonometrical ratio.

(a) $\dfrac{1 + \tan\theta}{1 - \tan\theta}$ (b) $\dfrac{1}{\sqrt{2}}\cos\theta + \dfrac{1}{\sqrt{2}}\sin\theta$

13 Find, without using a calculator, the values of $\cot\theta$ when $\cot\alpha = \frac{1}{2}$ and $\cot(\theta - \alpha) = 4$.

14 Show that $\dfrac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{\sin(\alpha + \beta) + \sin(\alpha - \beta)} = \tan\beta$.

15 If $\tan(\alpha + \beta) = b$ and $\tan\beta = 0.5$, show that $\tan\alpha = \dfrac{2b - 1}{b + 2}$.

16 In an alternating current circuit, the instantaneous power P at time t is given by this equation:

$$P = \mathrm{VI}\cos\phi\sin^2(\omega t) - \mathrm{VI}\sin\phi\sin(\omega t)\cos(\omega t)$$

Show that the equation is equivalent to this: $P = \mathrm{VI}\sin(\omega t)\sin(\omega t - \phi)$.

Double-angle formulae

Sin 2θ

We can show that $\sin 2\theta = 2\sin\theta\cos\theta$.

PROOF

We can use $\sin(A + B) = \sin A \cos B + \cos A \sin B$ where $A = \theta, B = \theta$:

$\sin(\theta + \theta) = \sin\theta\cos\theta + \cos\theta\sin\theta$

$\Rightarrow = 2\sin\theta\cos\theta = \text{right-hand side}$

Cos 2θ

We can show that $\cos 2\theta = \cos^2\theta - \sin^2\theta$

PROOF

Substitute $A = \theta, B = \theta$ into $\cos(A + B) = \cos A\cos B - \sin A\sin B$.

$\Rightarrow \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$

$\Rightarrow = \cos^2\theta - \sin^2\theta \quad (1)$

Also, $\cos 2\theta = (1 - \sin^2\theta) - \sin^2\theta \quad$ (Since $\cos^2\theta = 1 - \sin^2\theta$)

$= 1 - 2\sin^2\theta$

Further, substituting $\sin^2\theta = 1 - \cos^2\theta$ into (1), we have:

$\cos 2\theta = \cos^2\theta - (1 - \cos^2\theta)$

$= 2\cos^2\theta - 1$

$\therefore$ $\cos 2\theta$ has three different forms:

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

Tan 2θ

We can show that $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$.

PROOF

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Substituting $A = \theta$ and $B = \theta$, we get:

$$\tan(\theta + \theta) = \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}$$

$$= \frac{2\tan\theta}{1 - \tan^2\theta}$$

Double-angle formulae

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

Half-angle formulae

We can also derive a set of half-angle results by replacing $\theta = \frac{\theta}{2}$ into each double-angle result and obtain half-angle formulae.

$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

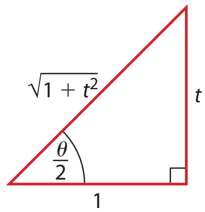
$\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

$\tan\theta = \dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

t-formulae

Let $t = \tan\frac{\theta}{2}$, and we draw a right-angled triangle:



We can use the diagram to derive half-angle formulae in terms of t, where $t = \tan\frac{\theta}{2}$.

$$\sin\frac{\theta}{2} = \frac{t}{\sqrt{1+t^2}}$$

$$\cos\frac{\theta}{2} = \frac{1}{\sqrt{1+t^2}}$$

We can also use t to derive formulae for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

$$\begin{aligned}\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\\ &= 2\times\frac{t}{\sqrt{1+t^2}}\times\frac{1}{\sqrt{1+t^2}}\\ &= \frac{2t}{1+t^2}\end{aligned}$$

$$\begin{aligned}\cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\\ &= \left(\frac{1}{\sqrt{1+t^2}}\right)^2 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2\\ &= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}\\ &= \frac{1-t^2}{1+t^2}\end{aligned}$$

$$\begin{aligned}\tan\theta &= \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\\ &= \frac{2t}{1-t^2}\end{aligned}$$

If $t = \tan\frac{\theta}{2}$, then:

$\sin\theta = \dfrac{2t}{1+t^2}$

$\cos\theta = \dfrac{1-t^2}{1+t^2}$

$\tan\theta = \dfrac{2t}{1-t^2}$

Proving identities using the addition theorems and the double-angle formulae

EXAMPLE 38 Prove the identity $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$.

SOLUTION

PROOF

Start with the left-hand side and convert the double angles to single angles, using:

$\cos 2\theta = 1 - 2\sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

So we have:

$$\frac{1-\cos 2\theta}{\sin 2\theta} = \frac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta}$$

$$= \frac{\cancel{2\sin^2\theta}}{\cancel{2\sin\theta}\cos\theta} \qquad \text{(Divide numerator and denominator by } 2\sin\theta\text{)}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta = \text{right-hand side} \qquad \text{Q.E.D.}$$

EXAMPLE 39 Express $\cos 3\theta$ in terms of $\cos\theta$ only.

SOLUTION

Using $\cos(A+B) = \cos A\cos B - \sin A\sin B$, where $A = 2\theta$ and $B = \theta$:

$$\cos 3\theta = \cos(2\theta + \theta)$$

$$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$$

Using $\cos 2\theta = 2\cos^2\theta - 1$ and $\sin 2\theta = 2\sin\theta\cos\theta$, we have:

$$\cos 3\theta = (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$$

$$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$$

Using $\sin^2\theta = 1 - \cos^2\theta$, we have:

$$\cos 3\theta = 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$$

$$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$$

$$= 4\cos^3\theta - 3\cos\theta$$

$$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta \qquad \text{Q.E.D.}$$

EXAMPLE 40 Prove that $\cot A - \tan A = 2\cot 2A$.

SOLUTION

PROOF

$$\cot A - \tan A = \frac{\cos A}{\sin A} - \frac{\sin A}{\cos A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\sin A\cos A} \qquad \text{(LCM is } \sin A\cos A\text{)}$$

$$= \frac{\cos 2A}{\frac{1}{2}\sin 2A} \qquad \text{(Since } \cos 2A = \cos^2 A - \sin^2 A,\ \tfrac{1}{2}\sin 2A = \sin A\cos A\text{)}$$

$$= 2\cot 2A = \text{right-hand side}$$

EXAMPLE 41 Prove that $\dfrac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \cot\theta$.

SOLUTION

PROOF

Start with the left-hand side, and use:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta + 1 = 2\cos^2\theta$$

$$1 - \cos 2\theta = 2\sin^2\theta$$

We have:

$$\frac{\sin 2\theta + \cos 2\theta + 1}{\sin 2\theta - \cos 2\theta + 1} = \frac{2\sin\theta\cos\theta + 2\cos^2\theta}{2\sin\theta\cos\theta + 2\sin^2\theta}$$
$$= \frac{2\cos\theta(\sin\theta + \cos\theta)}{2\sin\theta(\cos\theta + \sin\theta)}$$
$$= \frac{\cos\theta}{\sin\theta}$$
$$= \cot\theta = \text{right-hand side}$$

EXAMPLE 42 Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

SOLUTION

$$\sin 3\theta = \sin(2\theta + \theta)$$
$$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$$

Use: $\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = 1 - 2\sin^2\theta$

We get:

$$\sin 3\theta = (2\sin\theta\cos\theta)\cos\theta + (1 - 2\sin^2\theta)\sin\theta$$
$$= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta(1 - \sin^2\theta) + \sin\theta - 2\sin^3\theta$$
$$= 2\sin\theta - 2\sin^3\theta + \sin\theta - 2\sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta \qquad \text{Q.E.D.}$$

EXAMPLE 43 Prove that $\dfrac{\sin 4\theta}{1 + \cos 4\theta} = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

SOLUTION

PROOF

Start with the left-hand side and convert the numerator to double angles.

Since $\sin 2\theta = 2\sin\theta\cos\theta$

$\Rightarrow \sin 2(2\theta) = 2\sin 2\theta\cos 2\theta$

$\Rightarrow \sin 4\theta = 2\sin 2\theta\cos 2\theta$

Also $\cos 2\theta = 2\cos^2\theta - 1$

$\Rightarrow \cos 2(2\theta) = 2\cos^2 2\theta - 1$

$\Rightarrow \cos 4\theta = 2\cos^2 2\theta - 1$

$$\therefore \frac{\sin 4\theta}{1 + \cos 4\theta} = \frac{2\sin 2\theta\cos 2\theta}{1 + 2\cos^2 2\theta - 1}$$
$$= \frac{2\sin 2\theta\cos 2\theta}{2\cos^2 2\theta}$$
$$= \frac{\sin 2\theta}{\cos 2\theta}$$

$$= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} \qquad (\text{Since } \sin 2\theta = 2\sin\theta\cos\theta,\ \cos 2\theta = \cos^2\theta - \sin^2\theta)$$

$$= \frac{\dfrac{2\sin\theta\cos\theta}{\cos^2\theta}}{\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}} \qquad (\text{Divide the numerator and denominator by } \cos^2\theta)$$

$$= \frac{2\tan\theta}{1-\tan^2\theta}$$

Hence, $\dfrac{\sin 4\theta}{1+\cos 4\theta} = \dfrac{2\tan\theta}{1-\tan^2\theta}$.

Alternative method:

Since $\dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

$$\frac{\sin 2\theta}{\cos 2\theta} = \frac{2\tan^2\theta}{1-\tan^2\theta}$$

The form $a\cos\theta + b\sin\theta$

EXAMPLE 44 Express $a\cos\theta + b\sin\theta$ in the form $r\cos(\theta - \alpha)$ where $r > 0$ and $0° < \alpha < 90°$.

SOLUTION Let $a\cos\theta + b\sin\theta = r\cos(\theta - \alpha)$.

Using the expansion of $\cos(A - B) = \cos A\cos B + \sin A\sin B$ where $A = \theta$, $B = \alpha$, we get:

$$a\cos\theta + b\sin\theta = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$$
$$= r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$$

For the two sides to be equal the terms in $\cos\theta$ and $\sin\theta$ must be equal. Equating coefficients of $\cos\theta$ and $\sin\theta$ gives:

$\Rightarrow a\cos\theta = r\cos\theta\cos\alpha$

$\Rightarrow a = r\cos\alpha$ [1]

$\Rightarrow b\sin\theta = r\sin\theta\sin\alpha$

$\Rightarrow b = r\sin\alpha$ [2]

We can now find r and α in terms of a and b:

$\dfrac{b}{a} = \dfrac{r\sin\alpha}{r\cos\alpha}$ [2] ÷ [1]

$= \tan\alpha$

$\Rightarrow \alpha = \tan^{-1}\dfrac{b}{a}$

$r^2\cos^2\alpha + r^2\sin^2\alpha = a^2 + b^2$ $[1]^2 + [2]^2$

$r^2(\cos^2\alpha + \sin^2\alpha) = a^2 + b^2$

$r^2 = a^2 + b^2$ (Since $\cos^2\alpha + \sin^2\alpha = 1$)

$r = \pm\sqrt{a^2 + b^2}$

Since $r > 0$, $r = \sqrt{a^2 + b^2}$

$\therefore\ a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\cos\left(\theta - \tan^{-1}\left(\dfrac{b}{a}\right)\right)$

Note

Do not learn this result. Understand the procedure and use the procedure to convert from one form to the other.

EXAMPLE 45 Express $2\cos\theta + \sin\theta$ in the form $r\cos(\theta - \alpha)$ where $r > 0, 0° < \alpha < 90°$.

SOLUTION Let $2\cos\theta + \sin\theta = r\cos(\theta - \alpha)$.

Expand the right-hand side, using $\cos(A - B) = \cos A\cos B + \sin A\sin B$ where $A = \theta$, $B = \alpha$.

$\therefore\ 2\cos\theta + \sin\theta = r(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$2\cos\theta + 1\sin\theta = r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$ gives:

$r\cos\alpha = 2$ [1]

Equating coefficients of $\sin\theta$ gives:

$r\sin\alpha = 1$ [2]

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{1}{2}$ [2] ÷ [1]

$\tan\alpha = \frac{1}{2}$

$\Rightarrow \alpha = \tan^{-1}\frac{1}{2} = 26.6°$

$r^2\cos\alpha + r^2\sin^2\alpha = 1^2 + 2^2$ $[1]^2 + [2]^2$

$r^2 = 5$

$r = \sqrt{5}$, since $r > 0$

Hence, $2\cos\theta + \sin\theta = \sqrt{5}\cos(\theta - 26.6°)$.

EXAMPLE 46 Express $2\sin x + 3\cos x$ in the form $r\sin(x + \alpha)$, $r > 0, 0° < \alpha < 90°$.

SOLUTION This form is different from that in Example 45 but we can use the same procedure to make the conversion.

In this case, we expand $\sin(x + \alpha)$ by using $\sin(A + B) = \sin A\cos B + \cos A\sin B$:

$2\sin x + 3\cos x = r\sin(x + \alpha)$

$2\sin x + 3\cos x = r(\sin x\cos\alpha + \cos x\sin\alpha)$

$2\sin x + 3\cos x = r\sin x\cos\alpha + r\cos x\sin\alpha$

Equating coefficients of $\sin x$:

$r\cos\alpha = 2$ [1]

Equating coefficients of $\cos x$:

$r\sin\alpha = 3$ [2]

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{3}{2}$ [2] ÷ [1]

$\tan\alpha = \frac{3}{2}$

$\alpha = \tan^{-1}\frac{3}{2} = 56.3°$

$r^2 = 2^2 + 3^2$ $[1]^2 + [2]^2$

$r = \sqrt{13}$, since $r > 0$

$\therefore\ 2\cos x + 3\cos x = \sqrt{13}\sin(x + 56.3°)$

There are four different forms that we can use in our conversion. The four forms are:

$r\cos(x + \alpha)$

$r\cos(x - \alpha)$

$r\sin(x + \alpha)$

$r\sin(x - \alpha)$

The conversion is convenient for solving equations and to find the maximum and minimum of some functions without using calculus.

Follow these steps to find the maximum and minimum of $f(\theta) = a\cos\theta + b\sin\theta$.

(Recall that the maximum of cos A is 1 and the minimum of cos A is -1.)

(i) Write a $\cos\theta + b\sin\theta = r\cos(\theta - \alpha)$.

(ii) The maximum value of $f(\theta)$ occurs when $\cos(\theta - \alpha) = 1$
$\therefore$ maximum of $f(\theta) = r$.

(iii) The minimum value of $f(\theta)$ occurs when $\cos(\theta - \alpha) = -1$.
$\therefore$ the minimum value of $f(\theta) = -r$.

EXAMPLE 47

Express $f(\theta) = 4\cos\theta + 3\sin\theta$ in the form $r\cos(\theta - \alpha)$ where $r > 0$, $0° < \alpha < 90°$. Hence, find the maximum value of $f(\theta)$ and the value of θ for which $f(\theta)$ is minimum.

SOLUTION

$4\cos\theta + 3\sin\theta = r\cos(\theta - \alpha)$

$4\cos\theta + 3\sin\theta = r\cos\theta\cos\alpha + r\sin\theta\sin\alpha$

Equating coefficients of $\cos\theta$ gives:

$r\cos\alpha = 4 \qquad [1]$

Equating coefficients of $\sin\theta$ gives:

$r\sin\alpha = 3 \qquad [2]$

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{3}{4} \qquad [2] \div [1]$

$\tan\alpha = \frac{3}{4}$

$\Rightarrow \alpha = \tan^{-1}\frac{3}{4}$

$= 36.9°$

$r^2 = 4^2 + 3^2 \qquad [1]^2 + [2]^2$

$r = \sqrt{25} = 5$, since $r > 0$

$\therefore\ 4\cos\theta + 3\sin\theta = 5\cos(\theta - 36.9°)$

The maximum value of the function $= 5$ when $\cos(\theta - 36.9°) = 1$.

$f(\theta)$ is at a minimum when $\cos(\theta - 36.9°) = -1$.

When $\cos(\theta - 36.9°) = -1$:

$\theta - 36.9° = 180°$

$\theta = 216.9°$

EXAMPLE 48 **(a)** Given that $3\sin x - \cos x = r\sin(x - \alpha)$, $r > 0$ and $0° < \alpha < 90°$, find r and α.

(b) Find the maximum value of $3\sin x - \cos x$ and state the value of x $(0° < x < 180°)$ for which the maximum occurs.

(c) State the minimum value of $3\sin x - 4\cos x$ and state the value of x $(0° < x < 360°)$ for which the minimum occurs.

SOLUTION **(a)** $3\sin x - \cos x = r\sin(x - \alpha)$

$3\sin x - \cos x = r\sin x\cos\alpha - r\cos x\sin\alpha$

Equating coefficients of $\sin x$ gives:

$r\cos\alpha = 3$ [1]

Equating coefficients of $-\cos x$ gives:

$r\sin\alpha = 1$ [2]

$\dfrac{r\sin\alpha}{r\cos\alpha} = \dfrac{1}{3}$ [2] ÷ [1]

$\tan\alpha = \dfrac{1}{3}$

$\alpha = \tan^{-1}\dfrac{1}{3}$

$= 18.4°$

$r^2 = 1^2 + 3^2$ $[1]^2 + [2]^2$

$r = \sqrt{10}$, $r > 0$

$\therefore\ 3\sin x - \cos x = \sqrt{10}\sin(x - 18.4°)$

(b) When $\sin(x - 18.4°) = 1$, maximum value $= \sqrt{10} \times 1 = \sqrt{10}$.

$\sin(x - 18.4°) = 1$

$x - 18.4° = 90°$

$x = 108.4°$

(c) When $\sin(x - 18.4°) = -1$, minimum value $= \sqrt{10} \times -1 = -\sqrt{10}$.

$\sin(x - 18.4°) = -1$

$x - 18.4° = 270°$

$x = 288.4°$

Solving equations of the form $a\cos\theta + b\sin\theta = c$

To solve an equation of this form, we convert to one of the forms $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$ and then solve.

EXAMPLE 49 Find the general solution of the equation $\sin\theta + \cos\theta = 1$.

SOLUTION $\sin\theta + \cos\theta = r\sin(\theta + \alpha)$ where $r > 0$, $0° < \alpha < 90°$

$\sin\theta + \cos\theta = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$:

$r\cos\alpha = 1$ [1]

Equating coefficients of $\cos\theta$ gives:

$r\sin\alpha = 1$ [2]

$\dfrac{r\sin\alpha}{r\cos\alpha} = \dfrac{1}{1}$ [2] ÷ [1]

$\tan\alpha = 1$

$\alpha = \tan^{-1}1$

$= 45°$

$r^2 = 1^2 + 1^2$ $[1]^2 + [2]^2$

$r = \sqrt{2}$

$\therefore\ \sin\theta + \cos\theta = \sqrt{2}\sin(\theta + 45°)$

We need to solve $\sin\theta + \cos\theta = 1$:

Using $\sin\theta + \cos\theta = \sqrt{2}\sin(\theta + 45°)$, we have:

$\sqrt{2}\sin(\theta + 45°) = 1$

$\sin(\theta + 45°) = \dfrac{1}{\sqrt{2}}$

$\theta + 45° = 180°n + (-1)^n 45°,\ n \in \mathbb{Z}$

$\theta = 180°n + (-1)^n 45° - 45°$

Let $n = 2p$, $\theta = 360°p + 45° - 45°$

$= 360°p,\ p \in \mathbb{Z}$

Let $n = 2p + 1$, $\theta = 180°(2p + 1) - 45° - 45°$

$= 360°p + 180° - 90°$

$= 360°p + 90°,\ p \in \mathbb{Z}$

EXAMPLE 50

(a) Write the function $f(x) = 7\cos x - 24\sin x$ in the form $r\cos(x + \alpha)$ where $r > 0$, $0° < \alpha < 90°$.

(b) Hence, solve the equation $7\cos x - 24\sin x = 4$ for $0° < x < 360°$.

(c) Identify the maximum value of $f(x)$.

SOLUTION

(a) $7\cos x - 24\sin x = r\cos(x + \alpha)$

$7\cos x - 24\sin x = r\cos x\cos\alpha - r\sin x\sin\alpha$

Equating coefficients of $\cos x$ gives:

$r\cos\alpha = 7$ [1]

Equating coefficients of $-\sin x$ gives:

$r\sin\alpha = 24$ [2]

$\dfrac{r\sin\alpha}{r\cos\alpha} = \dfrac{24}{7}$ [2] ÷ [1]

$\tan\alpha = \dfrac{24}{7}$

$\alpha = \tan^{-1}\dfrac{24}{7} = 73.7°$

$r^2 = 7^2 + 24^2$ $\qquad [1]^2 + [2]^2$

$r = \sqrt{49 + 576} = 25$

$\therefore 7\cos x - 24\sin x = 25\cos(x + 73.7°)$

(b) We need to solve $7\cos x - 24\sin x = 4$.

Since $7\cos x - 24\sin x = 25\cos(x - 73.7°)$, we solve:

$25\cos(x - 73.3°) = 4$

$\cos(x - 73.3°) = \frac{4}{25}$

$x - 73.7° = \cos^{-1}\frac{4}{25}$

$x - 73.7° = 80.8°, 279.2°$

$x = 80.8° + 73.7°, 279.2° + 73.7°$

$= 154.5°, 352.9°$

(c) Since $7\cos x - 24\sin x = 25\cos(x - 73.7°)$

The maximum value is 25.

EXAMPLE 51 Find the solution of the equation $2\cos 2x - 3\sin 2x = 2$ for $0° \leq x \leq 360°$.

SOLUTION First we write the equation as:

$2\cos 2x - 3\sin 2x = r\cos(2x + \alpha), r > 0, 0° < \alpha < 90°$

Expanding the right-hand side:

$2\cos 2x - 3\sin 2x = r\cos 2x\cos\alpha - r\sin 2x\sin\alpha$.

Equating coefficients of $\cos 2x$

$r\cos\alpha = 2$ $\qquad [1]$

Equating coefficients of $-\sin 2x$:

$r\sin\alpha = 3$ $\qquad [2]$

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{3}{2}$ $\qquad [2] \div [1]$

$\tan\alpha = \frac{3}{2}$

$\alpha = \tan^{-1}\frac{3}{2}$

$= 56.3°$

$r^2 = 2^2 + 3^2$ $\qquad [1]^2 + [2]^2$

$r = \sqrt{13}$

$\therefore 2\cos 2x - 3\sin 2x = \sqrt{13}\cos(2x - 56.3°)$

Substituting into $2\cos 2x - 3\sin 2x = 2$:

$\sqrt{13}\cos(2x - 56.3°) = 2$

$\cos(2x - 56.3°) = \frac{2}{\sqrt{13}}$

$2x - 56.3° = \cos^{-1}\frac{2}{\sqrt{13}}$

We identify all solutions for $2x - 56.3°$.

$2x - 56.3° = 56.3°, 303.7°, 416.3°, 663.7°$

$2x = 112.6°, 360°, 472.6°, 720°$

$x = 56.3°, 180°, 236.2°, 360°$

EXAMPLE 52 Solve the equation $12 \sin\theta - 5\cos\theta = 4$ for $0° \leq \theta \leq 360°$. Use two different methods.

SOLUTION

Method 1

Convert $12 \sin\theta - 5\cos\theta$ to the form $r\sin(\theta - \alpha)$:

$12\sin\theta - 5\cos\theta = r\sin(\theta - \alpha)$

$12\sin\theta - 5\cos\theta = r\sin\theta\cos\alpha - r\cos\theta\sin\alpha$

Equating coefficients of $\sin\theta$ gives:

$r\cos\alpha = 12$ [1]

Equating coefficients of $-\cos\theta$ gives:

$r\sin\alpha = 5$ [2]

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{5}{12}$ [2] ÷ [1]

$\tan\alpha = \frac{5}{12}$

$\alpha = \tan^{-1}\frac{5}{12} = 22.6°$

$r^2 = 5^2 + 12^2$ $[1]^2 + [2]^2$

$r = \sqrt{169} = 13$

$\therefore 12\sin\theta - 5\cos\theta = 13\sin(\theta - 22.6°)$

$\Rightarrow 13\sin(\theta - 22.6°) = 4$

$\sin(\theta - 22.6°) = \frac{4}{13}$

$\theta - 22.6° = \sin^{-1}\frac{4}{13}$

$\theta - 22.6° = 17.9°, 162.1°$

$\theta = 40.5°, 184.7°$

Method 2

$12\sin\theta - 5\cos\theta = 4$

$\Rightarrow 12\sin\theta = 4 + 5\cos\theta$

Square both sides:

$144\sin^2\theta = 16 + 40\cos\theta + 25\cos^2\theta$

Use $\sin^2\theta = 1 - \cos^2\theta$:

$144(1 - \cos^2\theta) = 16 + 40\cos\theta + 25\cos^2\theta$

$144 - 144\cos^2\theta = 16 + 40\cos\theta + 25\cos^2\theta$

$169\cos^2\theta + 40\cos\theta - 128 = 0$

Let $y = \cos\theta$.

$\Rightarrow 169y^2 + 40y - 128 = 0$

$$y = \frac{-40 \pm \sqrt{40^2 - 4 \times 169 \times -128}}{2 \times 169}$$

$$= \frac{-40 \pm 296.864}{338}$$

$y = 0.75995, -0.9966$

For $y = 0.75995$, $\cos\theta = 0.75995$

$\theta = \cos^{-1} 0.75995$

$\theta = 40.5°, 319.5°$

For $y = -0.9966$, $\cos\theta = -0.9966$

$\theta = \cos^{-1}(-0.9966)$

$\theta = 175.3°, 184.7°$

At this stage, you need to test all your values to identify which values satisfy the range in the question.

When $\theta = 40.5°$, $12\sin 40.5° - 5\cos 40.5° \approx 4$ (due to rounding off)

When $\theta = 319.5°$, $12\sin 319.5° - 5\cos 319.5° = -11.6$ (not a solution)

When $\theta = 175.3°$, $12\sin 175.3° - 5\cos 175.3° = 5.97$ (not a solution)

When $\theta = 184.7°$, $12\sin 184.7° - 5\cos 184.7° = 4$ (solution)

$\therefore\ \theta = 40.5°, 184.7°$

Equations of the form $a\cos\theta + b\sin\theta = c$ can be solved by squaring. Each solution must be tested when using this method for solving the equation.

Try these 9.5

(a) Solve the equation $3\sin\theta - \cos\theta = 2$ for $0° \leq \theta \leq 360°$.

(b) Find the general solution of the equation $\sqrt{3}\cos 2\theta - \sin 2\theta = -1$

(c) write down the maximum and minimum values of $f(x) = 2\sin x - \cos x$ and the values of x in the interval between $-180°$ and $180°$ inclusive, where the maximum and minimum occurs.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos 2\theta = 1 - 2\sin^2\theta$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

Equations involving double-angle or half-angle formulae

EXAMPLE 53 Solve the equation $\sin 2\theta + \sin\theta = 0$ for $0° \leq \theta \leq 360°$.

SOLUTION

$\sin 2\theta + \sin\theta = 0$

We have a combination of a double angle and a single angle. We can convert the double angle to a single angle and then solve.

Using $\sin 2\theta = 2\sin\theta\cos\theta$, we get:

$2\sin\theta\cos\theta + \sin\theta = 0$

Factorising gives:

$\sin\theta(2\cos\theta + 1) = 0$

$\therefore$ either $\sin\theta = 0$ or $2\cos\theta + 1 = 0$

$\theta = \sin^{-1} 0$

$\Rightarrow \theta = 0°, 180°, 360°$

$2\cos\theta + 1 = 0$

$\Rightarrow \cos\theta = -\frac{1}{2}$

$\Rightarrow \theta = 120°, 240°$

Hence, $\theta = 0°, 120°, 180°, 240°, 360°$.

EXAMPLE 54 Solve the equation $2\cos 2x + \sin x - 1 = 0$ for all values of x from $0°$ to $360°$ inclusive.

SOLUTION

We need to convert $\cos 2x$ to a single angle formula.

Since the equation contains $\sin x$, we use $\cos 2x = 1 - 2\sin^2 x$.

$2\cos 2x + \sin x - 1 = 0$

$\Rightarrow 2(1 - 2\sin^2 x) + \sin x - 1 = 0$

$\Rightarrow 2 - 4\sin^2 x + \sin x - 1 = 0$

$\Rightarrow -4\sin^2 x + \sin x + 1 = 0$

$\Rightarrow 4\sin^2 x - \sin x - 1 = 0$ (Multiply by -1)

Let $y = \sin x$

$\therefore 4y^2 - y - 1 = 0$

$$y = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 4 \times -1}}{2 \times 4}$$

$$y = \frac{1 \pm \sqrt{17}}{8}$$

$y = 0.6404, -0.3904$

$\sin x = 0.6404, \Rightarrow x = 39.8°, 140.2°$

$\sin x = -0.3904 \Rightarrow x = 203.0°, 337.0°$

Therefore, $x = 39.8°, 140.2°, 203.0°, 337.0°$, for $0 \leq x \leq 360°$.

EXAMPLE 55 Solve the equation $3\tan x = \tan 2x$ for all values of x from $0°$ to $360°$.

SOLUTION We form a quadratic in $\tan x$ by using $\tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$. We have:

$3\tan x = \dfrac{2\tan x}{1-\tan^2 x}$

$\Rightarrow 3\tan x\,(1-\tan^2 x) = 2\tan x$

Expanding gives:

$3\tan x - 3\tan^3 x = 2\tan x$

$\Rightarrow 3\tan^3 x - \tan x = 0$

Factorising:

$\tan x(3\tan^2 x - 1) = 0$

$\therefore \tan x = 0,\ 3\tan^2 x - 1 = 0$

For $\tan x = 0$:

$x = \tan^{-1} 0$

$x = 0°, 180°$

For $3\tan^2 x - 1 = 0$:

$\tan^2 x = \dfrac{1}{3}$

$\tan x = \pm\dfrac{1}{\sqrt{3}}$

For $\tan x = +\dfrac{1}{\sqrt{3}}$:

$x = \tan^{-1}\dfrac{1}{\sqrt{3}}$

$x = 30°, 210°$

For $\tan x = -\dfrac{1}{\sqrt{3}}$:

$x = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

$x = 330°, 150°$

$\therefore x = 0°, 30°, 150°, 180°, 210°, 330°$ for $0° \leq x \leq 360°$.

EXAMPLE 56 Find the general solution of $3\cos 2x - \cos x = 2$.

SOLUTION We form a quadratic equation in $\cos x$. Converting $\cos 2x = 2\cos^2 x - 1$, we have:

$3(2\cos^2 x - 1) - \cos x = 2$

$\Rightarrow 6\cos^2 x - 3 - \cos x - 2 = 0$

$\Rightarrow 6\cos^2 x - \cos x - 5 = 0$

Let $y = \cos x$:

$6y^2 - y - 5 = 0$

Factorising gives:

$(6y + 5)(y - 1) = 0$

$\therefore 6y + 5 = 0, y - 1 = 0$

$y = -\frac{5}{6}, y = 1$

Since $y = \cos x$, for $y = -\frac{5}{6}$:

$\cos x = -\frac{5}{6}$

$x = \cos^{-1}\left(-\frac{5}{6}\right) = 146.4°$

General solution:

$x = 360°n \pm 146.4°, n \in \mathbb{Z}$

Since $y = \cos x$, for $y = 1$:

$x = \cos^{-1}1 = 0°$

General solution:

$x = 360°n \pm 0°$

$\quad = 360°n, n \in \mathbb{Z}$

General solution is $x = 360°n \pm 146.4°$ and $x = 360°n, n \in \mathbb{Z}$.

EXAMPLE 57 Solve the equation $\cos\theta + \sin\frac{\theta}{2} = 1$ for all θ between 0° and 180° inclusive.

SOLUTION

Convert $\cos\theta$ to a half angle result:

$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$

Substitute into the equation:

$1 - 2\sin^2\frac{\theta}{2} + \sin\frac{\theta}{2} = 1$

$2\sin^2\frac{\theta}{2} - \sin\frac{\theta}{2} = 0$

Factorising:

$\sin\frac{\theta}{2}\left(2\sin\frac{\theta}{2} - 1\right) = 0 \quad \therefore \sin\frac{\theta}{2} = 0, 2\sin\frac{\theta}{2} - 1 = 0$

For $\sin\frac{\theta}{2} = 0$

$\frac{\theta}{2} = \sin^{-1}0$

$\frac{1}{2}\theta = 0°, 180°$

$\theta = 0°, 360°$

For $2\sin\frac{\theta}{2} - 1 = 0$

$\sin\frac{\theta}{2} = \frac{1}{2}$

$\frac{\theta}{2} = \sin^{-1}\frac{1}{2}$

$\frac{\theta}{2} = 30°, 150°$

$\theta = 60°, 300°$

Since $0° \leq \theta \leq 180°$, $\theta = 0°, 60°$.

Try these 9.6 Solve the following equations for θ, giving your answers in the interval $0° \le \theta \le 360°$.

(a) $\cos 2\theta - \cos\theta = 0$

(b) $\cos\theta = \sin 2\theta$

(c) $\cos 2\theta - 2\cos\theta = 3$

EXERCISE 9D

1 Show that $\dfrac{\sin 2x + \cos x}{2 - 2\cos^2 x + \sin x} = \cot x$.

2 Prove that $\dfrac{1 - \cos 2x}{1 + \cos 2x} = \tan^2 x$.

3 Prove that $\tan x - \cot x = -2\cot 2x$.

4 Prove that $\dfrac{\cos 2x}{\cos x - \sin x} = \cos x + \sin x$.

5 Prove that $\dfrac{1 - \cos 2A + \sin A}{\sin 2A + \cos A} = \tan A$.

6 Prove that $\dfrac{1 - \cos 4\theta}{\sin 4\theta} = \tan 2\theta$.

7 Given that $\tan 2x = \frac{1}{4}$ and that angle x is acute, calculate, without using a calculator, the values of these.

(a) $\cos 2x$ (b) $\sin x$

8 Prove that $\dfrac{\sin(x+y) - \sin(x-y)}{\cos(x+y) + \cos(x-y)} = \tan y$.

9 Given that $\dfrac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} = \dfrac{4}{5}$, show that $\tan\theta = 9\tan\alpha$. If $\tan\alpha = \frac{1}{3}$, calculate without using a calculator, the values of $\tan\theta$ and $\tan 2\theta$.

10 Given that $\tan\alpha = \frac{5}{12}$ and that $0° < \alpha < 90°$, calculate, without using a calculator, the values of $\cos 2\alpha$ and $\cos 4\alpha$.

11 If $\cos\theta = p$, express the following, in terms of p.

(a) $\sin 2\theta$ (b) $\tan^2\theta$ (c) $\sin 4\theta$

12 Given that $\tan 2\alpha = 1$ and that α is acute, show that $\tan\alpha = \sqrt{2} - 1$. Do not use a calculator.

13 Express $\tan 2\alpha$ in terms of $\tan\alpha$ and hence, find, without using a calculator the values of $\tan 67\frac{1}{2}°$ in surd form.

14 Two acute angles θ and β, are such that $\tan\theta = \frac{3}{4}$ and $\tan(\theta + \beta) = -2$. Without evaluating θ or β, carry out the following.

(a) Show that $\tan\beta = \frac{11}{2}$.

(b) Find $\sin\theta$ and $\sin\beta$.

15 Given that $\dfrac{\cos(A - B)}{\cos(A + B)} = \dfrac{5}{2}$, show that $7\tan A = 3\cot B$. Given further that A is acute and that $\tan B = 3$ find, without using a calculator, the value of the following.

(a) $\tan(A + B)$ (b) $\sin A$ (c) $\cos 2A$

16 Given that α, β and θ are the angles of a triangle, show that

$$\tan\theta = \frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta - 1}$$

17 Given that $\sin\theta = \frac{1}{4}$, find, without using a calculator, the value of the following.

(a) $\cos 2\theta$ (b) $\cos 4\theta$

18 Express $3\cos x + 2\sin x$ in the form $r\cos(x - \alpha)$ where $r > 0$ and $0° < \alpha < 90°$.

(a) Write down the maximum value of $3\cos x + 2\sin x$ and the value of x where $3\cos x + 2\sin x$ is maximum.

(b) Find the general solution of $3\cos x + 2\sin x = 2$.

19 (a) Express $f(x) = 2\sin x + 4\cos x$ in the form $r\sin(x + \alpha)$ where $r > 0$ and $0° < \alpha < 90°$.

(b) Write down the maximum value of $\frac{2}{2\sin x + 4\cos x}$ in surd form.

20 (a) Express $4\cos x - 3\sin x$ in the form $r\cos(x - \alpha)$ where $r > 0$ and $0° < \alpha < 90°$.

(b) Solve the equation $4\cos x - 3\sin x = 2$ for and $0° < x < 360°$.

(c) Write down the maximum value of $4 - 4\cos x + 3\sin x$.

(d) Sketch the graph of $y = 4\cos x - 3\sin x$ for and $0° \leq x \leq 360°$.

Products as sums and differences

The following results can be used to convert a product of terms to a sum or difference of terms. These are particularly helpful in proving other identities, solving equations and integrating some products of sine and cosine.

Recall:

$\sin(A + B) = \sin A\cos B + \cos A\sin B$ [1]

$\sin(A - B) = \sin A\cos B - \cos A\sin B$ [2]

$\Rightarrow \sin(A + B) - \sin(A - B) = 2\cos A\sin B$ [1] − [2]

$\Rightarrow \sin(A + B) + \sin(A - B) = 2\sin A\cos B$ [1] + [2]

Using these:

$\cos(A + B) = \cos A\cos B - \sin A\sin B$ [3]

$\cos(A - B) = \cos A\cos B + \sin A\sin B$ [4]

$\Rightarrow \cos(A + B) + \cos(A - B) = 2\cos A\cos B$ [3] + [4]

$\Rightarrow \cos(A + B) - \cos(A - B) = -2\sin A\sin B$ [3] + [4]

Let us use these results.

$2\sin A\cos B = \sin(A + B) + \sin(A - B)$

$2\cos A\sin B = \sin(A + B) - \sin(A - B)$

$2\cos A\cos B = \cos(A + B) + \cos(A - B)$

$-2\sin A\sin B = \cos(A + B) - \cos(A - B)$

EXAMPLE 58 Convert $\sin 8\theta \sin 6\theta$ to a sum or difference of terms.

SOLUTION Using $-2\sin A \sin B = \cos(A+B) - \cos(A-B)$, where $A = 8\theta$, $B = 6\theta$ we have:

$$-2\sin 8\theta \sin 6\theta = \cos(8\theta + 6\theta) - \cos(8\theta - 6\theta)$$
$$-2\sin 8\theta \sin 6\theta = \cos 14\theta - \cos 2\theta$$
$$\therefore \sin 8\theta \sin 6\theta = -\frac{1}{2}(\cos 14\theta - \cos 2\theta)$$
$$= \frac{1}{2}(\cos 2\theta - \cos 14\theta)$$

EXAMPLE 59 Convert $\cos 7\theta \cos 6\theta$ to a sum of terms.

SOLUTION Using $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, where $A = 7\theta$, $B = 6\theta$, we get:

$$2\cos 7\theta \cos 6\theta = \cos(7\theta + 6\theta) + \cos(7\theta - 6\theta)$$
$$2\cos 7\theta \cos 6\theta = \cos 13\theta - \cos\theta$$
$$\therefore \cos 7\theta \cos 6\theta = \frac{1}{2}(\cos 13\theta - \cos\theta)$$

EXAMPLE 60 Express $2\sin\theta \cos 3\theta$ as a sum or difference of two ratios.

SOLUTION Using $2\sin A \cos B = \sin(A+B) + \sin(A-B)$, where $A = \theta$, $B = 3\theta$ gives:

$$2\sin\theta \cos 3\theta = \sin(\theta + 3\theta) + \sin(\theta - 3\theta)$$
$$2\sin\theta \cos 3\theta = \sin 4\theta + \sin(-2\theta)$$

Recall $\sin(-x) = -\sin x$:

$$\therefore \sin(-2\theta) = -\sin 2\theta$$

Hence, $2\sin\theta \cos 3\theta = \sin 4\theta - \sin 2\theta$.

EXAMPLE 61 Express $6\cos 5\theta \cos 3\theta$ as a sum or difference of two ratios.

SOLUTION Using $2\cos A \cos B = \cos(A+B) + \cos(A-B)$, where $A = 5\theta$ and $B = 3\theta$, we have:

$$2\cos 5\theta \cos 3\theta = \cos(5\theta + 3\theta) + \cos(5\theta - 3\theta)$$
$$2\cos 5\theta \cos 3\theta = \cos 8\theta + \cos 2\theta$$
$$6\cos 5\theta \cos 3\theta = 3\cos 8\theta + 3\cos 2\theta \qquad \text{(Multiplying by 3)}$$

Converting sums and differences to products

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$
$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Let us prove $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

PROOF

Starting with right-hand side, and using $2\sin A\cos B = \sin(A+B)+\sin(A-B)$:

$$2\sin\frac{C+D}{2}\cos\frac{C-D}{2} = \sin(A+B)+\sin(A-B) \text{ where } A=\frac{C+D}{2} \text{ and } B=\frac{C-D}{2}$$
$$= \sin\left(\frac{C+D}{2}+\frac{C-D}{2}\right)+\sin\left(\frac{C+D}{2}-\frac{C-D}{2}\right)$$
$$= \sin C+\sin D = \text{left-hand side}$$

Similarly you can prove the other identities.

Try these 9.7

Prove that:

(a) $\sin A - \sin B = 2\sin\frac{A-B}{2}\cos\frac{A+B}{2}$

(b) $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

(c) $\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$

EXAMPLE 62

Convert $\sin 9\theta + \sin 7\theta$ to a product of terms.

SOLUTION

Using $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ where $A = 9\theta$, $B = 7\theta$ gives:

$$\sin 9\theta + \sin 7\theta = 2\sin\left(\frac{9\theta+7\theta}{2}\right)\cos\left(\frac{9\theta-7\theta}{2}\right)$$
$$= 2\sin 8\theta\cos\theta$$

$\therefore \sin 9\theta + \sin 7\theta = 2\sin 8\theta\cos\theta$

EXAMPLE 63

Convert $\cos 7x + \cos x$ to a product of terms.

SOLUTION

Using $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ with $A = 7x$, $B = x$:

$$\cos 7x + \cos x = 2\cos\left(\frac{7x+x}{2}\right)\cos\left(\frac{7x-x}{2}\right)$$
$$= 2\cos 4x\cos 3x$$

$\therefore \cos 7x + \cos x = 2\cos 4x\cos 3x$

EXAMPLE 64

Prove that $\dfrac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta} = \tan\theta$.

SOLUTION

PROOF

We convert the numerator and denominator to a product. When we do this, both the numerator and denominator contain the factor $2\sin 2\theta$.

For the numerator, we use $\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$, where $A = \theta, B = 3\theta$:

$$\cos\theta - \cos 3\theta = -2\sin\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)$$
$$= -2\sin 2\theta\sin(-\theta)$$
$$= 2\sin 2\theta\sin\theta \quad (\text{Since } \sin(-\theta) = -\sin\theta)$$

For the denominator, use $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ where $A = \theta$, $B = 3\theta$:

$$\sin\theta + \sin 3\theta = 2\sin\left(\frac{\theta+3\theta}{2}\right)\cos\left(\frac{\theta-3\theta}{2}\right)$$

$$= 2\sin 2\theta\cos(-\theta)$$

$$= 2\sin 2\theta\cos\theta \quad (\text{Since } \cos(-\theta) = \cos\theta)$$

$$\therefore \frac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta} = \frac{\cancel{2\sin 2\theta}\sin\theta}{\cancel{2\sin 2\theta}\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

EXAMPLE 65 Prove that $\frac{\cos\theta - 2\cos 3\theta + \cos 5\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = -\tan^2\theta$.

SOLUTION

PROOF

We convert $\cos\theta + \cos 5\theta$ to a product and obtain $2\cos 3\theta$ common to both the numerator and denominator.

$$\frac{\cos\theta - 2\cos 3\theta + \cos 5\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \frac{\cos\theta + \cos 5\theta - 2\cos 3\theta}{\cos\theta + \cos 5\theta + 2\cos 3\theta}$$

Convert $\cos\theta + \cos 5\theta$ to a product, using $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

where $A = \theta$ and $B = 5\theta$:

$$\cos\theta + \cos 5\theta = 2\cos\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)$$

$$= 2\cos 3\theta\cos(-2\theta)$$

$$= 2\cos 3\theta\cos 2\theta \quad (\text{Since } \cos(-2\theta) = \cos 2\theta)$$

Substituting we have:

$$\frac{\cos\theta - 2\cos 3\theta + \cos 5\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = \frac{2\cos 3\theta\cos 2\theta - 2\cos 3\theta}{2\cos 3\theta\cos 2\theta + 2\cos 3\theta}$$

$$= \frac{2\cos 3\theta(\cos 2\theta - 1)}{2\cos 3\theta(\cos 2\theta + 1)}$$

$$= \frac{\cos 2\theta - 1}{\cos 2\theta + 1}$$

We next convert our double angle to a single angle.

$$\frac{\cos 2\theta - 1}{\cos 2\theta + 1} = \frac{1 - 2\sin^2\theta - 1}{2\cos^2\theta - 1 + 1} \quad (\text{Since } \cos 2\theta = 1 - 2\sin^2\theta \text{ and } \cos 2\theta = 2\cos^2\theta - 1)$$

$$= -\frac{2\sin^2\theta}{2\cos^2\theta}$$

$$= -\frac{\sin^2\theta}{\cos^2\theta}$$

$$= -\tan^2\theta \left(\text{Since } \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta\right)$$

Hence, $\frac{\cos\theta + 2\cos 3\theta + \cos 5\theta}{\cos\theta + 2\cos 3\theta + \cos 5\theta} = -\tan^2\theta$ Q.E.D.

EXAMPLE 66 Given that $A + B + C = 180°$, prove that $\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

SOLUTION

PROOF

We start with the left-hand side.

We use $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ and the half-angle formula $\sin C = 2\sin\frac{C}{2}\cos\frac{C}{2}$:

$$\sin A + \sin B + \sin C = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \qquad [1]$$

Since $A + B + C = 180°$

$$A + B = 180° - C$$

$$\frac{A+B}{2} = 90° - \frac{C}{2}$$

Substituting $\frac{A+B}{2} = 90° - \frac{C}{2}$ in (1) gives:

$$\begin{aligned}\sin A + \sin B + \sin C &= 2\sin\left(90° - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2}\\ &= 2\cos\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 2\sin\frac{C}{2}\cos\frac{C}{2} \quad \left(\text{Since } \sin\left(90° - \frac{C}{2}\right) = \cos\frac{C}{2}\right)\\ &= 2\cos\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right) + \sin\frac{C}{2}\right) \qquad [2]\end{aligned}$$

Since $A + B + C = 180°$

$$C = 180° - (A + B)$$

$$\frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)$$

Substituting $\frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)$ in [2] gives:

$$\begin{aligned}\sin A + \sin B + \sin C &= 2\cos\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right) + \sin\left(90° - \left(\frac{A+B}{2}\right)\right)\right]\\ &= 2\cos\frac{C}{2}\left(\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)\right) \quad [3] \quad \left(\text{Since } \sin\left(90° - \left(\frac{A+B}{2}\right)\right) = \cos\left(\frac{A+B}{2}\right)\right)\end{aligned}$$

Converting $\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right)$ into a product:

$$\begin{aligned}\cos\left(\frac{A-B}{2}\right) + \cos\left(\frac{A+B}{2}\right) &= 2\cos\left(\frac{\frac{A+B}{2} + \frac{A-B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2} - \frac{A-B}{2}}{2}\right)\\ &= 2\cos\left(\frac{A}{2}\right)\cos\left(-\frac{B}{2}\right)\\ &= 2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right) \quad \left(\text{Since } \cos\left(-\frac{B}{2}\right) = \cos\left(\frac{B}{2}\right)\right)\end{aligned}$$

Sustituting $2\cos\frac{A}{2}\cos\frac{B}{2}$ in (3) gives:

$$\begin{aligned}\sin A + \sin B + \sin C &= 2\cos\frac{C}{2} \times \left(2\cos\frac{A}{2}\cos\frac{B}{2}\right)\\ &= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \qquad \text{Q.E.D.}\end{aligned}$$

Solving equations using the sums and differences as products

EXAMPLE 67 Find all values of θ in the interval 0° and 180° satisfying the equation $\cos 4\theta + \cos 2\theta = 0$.

SOLUTION We convert $\cos 4\theta + \cos 2\theta$ to a product using $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$ where, $A = 4\theta$, $B = 2\theta$, and then solve:

$$\cos 4\theta + \cos 2\theta = 2\cos\left(\frac{4\theta + 2\theta}{2}\right)\cos\left(\frac{4\theta - 2\theta}{2}\right)$$

$$= 2\cos 3\theta \cos \theta$$

Since $\cos 4\theta + \cos 2\theta = 0$:

$2\cos 3\theta \cos\theta = 0$

$\Rightarrow \cos 3\theta = 0, \cos\theta = 0$

For $\cos 3\theta = 0$:

$3\theta = \cos^{-1} 0$

$\Rightarrow 3\theta = 90°, 270°, 450°$

$\Rightarrow \theta = 30°, 90°, 180°$

For $\cos\theta = 0$:

$\theta = \cos^{-1} 0$

$\Rightarrow \theta = 90°$

$\therefore$ the values of θ that satisfy the equation are $\theta = 30°, 90°, 180°$.

EXAMPLE 68 Find the general solution of $\cos\theta = -1$.

SOLUTION $\cos\theta = -1$

$\Rightarrow \theta = \cos^{-1}(-1)$

$= 2n\pi \pm \pi, n \in \mathbb{Z}$

The general solution is $\theta = \cos^{-1}(-1) = 2n\pi \pm \pi, n \in \mathbb{Z}$.

EXAMPLE 69 Find the general solution of $\cos 2x - \cos 5x = 0$.

SOLUTION Converting $\cos 2x - \cos 5x$ to a product:

$$\cos 2x - \cos 5x = -2\sin\left(\frac{2x + 5x}{2}\right)\sin\left(\frac{2x - 5x}{2}\right)$$

$$= -2\sin\frac{7x}{2}\sin\left(-\frac{3x}{2}\right)$$

$$= 2\sin\frac{7x}{2}\sin\frac{3x}{2} \quad \left(\text{Since } \sin\left(-\frac{3x}{2}\right) = -\sin\frac{3x}{2}\right)$$

Since $\cos 2x - \cos 5x = 0$:

$2\sin\frac{7x}{2}\sin\frac{3x}{2} = 0$

$\Rightarrow \sin\frac{7x}{2} = 0, \sin\frac{3x}{2} = 0$

For $\sin\frac{7x}{2} = 0$:

$\frac{7x}{2} = \sin^{-1} 0$

$\frac{7x}{2} = n\pi, n \in \mathbb{Z}$

$x = \frac{2n\pi}{7}, n \in \mathbb{Z}$

For $\sin\frac{3x}{2} = 0$:

$\frac{3x}{2} = \sin^{-1} 0$

$\frac{3x}{2} = n\pi, n \in \mathbb{Z}$

$x = \frac{2n\pi}{3}, n \in \mathbb{Z}$

The general solution is $x = \frac{2n\pi}{7}, \frac{2n\pi}{3}, n \in \mathbb{Z}$.

EXAMPLE 70 Find the general solution of the equation $\sin x + \sin 2x + \sin 3x = 0$.

SOLUTION We have three terms: taking two of them and converting to a product.

We use the first and the last term, since when combined we get $\sin 2x$ as a factor.

$$\begin{aligned}\sin x + \sin 3x &= 2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)\\ &= 2\sin 2x\cos(-x)\\ &= 2\sin 2x\cos x \quad (\text{Since } \cos(-x) = \cos x)\end{aligned}$$

Substituting $\sin x + \sin 3x = 2\sin 2x\cos x$ into $\sin x + \sin 2x + \sin 3x = 0$ gives:

$\sin 2x + 2\sin 2x\cos x = 0$

Factorising gives:

$\sin 2x(1 + 2\cos x) = 0$

$\Rightarrow \sin 2x = 0, 1 + 2\cos x = 0$

For $\sin 2x = 0$:

$2x = \sin^{-1} 0$

$2x = n\pi, n \in \mathbb{Z}$

$\therefore x = \frac{n\pi}{2}, n \in \mathbb{Z}$

For $1 + 2\cos x = 0$:

$2\cos x = -1$

$\cos x = -\frac{1}{2}$

$x = \cos^{-1}\left(-\frac{1}{2}\right) = 2n\pi \pm \frac{2\pi}{3}, n \in \mathbb{Z}$

Hence, the solutions are $x = 2n\pi \pm \frac{2\pi}{3}, x = \frac{n\pi}{2}, n \in \mathbb{Z}$.

EXAMPLE 71 Find the general solution of $\cos 4\theta + 2\cos 5\theta + \cos 6\theta = 0$.

SOLUTION

Convert to a product:

$$\cos 4\theta + \cos 6\theta = 2\cos\left(\frac{4\theta + 6\theta}{2}\right)\cos\left(\frac{4\theta - 6\theta}{2}\right)$$

$$= 2\cos 5\theta\cos(-\theta)$$

$$= 2\cos 5\theta\cos\theta \quad (\text{Since } \cos(-\theta) = \cos\theta)$$

Since $\cos 4\theta + 2\cos 5\theta + \cos 6\theta = 0$

$2\cos 5\theta\cos\theta + 2\cos 5\theta = 0$

$\Rightarrow 2\cos 5\theta(\cos\theta + 1) = 0$

$\Rightarrow \cos 5\theta = 0, \cos\theta + 1 = 0$

For $\cos 5\theta = 0$:

$5\theta = \cos^{-1} 0$

$5\theta = 2n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$

$\theta = \frac{1}{5}\left(2n\pi \pm \frac{\pi}{2}\right), n \in \mathbb{Z}$

Also $\cos\theta + 1 = 0$

$\theta = \cos^{-1}(-1) = 2n\pi \pm \pi, n \in \mathbb{Z}$.

Therefore, the general solution is $\theta = \frac{1}{5}\left(2n\pi \pm \frac{\pi}{2}\right), n \in \mathbb{Z}$

$\theta = 2n\pi \pm \pi, n \in \mathbb{Z}$.

EXERCISE 9E

In questions **1** to **10**, factorise each expression.

1 $\sin 4x - \sin x$

2 $\cos 3x + \cos 2x$

3 $\cos 5A - \cos A$

4 $\sin 4A + \sin 4B$

5 $\cos 6A - \cos 4A$

6 $\cos 2A - \cos 8A$

7 $\sin 7x + \sin 3x$

8 $\cos 5x + \cos 3x$

9 $\sin 6x - \sin 2x$

10 $\sin 7x + \sin 5x$

11 Evaluate the following without the use of calculators:

(a) $\cos\frac{5\pi}{12} + \cos\frac{\pi}{12}$

(b) $\cos\frac{5\pi}{12} - \cos\frac{\pi}{12}$

(c) $\sin\frac{5\pi}{12} - \sin\frac{\pi}{12}$

12 Prove that $\frac{\cos\alpha + \cos}{\sin\alpha + \sin\beta} = \cot\left(\frac{\alpha + \beta}{2}\right)$.

13 Prove that $\sin 40° + \cos 70° = 2\cos 60°\cos 10°$.

In questions **14** to **23**, find the general solution of the following equations.

14 $\cos 5x + \cos x = 0$

15 $\sin 6x + \sin 2x = 0$

16 $\cos 6x - \cos 4x = 0$

17 $\sin 3x = \sin x$

18 $\cos 6x + \cos 2x = \cos 4x$

19 $\sin 7x + \sin x = \sin 4x$

20 $\cos 5x - \sin 3x - \cos x = 0$

21 $\sin 3x + \sin 4x + \sin 5x = 0$

22 $\sin x + 2 \sin 2x + \sin 3x = 0$

23 $\cos 3x + \cos x + 2 \cos 2x = 0$

In questions **24** to **33**, simplify the fractions.

24 $\dfrac{\sin 4\theta + \sin \theta}{\cos 4\theta + \cos \theta}$

25 $\dfrac{\sin 6\theta - \sin 2\theta}{\cos 6\theta + \cos 2\theta}$

26 $\dfrac{\sin 8\theta + \sin 4\theta}{\cos 8\theta - \cos 4\theta}$

27 $\dfrac{\cos 7\theta + \cos \theta}{\sin 7\theta + \sin \theta}$

28 $\dfrac{\sin x + 2 \sin 3x + \sin 5x}{\sin 3x + 2 \sin 5x + \sin 7x}$

29 $\dfrac{\sin x + \sin 2x}{\cos x - \cos 2x}$

30 $\dfrac{\sin 3x + \sin 2x}{\sin 3x - \sin 2x}$

31 $\dfrac{\cos 3x + \cos x}{\sin 3x + \sin x}$

32 $\dfrac{\sin 7\theta + \sin \theta}{\cos 7\theta + \cos \theta}$

33 $\dfrac{\cos \theta + 2 \cos 2\theta + \cos 3\theta}{\cos \theta - 2 \cos 2\theta + \cos 3\theta}$

In questions **34** to **38**, prove each identity.

34 $\dfrac{\sin 4\theta + \sin 6\theta + \sin 5\theta}{\cos 4\theta + \cos 6\theta + \cos 5\theta} = \tan 5\theta$

35 $\dfrac{\sin 6\theta + \sin 7\theta + \sin \theta + \sin 2\theta}{\cos 2\theta + \cos \theta + \cos 6\theta + \cos 7\theta} = \tan 4\theta$

36 $\dfrac{\sin \theta + \sin 3\theta + \cos 5\theta + \cos 7\theta}{\sin 4\theta + \cos 8\theta + \cos 4\theta} = \dfrac{\sec \theta}{2 - \sec^2 \theta}$

37 $\dfrac{\cos\theta - 2\cos 3\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 7\theta} = -\tan^2 2\theta$

38 $\dfrac{\cos 5\theta + 2\cos 7\theta + \cos 9\theta}{\cos 5\theta - 2\cos 7\theta + \cos 9\theta} = -\cot^2\theta$

In questions **39** to **44**, express the following as the sum or difference of two ratios.

39 $2\sin 6\theta\cos\theta$

40 $-2\sin 8\theta\cos 4\theta$

41 $2\cos 6\theta\cos 2\theta$

42 $2\sin 7\theta\sin\theta$

43 $2\cos 7\theta\cos 3\theta$

44 $-2\sin 7\theta\cos 3\theta$

45 Prove that $2\cos x(\sin 3x - \sin x) = \sin 4x$.

46 Show that $\sin 5x + \sin x = 2\sin 3x\cos 2x$. Hence, find the general solution of $\sin 5x + \sin x + \cos 2x = 0$.

47 If $\mathrm{P} + \mathrm{Q} + \mathrm{R} = 180°$, prove the following.

(a) $\sin 2\mathrm{P} + \sin 2\mathrm{Q} + \sin 2\mathrm{R} = 4\sin\mathrm{P}\sin\mathrm{Q}\sin\mathrm{R}$
(b) $\sin 2\mathrm{P} + \sin 2\mathrm{Q} - \sin 2\mathrm{R} = 4\cos\mathrm{P}\cos\mathrm{Q}\sin\mathrm{R}$

48 If P, Q and R are the angles of a triangle, prove the following.
(a) $\sin(\mathrm{Q} + \mathrm{R}) = \sin\mathrm{P}$
(b) $\cos(\mathrm{Q} + \mathrm{R}) = -\cos\mathrm{P}$

49 If α, β and γ are the angles of a triangle, prove the following.

(a) $\sin\beta\cos\gamma + \cos\beta\sin\gamma = \sin\alpha$
(b) $\cos\gamma + \cos\beta\cos\alpha = \sin\alpha\sin\beta$
(c) $\sin\alpha - \cos\beta\sin\gamma = \sin\beta\cos\gamma$

50 Show that $1 + \cos 2\theta + \cos 4\theta + \cos 6\theta = 4\cos\theta\cos 2\theta\cos 3\theta$. Hence, find the general solution of the equation $1 + \cos 2\theta + \cos 4\theta + \cos 6\theta = 0$.

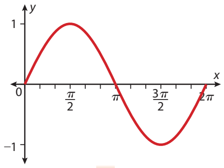
51 Prove that $1 - \cos 2\theta + \cos 4\theta - \cos 6\theta = 4\sin\theta\cos 2\theta\sin 3\theta$. Hence, find the general solution of the equation $1 - \cos 2\theta + \cos 4\theta - \cos 6\theta = 0$.

SUMMARY

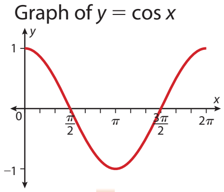
Trigonometry

Graphs and general solutions

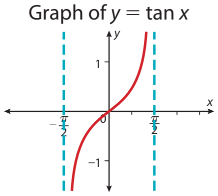
Graph of $y = \sin x$



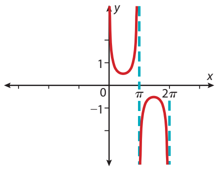
Graph of $y = \cos x$



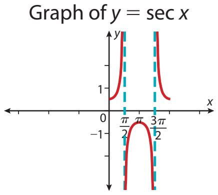
Graph of $y = \tan x$



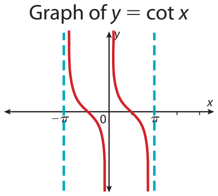
Graph of $y = \operatorname{cosec} x$



Graph of $y = \sec x$



Graph of $y = \cot x$



General solution of $\sin x$:
$x = n\pi + (-1)^n\alpha, n \in \mathbb{Z}$

General solution of $\cos x$:
$x = 2n\pi \pm \alpha, n \in \mathbb{Z}$

General solution of $\tan x$:
$x = n\pi + \alpha, n \in \mathbb{Z}$

Trigonometric identities

$\sin^2\theta + \cos^2\theta = 1$
$\tan^2\theta + 1 = \sec^2\theta$
$1 + \cot^2\theta = \operatorname{cosec}^2\theta$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\cos 2\theta = 2\cos^2\theta - 1$
$\cos 2\theta = 1 - 2\sin^2\theta$
$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

If $t = \tan\dfrac{\theta}{2}$, then:

$\sin\theta = \dfrac{2t}{t^2 + 1}$

$\cos\theta = \dfrac{1 - t^2}{1 + t^2}$

$\tan\theta = \dfrac{2t}{1 - t^2}$

$2\sin A\cos B = \sin(A + B) + \sin(A - B)$
$2\cos A\sin B = \sin(A + B) - \sin(A - B)$
$2\cos A\cos B = \cos(A + B) + \cos(A - B)$
$-2\sin A\sin B = \cos(A + B) - \cos(A - B)$

$\sin A + \sin B = 2\sin\dfrac{A + B}{2}\cos\dfrac{A - B}{2}$

$\sin A - \sin B = 2\cos\dfrac{A + B}{2}\sin\dfrac{A - B}{2}$

$\cos A + \cos B = 2\cos\dfrac{A + B}{2}\cos\dfrac{A - B}{2}$

$\cos A - \cos B = -2\sin\dfrac{A + B}{2}\sin\dfrac{A - B}{2}$

$a\cos\theta + b\sin\theta \equiv r\cos\theta$
where $r > 0$ and $0° < \alpha < 90°$

Maximum $(a\cos\theta + b\sin\theta) = r$,
when $\cos(\theta - \alpha) = 1$

Minimum $(a\cos\theta + b\sin\theta) = -r$,
when $\cos(\theta - \alpha) = -1$

Checklist

Can you do these?

- Prove and use the identity $\sin^2\theta + \cos^2\theta = 1$.
- Prove and use the identity $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.
- Prove and use the identity $\tan^2\theta + 1 = \sec^2\theta$.
- Solve trigonometric equations involving quadratics.
- Prove and use $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$.
- Use $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$.
- Use $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$.
- Prove and use the double angle results.
- Convert $a\cos + b\sin\theta$ to $R\cos(\theta - \alpha)$, $R > 0$, $0° < \alpha < 90°$.
- Identify the maximum and minimum of $a\cos\theta + b\sin\theta$.
- Identify the angle at which $a\cos\theta + b\sin\theta$ is a maximum or minimum.
- Solve equations of the form $a\cos\theta + b\sin\theta = c$, $c \neq 0$.
- Solve equations using the double angle results.
- Convert products to sums and differences.
- Convert sums and differences to products.

Review Exercise 9

1 Prove the following identities.

(a) $\dfrac{\cos\theta}{1-\sin\theta} = \sec\theta + \tan\theta$

(b) $\dfrac{1}{\sec x - \tan x} = \sec x + \tan x$

2 Find the general solution of

(a) $3\cos^2 x = 1 + \sin x$

(b) $3\cos x = 2\sin^2 x$

3 Simplify the expression $\sqrt{\dfrac{9 + 9\cos 6\alpha}{2}}$.

4 Show that $\dfrac{\cos 3\alpha}{\cos\alpha} + \dfrac{\sin 3\alpha}{\sin\alpha} = 4\cos 2\alpha$.

5 Show that $\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin^2\alpha - \sin^2\beta$.

6 Given that $\sin\theta = \frac{5}{13}$, $0° < \theta < 90°$ and $\sin\alpha = -\frac{3}{5}$, $180° < \alpha < 270°$. Find

(a) $\sin(\theta + \alpha)$

(b) $\cos(\theta + \alpha)$.

7 Prove each of the following identities

(a) $\dfrac{\cos\theta - 2\cos 3\theta + \cos 7\theta}{\cos\theta + 2\cos 3\theta + \cos 7\theta} = -\tan^2 2\theta$

(b) $\dfrac{\cos 5\theta + 2\cos 7\theta + \cos 9\theta}{\cos 5\theta - 2\cos 7\theta + \cos 9\theta} = -\cot^2\theta$

8 Find the exact value of $\cos 15°$.

9 Show that $\dfrac{\sin(\theta - \alpha)}{\sin\theta\sin\alpha} = \cot\alpha - \cot\theta$.

10 Show that $\sin\left(\dfrac{\pi}{4} + x\right)\cos\left(\dfrac{\pi}{4} + x\right) = \dfrac{1}{2}\cos 2x$.

11 Given that $f(\theta) = 3\sin\theta + 4\cos\theta$, express $f(\theta)$ in the form $r\sin(\theta + \alpha)$ where $r > 0$ and $0° < \alpha < 90°$.

Find the maximum value of $f(\theta)$. Hence, find the minimum value of $\dfrac{1}{10 + f(\theta)}$.

12 f is a function given by $f(x) = \dfrac{1}{2}\sin\left(4x + \dfrac{\pi}{2}\right)$.

(a) Find the range of f. (b) Find the period of f.

(c) Sketch the graph of f over one period.

13 (a) Express $2\cos(2x) + 3\sin(2x)$ in the form $r\cos(2x - \theta)$ where $r > 0$ and $0° < \theta < 90°$.

(b) Hence, find the general solution of $2\cos(2x) + 3\sin(2x) = 2$.

(c) Find also the maximum value of $2\cos(2x) + 3\sin(2x)$ and the value of x for which $2\cos(2x) + 3\sin(2x)$ is a maximum.

In questions **14** to **16**, express the following as the sum or difference of two ratios.

14 $2\sin 6\theta\cos\theta$

15 $-2\sin 8\theta\cos 4\theta$

16 $2\cos 6\theta\cos 2\theta$

17 Find the general solution of

(a) $2\tan x - 1 = 3\cot x$ (b) $6\sec^2 Z = \tan Z + 8$.

18 Prove each of the following.

(a) $\dfrac{\sin x}{-\sec x + 1} + \dfrac{\sin x}{\sec x + 1} = -2\cos x\cot x$

(b) $\dfrac{1 - \sin x}{\cos x} = \dfrac{\cos x}{1 + \sin x}$ (c) $\cot\theta + \tan\theta = \sec\theta\operatorname{cosec}\theta$

19 Find the general solution of the following.

(a) $\cos 5x - \sin 3x - \cos x = 0$

(b) $\sin 3x + \sin 4x + \sin 5x = 0$

20 Simplify the following as far as possible.

(a) $\dfrac{\sin 7\theta - \sin\theta}{\cos 7\theta + \cos\theta}$

(b) $\dfrac{\cos\theta + 2\cos 2\theta + \cos 3\theta}{\cos\theta - 2\cos 2\theta + \cos 3\theta}$

CHAPTER 10

Coordinate Geometry

At the end of this chapter you should be able to:

- Recognise the Cartesian equation of a circle, ellipse and parabola
- Identify the centre and radius of a circle
- Find the equation of a circle given its centre and radius
- Find the equation of a tangent to a circle, ellipse and parabola
- Find the equation of a normal to a circle, ellipse and parabola
- Find the point of intersection of a curve and a straight line
- Find the points of intersection of two curves
- Find the Cartesian equation of a circle, ellipse or parabola given its parametric equations
- Find the parametric equations of a circle, ellipse or parabola given its Cartesian equation
- Find the foci of an ellipse
- Find the length of the major axis and the length of the minor axis of an ellipse
- Draw the graph of an ellipse
- Draw the graph of a parabola

KEYWORDS/TERMS

coordinates • geometry • circle • radius • centre • intersection • tangent • normal • Cartesian equation • parametric equation • ellipse • parabola • focus • directrix • symmetry • major axis • minor axis • vertex • latus rectum

Review of coordinate geometry

Let A be the point with coordinates (x_1, y_1) and B be the point with coordinates (x_2, y_2).

The length of line segment AB $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The midpoint of AB $= \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

The gradient of AB $= \dfrac{y_2 - y_1}{x_2 - x_1}$

Equation of a straight line

(i) Let m be the gradient of a line and (x_1, y_1) be a point on the line.

The equation of the line is given by:

$y - y_1 = m(x - x_1)$

(ii) Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points on a line. The equation of the line passing through A and B is given by:

$y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

A line parallel to the x-axis has equation $y = c$, where c is a constant. The gradient of the line is 0.

A line parallel to the y-axis has equation $x = c$, where c is a constant. The gradient of the line approaches infinity.

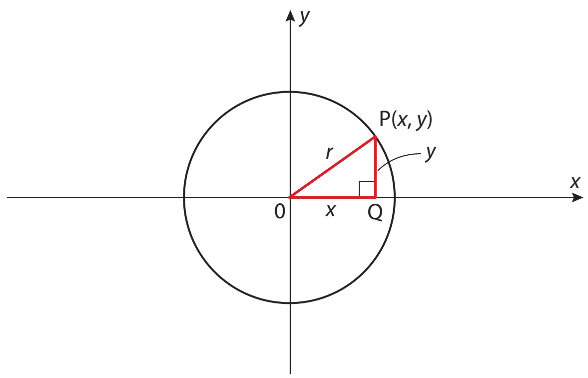
Parallel lines

Two lines are parallel if and only if their gradients are the same.

Perpendicular lines

Two lines are perpendicular if and only if the product of their gradients is -1.

The equation of a circle



Let P(x, y) be any point on the circle with centre (0, 0).

By Pythagoras' theorem:

$OP^2 = OQ^2 + QP^2$

$\Rightarrow r^2 = x^2 + y^2$

$\therefore$ the equation of a circle with its centre at the origin (0, 0) and radius r is $x^2 + y^2 = r^2$.

The equation $x^2 + y^2 = 4$ represents a circle with centre (0, 0) and radius 2.

EXAMPLE 1 Find the radius of the circle $2x^2 + 2y^2 = 18$.

SOLUTION We first write the equation in the form $x^2 + y^2 = r^2$.

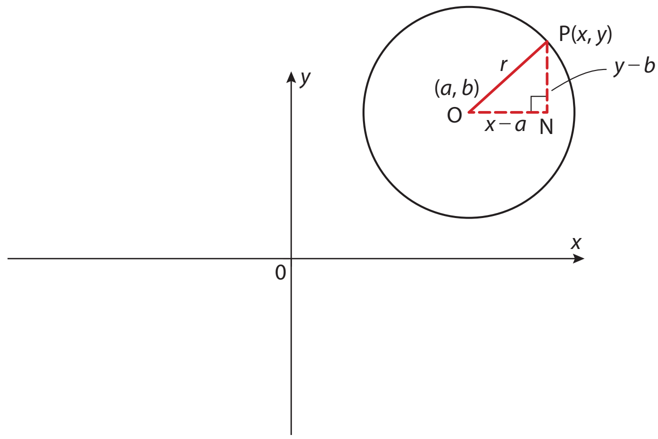
If $2x^2 + 2y^2 = 18$

$\Rightarrow x^2 + y^2 = 9$

$\Rightarrow x^2 + y^2 = 3^2$

$\therefore$ the radius of the circle is 3.

Equation of a circle with centre (a, b) and radius r



Let P(x, y) be any point on the circle.

From the triangle:

$r^2 = (x - a)^2 + (y - b)^2$ (Pythagoras' theorem)

$\therefore (x - a)^2 + (y - b)^2 = r^2$ represents the equation of a circle with centre (a, b) and radius r.

EXAMPLE 2 Find the equation of the circle with centre (2, 3) and radius 1.

SOLUTION Use $(x - a)^2 + (y - b)^2 = r^2$ where $a = 2$, $b = 3$ and $r = 1$.

The equation on the circle is:

$(x - 2)^2 + (y - 3)^2 = 1^2$

Expanding we have:

$x^2 - 4x + 4 + y^2 - 6y + 9 = 1$

$\Rightarrow x^2 + y^2 - 4x - 6y + 12 = 0$

EXAMPLE 3 Given that the radius of a circle is 4 units and the centre is $(1, -2)$, find the equation of the circle.

SOLUTION Using $(x - a)^2 + (y - b)^2 = r^2$ where $a = 1$, $b = -2$ and $r = 4$, we have:

$(x - 1)^2 + (y + 2)^2 = 4^2$

$\Rightarrow x^2 - 2x + 1 + y^2 + 4y + 4 = 16$

$\Rightarrow x^2 + y^2 - 2x + 4y - 11 = 0$

$\therefore$ $x^2 + y^2 - 2x + 4y - 11 = 0$ is the equation of the circle with centre $(1, -2)$ and radius 4 units.

General equation of the circle

The equation $(x - a)^2 + (y - b)^2 = r^2$ is the equation of a circle with centre (a, b) and radius r.

Expanding gives:

$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$

$\Rightarrow x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$

For an equation to represent a circle:

(i) The equation must be of second degree in x and y.

(ii) The coefficients of x^2 and y^2 must be the same.

(iii) There are no terms in the product xy.

From the equation $x^2 + y^2 - 2ax - 2by + a^2 + b^2 - r^2 = 0$, we can say the following:

(i) The centre of the circle is $(-\frac{1}{2}$ coefficient of x, $-\frac{1}{2}$ coefficient of $y)$.

(ii) The constant $c = a^2 + b^2 - r^2$ where r is the radius and (a, b) is the centre.

$\Rightarrow r^2 = a^2 + b^2 - c$

$\Rightarrow r = \sqrt{a^2 + b^2 - c}$

(iii) The coefficient of x^2 and y^2 are both 1.

EXAMPLE 5 Find the centre and radius of the circle $x^2 + y^2 - 4x - 6y - 3 = 0$.

SOLUTION $x^2 + y^2 - 4x - 6y - 3 = 0$

Since the coefficient of x^2 and y^2 is 1, the centre is:

$\left(\frac{-1}{2} \text{ coefficient of } x, \frac{-1}{2} \text{ coefficient of } y\right) = \left(-\frac{1}{2}(-4), -\frac{1}{2}(-6)\right) = (2, 3)$

Since $c = -3$

$$\begin{aligned} r &= \sqrt{a^2 + b^2 - c} \\ &= \sqrt{2^2 + 3^2 - (-3)} \\ &= \sqrt{4 + 9 + 3} \\ &= \sqrt{16} \\ &= 4 \end{aligned}$$

Hence, the centre is (2, 3) and the radius is 4.

We can also convert the equation $(x - a)^2 + (y - b)^2 = r^2$ as follows:

$x^2 + y^2 - 4x - 6y - 3 = 0$

Take the constant to the right-hand side and rearrange the equation as:

$x^2 - 4x + y^2 - 6y = 3$

Now complete the square of the quadratic in x and the quadratic in y:

$(x^2 - 4x + (-2)^2) - (-2)^2 + (y^2 - 6y + (-3)^2) - (-3)^2 = 3$

$\Rightarrow (x - 2)^2 - 4 + (y - 3)^2 - 9 = 3$

$\Rightarrow (x - 2)^2 + (y - 3)^2 = 16$

$\therefore (x - 2)^2 + (y - 3)^2 = 4^2$

The centre of the circle in (2, 3) and the radius is 4.

EXAMPLE 6 Find the radius and the coordinates of the centre of circle $2x^2 + 2y^2 - 12x - 8y + 18 = 0$.

SOLUTION Divide by 2 to make the coefficients of x^2 and y^2 each 1:

$\Rightarrow x^2 + y^2 - 6x - 4y + 9 = 0$

Centre of the circle is $\left(-\frac{1}{2}(-6), -\frac{1}{2}(-4)\right) = (3, 2)$.

Since $c = 9$

$$\begin{aligned} r &= \sqrt{a^2 + b^2 - c} \\ &= \sqrt{(3)^2 + (2)^2 - 9} \\ &= \sqrt{9 + 4 - 9} \\ &= \sqrt{4} \\ &= 2 \end{aligned}$$

$\therefore$ circle has centre (3, 2) and radius 2.

Or: $x^2 + y^2 - 6x - 4y + 9 = 0$

$\Rightarrow x^2 - 6x + y^2 - 4y = -9$

Complete the square:

$x^2 - 6x + (-3)^2 - (-3)^2 + y^2 - 4y + (-2)^2 - (-2)^2 = -9$

$\Rightarrow (x - 3)^2 - 9 + (y - 2)^2 - 4 = -9$

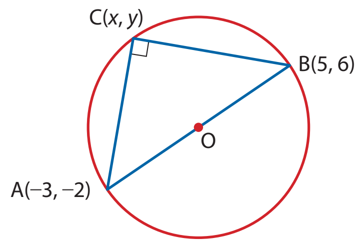
$(x - 3)^2 + (y - 2)^2 = 4$

$\Rightarrow (x - 3)^2 + (y - 2)^2 = 2^2$

The centre of the circle is (3, 2) and the radius is 2.

EXAMPLE 7 If the line joining A(−3, −2) and B(5, 6) is the diameter of a circle, find the equation of the circle.

SOLUTION



Let C(x, y) be a point on the circle.

Since AB is a diameter, △ABC is right-angled.

Gradient of AC × gradient of BC = −1

Gradient of AC $= \dfrac{y+2}{x+3}$

Gradient of BC $= \dfrac{y-6}{x-5}$

Since the product of the gradients is −1:

$$\frac{y+2}{x+3} \times \frac{y-6}{x-5} = -1$$

$$\Rightarrow \frac{y^2 - 4y - 12}{x^2 - 2x - 15} = -1$$

$$\Rightarrow y^2 - 4y - 12 = -x^2 + 2x + 15$$

$$\Rightarrow x^2 + y^2 - 2x - 4y - 27 = 0$$

The equation of the circle is $x^2 + y^2 - 2x - 4y - 27 = 0$.

Or: The centre of the circle is the midpoint of AB which is:

$$\left(\frac{5-3}{2}, \frac{6-2}{2}\right) = (1, 2)$$

Since AB is a diameter

Length of AB $= \sqrt{(6-(-2)^2 + (5-(-3))^2}$

$$r = \frac{1}{2}\sqrt{(6-(-2))^2 + (5-(-3))^2}$$

$$= \frac{1}{2}\sqrt{64+64}$$

$$= \frac{1}{2}\sqrt{128}$$

$$= \frac{1}{2}\sqrt{4 \times 32}$$

$$= \sqrt{32}$$

Using $(x-a)^2 + (y-b)^2 = r^2$, where centre is (a, b) and radius is r, gives $a = 1$, $b = 2$, $r = \sqrt{32}$.

∴ the equation of the circle is:

$$(x-1)^2 + (y-2)^2 = (\sqrt{32})^2$$

$$\Rightarrow x^2 - 2x + 1 + y^2 - 4y + 4 = 32$$

$$x^2 + y^2 - 2x - 4y - 27 = 0$$

EXAMPLE 8 Find the equation of the circle that passes through the points P(2, 3), Q(4, −1) and R(2, −1).

SOLUTION Let the centre of the circle be (a, b) and radius r.

The equation of the circle is $(x - a)^2 + (y - b)^2 = r^2$.

Since the points (2, 3), (4, −1) and (2, −1) are on the circle, they must satisfy the equation of the circle.

When $x = 2, y = 3 \Rightarrow (2 - a)^2 + (3 - b)^2 = r^2$ [1]

When $x = 4, y = -1 \Rightarrow (4 - a)^2 + (-1 - b)^2 = r^2$ [2]

When $x = 2, y = -1 \Rightarrow (2 - a)^2 + (-1 - b)^2 = r^2$ [3]

From [2] $\Rightarrow (4 - a)^2 = r^2 - (-1 - b)^2$

From [3] $\Rightarrow (2 - a)^2 = r^2 - (-1 - b)^2$

$\Rightarrow (4 - a)^2 = (2 - a)^2$ (Since [2] = [3])

$\Rightarrow 16 - 8a + a^2 = 4 - 4a + a^2$ (Expanding brackets)

$\Rightarrow 4a = 12$

$\Rightarrow a = 3$

Substitute $a = 3$ into [1] and [2] and equating:

$\Rightarrow (2 - 3)^2 + (3 - b)^2 = (4 - 3)^2 + (-1 - b)^2$

$\Rightarrow 1 + 9 - 6b + b^2 = 1 + 1 + 2b + b^2$

$8b = 8$

$b = 1$

Substitute $a = 3, b = 1$ into [1]:

$\Rightarrow (2 - 3)^2 + (3 - 1)^2 = r^2$

$\Rightarrow r^2 = 5$

Using $(x - a)^2 + (y - b)^2 = r^2$, where $a = 3, b = 1$ and $r^2 = 5$, the equation of the circle:

$(x - 3)^2 + (y - 1)^2 = 5$

$\Rightarrow x^2 - 6x + 9 + y^2 - 2y + 1 = 5$

$\Rightarrow x^2 + y^2 - 6x - 2y + 5 = 0$

As an alternative solution, we can find the centre of the circle by finding the point of intersection of the perpendicular bisector of PQ and QR as follows.

Finding the equation of the perpendicular bisector of PQ:

The mid-point of PQ is $\left(\frac{4 + 2}{2}, \frac{3 - 1}{2}\right) = (3, 1)$.

The gradient of PQ $= \frac{3 - (-1)}{2 - 4} = \frac{4}{-2} = -2$.

Since the product of the gradients of perpendicular lines is −1. The gradient of the perpendicular bisector $= \frac{1}{2}$.

The equation of the perpendicular bisector is $y - 1 = \frac{1}{2}(x - 3)$.

$y = \frac{1}{2}x - \frac{3}{2} + 1$

$= \frac{1}{2}x - \frac{1}{2}$

Now, we find the equation of the perpendicular bisector of QR.

The mid-point of QR $= \left(\frac{4+2}{2}, \frac{-1-1}{2}\right) = (3, -1)$.

Gradient of QR $= \frac{-1-(-1)}{4-2} = 0$.

The gradient of the perpendicular bisector $\to \infty$.

Since the gradient tends to infinity, the line is parallel to the x-axis and hence the equation is $x = 3$.

The equation of the perpendicular bisector is $x = 3$.

Solving the equations we have:

$y = \frac{1}{2}x - \frac{1}{2}$

$x = 3$

$\Rightarrow y = \frac{1}{2}(3) - \frac{1}{2}$

$= 1$

Therefore the point of intersection is (3, 1) which is the centre of the circle.

Using P(2, 3) and (3, 1) the radius of the circle is $r = \sqrt{(3-2)^2 + (1-3)^2} = \sqrt{5}$

The equation of the circle is:

$(x - 3)^2 + (y - 1)^2 = (\sqrt{5})^2$

$x^2 + y^2 - 6x - 2y + 5 = 0$

Try these 10.1

(a) (i) Find the equation of the perpendicular bisector of the line joining the points P(4, 5) and Q(5, 4).

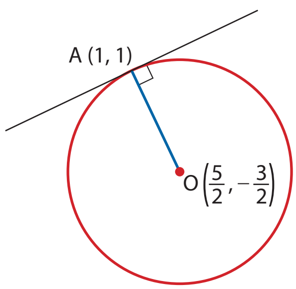
(ii) Find also the equation of the perpendicular bisector of the line joining the points Q(5, 4) and R(6, 1).

(iii) Hence, find the equation of the circle passing through the points P, Q and R.

(b) Find the equation of the circle passing through the points A(1, 3), B(−2, 6) and C(4, 2).

EXAMPLE 9

Find the equation of the tangent to the circle $3x^2 + 3y^2 - 15x + 9y = 0$ at the point A(1, 1).



SOLUTION

$3x^2 + 3y^2 - 15x + 9y = 0$

Divide by 3:

$\Rightarrow x^2 + y^2 - 5x + 3y = 0$

Centre of the circle is $\left(\frac{5}{2}, -\frac{3}{2}\right)$

$$\text{Gradient of OA} = \frac{-\frac{3}{2} - 1}{\frac{5}{2} - 1} = \frac{-\frac{5}{2}}{\frac{3}{2}} = -\frac{5}{3}$$

Since the tangent is perpendicular to the radius:

$$\text{gradient of the tangent} = \frac{-1}{-\frac{5}{3}} = \frac{3}{5}$$

Using $y - y_1 = m(x - x_1)$, the equation of the tangent at (1, 1) is:

$$y - 1 = \frac{3}{5}(x - 1)$$

$$y - 1 = \frac{3}{5}x - \frac{3}{5}$$

$$y = \frac{3}{5}x - \frac{3}{5} + 1$$

$$y = \frac{3}{5}x + \frac{2}{5}$$

EXAMPLE 10 Find the equation of the normal to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ at the point P(5, 1).

SOLUTION We first find the centre of the circle:

$$x^2 + y^2 - 4x + 6y - 12 = 0$$

$$x^2 - 4x + y^2 + 6y = 12$$

$$x^2 - 4x + (-2)^2 - (-2)^2 + y^2 + 6y + (+3)^2 - (+3)^2 = 12$$

$$(x - 2)^2 + (y + 3)^2 = 12 + 4 + 9$$

$$(x - 2)^2 + (y + 3)^2 = 25$$

Comparing with $(x - a)^2 + (y - b)^2 = r^2$, centre is $(2, -3)$.

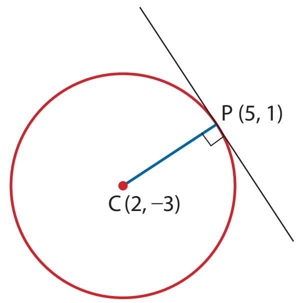
$$\text{Gradient of CP} = \frac{-3 - 1}{2 - 5} = \frac{-4}{-3} = \frac{4}{3}$$

Equation of the normal at (5, 1) is:

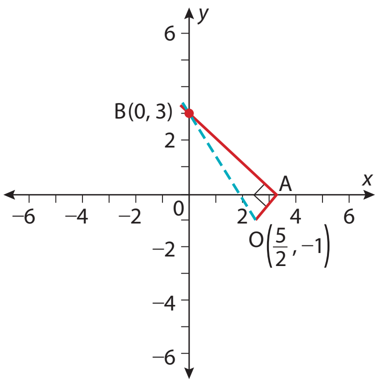
$$y - 1 = \frac{4}{3}(x - 5)$$

$$y - 1 = \frac{4}{3}x - \frac{20}{3}$$

$$y = \frac{4}{3}x - \frac{17}{3}$$



EXAMPLE 11 Given that the equation of a circle is $x^2 + y^2 - 5x + 2y + 1 = 0$, find the length of the tangent from the point B(0, 3) to the circle.



SOLUTION

$$x^2 + y^2 - 5x + 2y + 1 = 0$$

Centre of the circle is $\left(\frac{5}{2}, -1\right)$

$$r = \sqrt{\left(\frac{5}{2}\right)^2 + (-1)^2 - (1)}$$

$$= \frac{5}{2}$$

$$|OB| = \sqrt{\left(\frac{5}{2} - 0\right)^2 + (-1 - 3)^2}$$
$$= \sqrt{\frac{89}{4}}$$

$|OA| = \frac{5}{2}$ (radius of the circle)

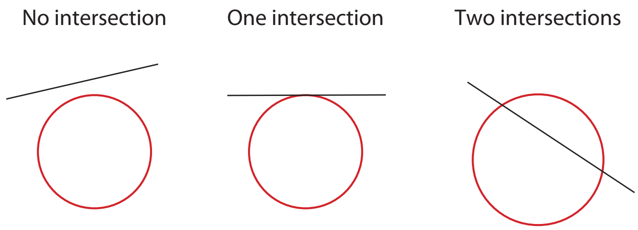
Using Pythagoras' theorem:

$$OB^2 = OA^2 + AB^2$$
$$\left(\sqrt{\frac{89}{4}}\right)^2 = \left(\frac{5}{2}\right)^2 + AB^2$$
$$\frac{89}{4} = \frac{25}{4} + AB^2$$
$$AB^2 = \frac{89}{4} - \frac{25}{4}$$
$$= \frac{64}{4} = 16$$
$$AB = \sqrt{16} = 4$$

$\therefore$ the length of the tangent from B is 4 units.

Intersection of a line and a circle

When the equation of the line is substituted into the equation of the circle, a quadratic equation is formed.



If $b^2 - 4ac < 0$, the line does not intersect the circle.

If $b^2 - 4ac = 0$, the line is a tangent to the circle.

If $b^2 - 4ac > 0$, the line and the circle intersect at two points.

EXAMPLE 12 Show that the line $y + 2x + 7 = 0$ does not intersect the circle $(x - 1)^2 + (y + 1)^2 = 9$.

SOLUTION

$$y + 2x + 7 = 0$$
$$\Rightarrow y = -7 - 2x$$

Substituting into the equation of the circle gives:

$$(x - 1)^2 + (y + 1)^2 = 9$$
$$\Rightarrow (x - 1)^2 + (-7 - 2x + 1)^2 = 9$$
$$\Rightarrow (x - 1)^2 + (-2x - 6)^2 = 9$$
$$\Rightarrow x^2 - 2x + 1 + 4x^2 + 24x + 36 - 9 = 0$$
$$\Rightarrow 5x^2 + 22x + 28 = 0$$

Using $b^2 - 4ac$, we get $b^2 - 4ac = 22^2 - 4(5)(28) = -76$

Since $b^2 - 4ac < 0$, the equation has no real roots.

Therefore, the line does not intersect the circle.

Intersection of two circles

EXAMPLE 13 Find the points of intersection of the circle with centre $(2, -3)$ and radius 5 and the circle with equation $x^2 + y^2 - 5x + 3y - 4 = 0$.

SOLUTION The equation of the circle with centre $(2, -3)$ and radius 5:

$(x - 2)^2 + (y + 3)^2 = 5^2$

$\Rightarrow x^2 - 4x + 4 + y^2 + 6y + 9 = 25$

$\Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$

Solving simultaneously gives:

$x^2 + y^2 - 4x + 6y - 12 = 0$ [1]

$x^2 + y^2 - 5x + 3y - 4 = 0$ [2]

$\Rightarrow x + 3y - 8 = 0$ [1] − [2]

$x = -3y + 8$

Substituting into [1]:

$\Rightarrow (8 - 3y)^2 + y^2 - 4(8 - 3y) + 6y - 12 = 0$

$\Rightarrow 64 - 48y + 9y^2 + y^2 - 32 + 12y + 6y - 12 = 0$

$10y^2 - 30y + 20 = 0$

$y^2 - 3y + 2 = 0$

$(y - 1)(y - 2) = 0$

$y = 1, 2$

When $y = 1, x = 8 - 3(1) = 5$

$y = 2, x = 8 - 3(2) = 2$

$\therefore$ the points of intersection are $(5, 1)$ and $(2, 2)$.

Try these 10.2

(a) Find the points of intersection of the two circles $x^2 + y^2 = 20$ and $(x - 6)^2 + (y - 3)^2 = 5$.

(b) The centre of a circle is $(-2, 3)$ and a point on the circumference is $(-5, -1)$.

(i) Find the equation of the line joining the two points.

(ii) Show that the radius of the circle is 5 units.

(iii) Write down the equation of the circle.

(iv) Determine the equation of the tangent to the circle at the point $(-5, -1)$.

(v) Find the points of intersection (if any exist) of the circle above with the circle $x^2 + y^2 + 6x - 7y - 10 = 0$.

Intersection of two curves

The points of intersection of two curves can be found by solving the equations simultaneously.

EXAMPLE 14 Find the points of intersection of the curves $y = \frac{6}{x}$ and $x^2 - y^2 = 5$.

SOLUTION Substituting $y = \frac{6}{x}$ into $x^2 - y^2 = 5$ we have:

$x^2 - \left(\frac{6}{x}\right)^2 = 5$

$x^2 - \frac{36}{x^2} = 5$

$\Rightarrow x^4 - 36 = 5x^2$

$\Rightarrow x^4 - 5x^2 - 36 = 0$

Factorising gives:

$(x^2 - 9)(x^2 + 4) = 0$

$\Rightarrow x^2 - 9 = 0, x^2 + 4 = 0$

Since x is real $x^2 = 9$.

$\Rightarrow x = 3, -3$

When $x = 3, y = \frac{6}{3} = 2$

When $x = -3, y = \frac{6}{-3} = -2$

Therefore the points of intersections of the two curves are $(3, 2)$ and $(-3, -2)$.

EXAMPLE 15 The curves $y = x^2$ and $4x^2 + y + 13x = 6$ intersect at the points A and B. Find the midpoint of AB.

SOLUTION Solving the equations simultaneously gives:

$y = x^2$ [1]

$4x^2 + y + 13x = 6$ [2]

$5x^2 + 13x = 6$ [1] into [2]

$\Rightarrow 5x^2 + 13x - 6 = 0$

$\Rightarrow (5x - 2)(x + 3) = 0$

$\Rightarrow x = \frac{2}{5}, -3$

When $x = \frac{2}{5}, y = \left(\frac{2}{5}\right)^2 = \frac{4}{25}$

When $x = -3, y = (-3)^2 = 9$

Therefore $A\left(\frac{2}{5}, \frac{4}{25}\right)$ and $B(-3, 9)$.

The midpoint of AB is $\left(\dfrac{\frac{2}{5} - 3}{2}, \dfrac{\frac{4}{25} + 9}{2}\right) = \left(-\frac{13}{10}, \frac{229}{50}\right)$.

EXERCISE 10A

1 Find the equation of the circle with centre (1, 1) and radius 4.

2 Find the equation of the circle with centre $(-2, 3)$ and radius 5.

3 Find the equation of the circle with centre (4, 2) and radius 7.

4 Find the equation of the circle with centre (0, 2) and radius 1.

5 Find the equation of the circle with centre $(-1, 1)$ and radius 2.

6 Find the equation of the circle with centre (3, 0) and radius $\sqrt{2}$.

7 Find the equation of the circle with centre $(-1, 2)$ and passing through (4, 1).

8 Find the equation of the circle with centre $(-3, 1)$ and passing through (2, 2).

9 Find the equation of the circle with centre (1, 1) and passing through (4, 6).

10 Find the equation of the circle with diameter AB where A is at (2, 4) and B is at $(-1, 6)$.

11 Find the centre and radius of each of these circles.

(a) $6x^2 + 6y^2 - 4x - 5y - 2 = 0$

(b) $x^2 + y^2 + 6x + 8y - 1 = 0$

(c) $3x^2 + 3y^2 - 4x + 8y - 2 = 0$

12 Find the equation of the tangent to the circle $x^2 + y^2 - 2x + 4y = 0$ at the point (2, 0).

13 Find the equation of the tangent to the circle $x^2 + y^2 - 6x + 4y + 3 = 0$ at the point $(0, -3)$.

14 Find the equation of the tangent to the circle $2x^2 + 2y^2 - x + 4y - 15 = 0$ at (3, 0).

15 Find the equation of the normal to the circle $x^2 + y^2 + 6x - 16 = 0$ at the point $(1, -3)$.

16 Find the equation of the normal to the circle $3x^2 + 3y^2 - 6x + 12y = 0$ at the point $(0, -4)$.

Parametric representation of a curve

Let $x = f(t)$ and $y = g(t)$, where f and g are two functions of t. These two equations are called the parametric equations of a curve with t being the parameter of the equation. The equation $y = f(x)$ is the Cartesian equation of the curve.

Cartesian equation of a curve given its parametric form

EXAMPLE 16 Given that $x = 2t - 1$, $y = t^2$, find the Cartesian equation of the curve.

SOLUTION The Cartesian equation represents an equation that connects x and y only. We can make t the subject of the formula in one of the equations and substitute into the next.

Since $x = 2t - 1$

$x + 1 = 2t$

$t = \dfrac{x+1}{2}$

Substituting into $y = t^2$, we get:

$y = \left(\dfrac{x+1}{2}\right)^2$

$y = \dfrac{1}{4}x^2 + \dfrac{1}{2}x + \dfrac{1}{4}$

This is the Cartesian equation of the curve.

Or, since $y = t^2$

$t = \sqrt{y}$

Substituting into $x = 2t - 1$, we have:

$x = 2\sqrt{y} - 1$

We can make y the subject of the formula:

$2\sqrt{y} = x + 1$

$\sqrt{y} = \dfrac{x+1}{2}$

$\Rightarrow y = \left(\dfrac{x+1}{2}\right)^2$

Hence, $y = \dfrac{1}{4}x^2 + \dfrac{1}{2}x + \dfrac{1}{4}$ is the Cartesian equation of the curve.

EXAMPLE 17 Find the Cartesian equation of $x = 2t^2 - 3t + 1$, $y = t + 1$.

SOLUTION In this case, it is easier to use $y = t + 1$ and thus $t = y - 1$.

Substitute into $x = 2t^2 - 3t + 1$:

$x = 2(y - 1)^2 - 3(y - 1) + 1$

$\Rightarrow x = 2y^2 - 7y + 6$

This is the Cartesian equation of the curve.

EXAMPLE 18 Given that $x = \dfrac{2t}{t+1}$ and $y = \dfrac{t+1}{t-1}$, find y in terms of x.

SOLUTION Using $x = \dfrac{2t}{t+1}$ and making t the subject of the formula:

$\Rightarrow xt + x = 2t$

$xt - 2t = -x$

$t(x - 2) = -x$

$t = \dfrac{-x}{x - 2}$

$\quad = \dfrac{x}{2 - x}$

Substituting into y gives:

$$y = \frac{\dfrac{x}{2-x} + 1}{\dfrac{x}{2-x} - 1}$$

$$= \frac{\dfrac{x+2-x}{2-x}}{\dfrac{x-2+x}{2-x}} \qquad \left(\text{Since } \frac{x}{2-x} + 1 = \frac{x+2-x}{2-x} \text{ and } \frac{x}{2-x} - 1 = \frac{x-2+x}{2-x}\right)$$

$$= \frac{x+2-x}{x-2+x}$$

$$= \frac{2}{2x-2}$$

$$= \frac{1}{x-1}$$

The Cartesian equation is $y = \dfrac{1}{x-1}$.

Parametric equations in trigonometric form

Trigonometric conversions can be done using the identities or right-angled triangle.

EXAMPLE 19 Find the Cartesian equation of the curve $x = \cos t$, $y = \sin t$.

SOLUTION

Method 1

$x = \cos t$

$x^2 = \cos^2 t \qquad [1]$

$y = \sin t$

$y^2 = \sin^2 t \qquad [2]$

$x^2 + y^2 = \sin^2 t + \cos^2 t \qquad [1] + [2]$

Recall that $\sin^2 t + \cos^2 t = 1$

$\Rightarrow x^2 + y^2 = 1$

Hence, the Cartesian equation is $x^2 + y^2 = 1$.

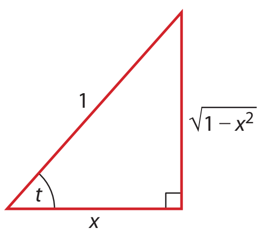
> **Note**
>
> $x^2 + y^2 = 1$ is a circle with centre $(0, 0)$ and radius 1 unit.

Method 2

Since $\cos t = x$

$\cos t = \dfrac{x}{1}$

Draw a triangle:



Pythagoras' theorem:

$\sin t = \dfrac{\sqrt{1 - x^2}}{1}$

$\sin t = \sqrt{1 - x^2}$

Since $y = \sin t$

$y = \sqrt{1 - x^2}$

$\Rightarrow y^2 = 1 - x^2$ (Squaring both sides)

$x^2 + y^2 = 1$

Hence, the Cartesian equation is $x^2 + y^2 = 1$.

EXAMPLE 20 Find the Cartesian equation of $x = 3 \cos t$, $y = 4 \sin t$.

SOLUTION

Method 1

$x = 3 \cos t$

Make $\cos t$ the subject:

$\cos t = \dfrac{x}{3}$

Square both sides:

$\cos^2 t = \dfrac{x^2}{9}$ [1]

$y = 4 \sin t$

$\sin t = \dfrac{y}{4}$

$\sin^2 t = \dfrac{y^2}{16}$ [2]

$\cos^2 t + \sin^2 t = \dfrac{x^2}{9} + \dfrac{y^2}{16}$ [1] + [2]

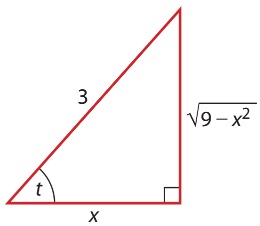
$\Rightarrow 1 = \dfrac{x^2}{9} + \dfrac{y^2}{16}$

$\Rightarrow 16x^2 + 9y^2 = 144$

Hence, the Cartesian equation is $16x^2 + 9y^2 = 144$.

Method 2

$\cos t = \dfrac{x}{3}$



$\sin t = \dfrac{\sqrt{9 - x^2}}{3}$

Since $y = 4 \sin t$

$\Rightarrow y = 4 \times \dfrac{\sqrt{9 - x^2}}{3}$

$\Rightarrow 3y = 4\sqrt{9 - x^2}$

$\Rightarrow 9y^2 = 16(9 - x^2)$ (Squaring both sides)

$\Rightarrow 9y^2 = 144 - 16x^2$

$9y^2 + 16x^2 = 144$

Therefore, the Cartesian equation is $9y^2 + 16x^2 = 144$.

EXAMPLE 21 Find the Cartesian equation of the curve $x = 3\sec t - 2$, $y = \tan t + 1$.

SOLUTION

$x = 3\sec t - 2$

$x + 2 = 3\sec t$

$\sec t = \dfrac{x+2}{3}$

$\sec^2 t = \dfrac{(x+2)^2}{9}$ [1]

$\tan t = y - 1$

$\tan^2 t = (y-1)^2$ [2]

Since $\sec^2 t - \tan^2 t = 1$:

$\sec^2 t - \tan^2 t = \dfrac{(x+2)^2}{9} - (y-1)^2$ [1] − [2]

$1 = \dfrac{(x+2)^2}{9} - (y-1)^2$

Therefore, the Cartesian equation is $\dfrac{(x+2)^2}{9} - (y-1)^2 = 1$.

Note

We can use the right-angled triangle to find the Cartesian equation.

EXAMPLE 22 Sketch the graph of $x = t^2 + 1$, $y = 2t - 1$.

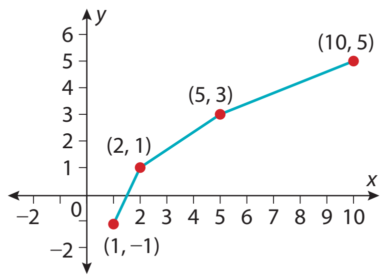
SOLUTION

When $t = 0$, $x = 1$, $y = -1$ $(1, -1)$

When $t = 1$, $x = 2$, $y = 1$ $(2, 1)$

When $t = 2$, $x = 5$, $y = 3$ $(5, 3)$

When $t = 3$, $x = 10$, $y = 5$ $(10, 5)$



Parametric equations of a circle

The parametric equations of a circle with centre (0, 0) and radius r are:

$x = r\cos\theta$ $y = r\sin\theta$

PROOF

Since $x = r\cos\theta$,

$x^2 = r^2\cos^2\theta$ [1]

Since $y = r\cos\theta$,

$y^2 = r^2\sin^2\theta$ [2]

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$ [1] + [2]

$\Rightarrow x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow x^2 + y^2 = r^2$ (Since $\cos^2\theta + \sin^2\theta = 1$)

This is the Cartesian equation of a circle with centre (0, 0) and radius r.

The parametric equations of a circle with centre (a, b) and radius r are:

$x = a + r\cos\theta$ $y = b + r\sin\theta$

PROOF

We use $x = a + r\cos\theta$, and make $\cos\theta$ the subject of the formula:

$$\cos\theta = \frac{x-a}{r}$$

$$\Rightarrow \cos^2\theta = \frac{(x-a)^2}{r^2} \qquad [1]$$

We use $y = b + r\sin\theta$, and make $\sin\theta$ the subject of the formula:

$$\sin\theta = \frac{y-b}{r}$$

$$\Rightarrow \sin^2\theta = \frac{(y-b)^2}{r^2} \qquad [2]$$

$$\cos^2\theta + \sin^2\theta = \frac{(x-a)^2}{r^2} + \frac{(y-b)^2}{r^2} \qquad [1] + [2]$$

$$\Rightarrow 1 = \frac{(x-a)^2}{r^2} + \frac{(y-b)^2}{r^2}$$

$$\Rightarrow (x-a)^2 + (y-b)^2 = r^2$$

This is the Cartesian equation of a circle with centre (a, b) and radius r.

EXAMPLE 23

Find the Cartesian equation of:

$x = 2 + 3\cos\theta \qquad y = 4 + 3\sin\theta$

Describe the curve in full.

SOLUTION

$$x = 2 + 3\cos\theta$$

$$\Rightarrow \cos\theta = \frac{x-2}{3}$$

$$\Rightarrow \cos^2\theta = \frac{(x-2)^2}{3^2} \qquad [1]$$

$$y = 4 + 3\sin\theta$$

$$\Rightarrow \sin\theta = \frac{y-4}{3}$$

$$\Rightarrow \sin^2\theta = \frac{(y-4)^2}{3^2} \qquad [2]$$

$$\cos^2\theta + \sin^2\theta = \frac{(x-2)^2}{3^2} + \frac{(y-4)^2}{3^2} \qquad [1] + [2]$$

$$\Rightarrow 1 = \frac{(x-2)^2}{3^2} + \frac{(y-4)^2}{3^2}$$

$$\Rightarrow (x-2)^2 + (y-4)^2 = 3^2$$

The curve is a circle with centre (2, 4) and radius 3.

EXAMPLE 24

Find the Cartesian equation of:

$x = 2 + 3\cos 2\theta \qquad y = 4 + 3\sin 2\theta$

SOLUTION

$$x = 2 + 3\cos 2\theta$$

$$\Rightarrow \cos 2\theta = \frac{x-2}{3}$$

Note

In Examples 23 and 24, we started off with two different parametric equations but ended with the same Cartesian equation. Can you identify the difference?

$$\Rightarrow \cos^2(2\theta) = \frac{(x-2)^2}{3^2} \qquad [1]$$

$y = 4 + 3\sin 2\theta$

$$\Rightarrow \sin 2\theta = \frac{y-4}{3}$$

$$\Rightarrow \sin^2(2\theta) = \frac{(y-4)^2}{3^2} \qquad [2]$$

$$\cos^2(2\theta) + \sin^2(2\theta) = \frac{(x-2)^2}{3^2} + \frac{(y-4)^2}{3^2} \qquad [1] + [2]$$

$$\Rightarrow 1 = \frac{(x-2)^2}{3^2} + \frac{(y-4)^2}{3^2}$$

$$\Rightarrow (x-2)^2 + (y-4)^2 = 3^2$$

The Cartesian equation is $(x-2)^2 + (y-4)^2 = 3^2$.

EXAMPLE 25 The line l has equation $3x - 4y = 0$ and the circle C has equation $x^2 + y^2 = 25$.

(a) If l intersects C at the points A and B, find the coordinates of the midpoint of AB.

(b) Find the values of a such that $x = a\cos\theta$ and $y = a\sin\theta$ are the parametric equations of C.

SOLUTION

(a) $3x - 4y = 0$

$\Rightarrow 4y = 3x$

$\Rightarrow y = \frac{3}{4}x$

Substituting $y = \frac{3}{4}x$ into $x^2 + y^2 = 25$ gives:

$$x^2 + \left(\frac{3}{4}x\right)^2 = 25$$

$$\Rightarrow x^2 + \frac{9}{16}x^2 = 25$$

$$\Rightarrow \frac{25}{16}x^2 = 25$$

$$\Rightarrow x^2 = 16 \times \frac{25}{25}$$

$$\Rightarrow x^2 = 16$$

Therefore, $x = \pm 4$

When $x = 4$, $y = \frac{3}{4} \times 4 = 3$

When $x = -4$, $y = \frac{3}{4} \times -4 = -3$

Hence, A$(-4, -3)$ and B$(4, 3)$.

$$\text{Midpoint of AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$= \left(\frac{-4+4}{2}, \frac{-3+3}{2}\right)$$

$$= (0, 0)$$

The midpoint of AB is $(0, 0)$.

(b) $x = a\cos\theta \qquad y = a\sin\theta$

$\Rightarrow x^2 = a^2\cos^2\theta \qquad y^2 = a^2\sin^2\theta$

$x^2 + y^2 = a^2\cos^2\theta + a^2\sin^2\theta$

$\Rightarrow x^2 + y^2 = a^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow x^2 + y^2 = a^2$

Since we have $x^2 + y^2 = 25$,

$a^2 = 25$

$\Rightarrow a = \pm 5$

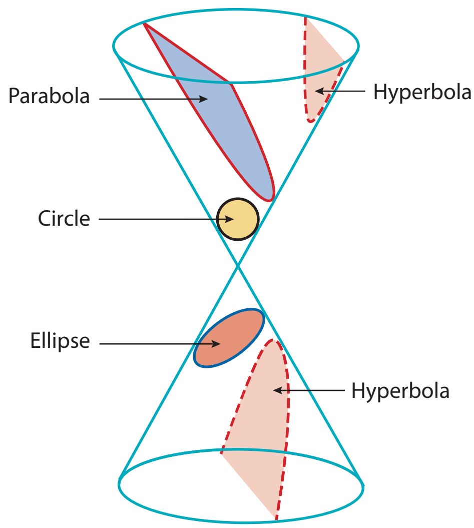
a must be positive.

Hence, $a = 5$.

Conic sections

A conic section is a section made from a cone. A conic section is formed by the intersection of a plane with a right circular cone. Conic sections are popular ways to describe light, motion and other occurrences.

There are four main conic sections: the circle, the parabola, the ellipse and the hyperbola. The angle at which the plane intersects the surface will determine the curve produced. From the diagram, we can see the circle, ellipse, hyperbola and parabola cut from the cone.



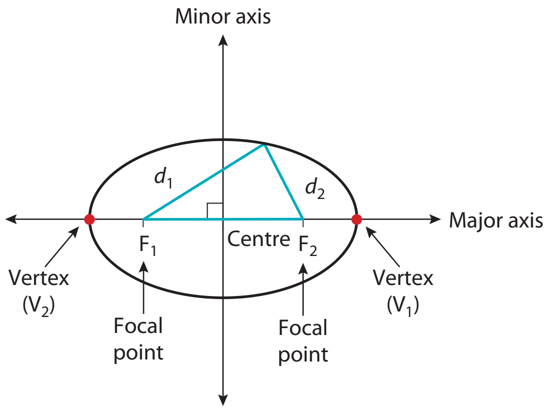
There are many real-world applications of conic sections. Some of examples of ellipses are: the paths of the planets around the sun are ellipses with the sun being a focus; electrons of an atom move in approximate elliptic orbit with the nucleus as a focus; and whispering galleries in cathedrals and other buildings are elliptical in shape. If you throw a ball in the air at some angle to the horizontal, the trajectory of the ball follows the path of a parabola. Parabolic surfaces are also used as headlamp reflectors.

Ellipses

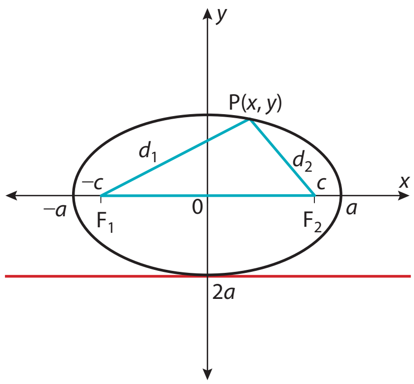
An ellipse is the locus of points the sum of whose distances from two fixed points is constant. The two fixed points are the **foci** (plural of focus) of the ellipse.

The diagram shows an ellipse.

- F_1 and F_2 show the position of the foci.
- The foci lie on the **major axis**.
- The midpoint of F_1F_2 is the centre of the ellipse.
- The **vertices** V_1 and V_2 are the points of intersection of the ellipse and the major axis.
- The length of V_1V_2 is equal to the sum of d_1 and d_2.
- The **minor axis** is a line perpendicular to the major axis and passing through the centre of the ellipse.



Equation of an ellipse



Let the point $P(x, y)$ be any point on the ellipse.

$F_1(-c, 0)$ and $F_2(c, 0)$ are the foci and $(0, 0)$ is the centre.

The vertices are at $(-a, 0)$ and $(a, 0)$. The distance between the vertices is $2a$. (This distance is equal to the sum of d_1 and d_2, which are the distances of P from the foci.)

From the diagram on the previous page:

$$d_1 = \sqrt{(x+c)^2 + y^2}$$
$$d_2 = \sqrt{(x-c)^2 + y^2}$$

Since $d_1 + d_2 = 2a$,

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\Rightarrow \sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$(x+c)^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2 \qquad \text{(Squaring both sides)}$$

$$x^2 + 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + x^2 - 2cx + c^2 + y^2$$

$$\Rightarrow 4a^2 - 4cx = 4a\sqrt{(x-c)^2 + y^2} \qquad \text{(Rearranging)}$$

$$\Rightarrow a^2 - cx = a\sqrt{(x-c)^2 + y^2} \qquad \text{(Dividing both sides by 4)}$$

$$a^4 - 2cxa^2 + c^2x^2 = a^2((x-c)^2 + y^2) \qquad \text{(Squaring both sides)}$$

$$\Rightarrow a^4 - 2cxa^2 + c^2x^2 = a^2(x^2 - 2cx + c^2 + y^2)$$

$$\Rightarrow a^4 - a^2c^2 = a^2x^2 - c^2x^2 + a^2y^2 \qquad \text{(Rearranging)}$$

$$a^2(a^2 - c^2) = x^2(a^2 - c^2) + a^2y^2$$

$$\Rightarrow 1 = \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} \qquad \text{(Dividing by } a^2(a^2 - c^2))$$

Let $b^2 = a^2 - c^2$.

$$1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

The equation of an ellipse with centre (0, 0) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

This ellipse is symmetric about the x-axis, the y-axis and the origin. The long axis is the major axis and the short axis is the minor axis. If the foci are at $(0, c)$ and $(0, -c)$, then the minor axis will be along the x-axis.

A chord through the focus of an ellipse and the perpendicular to the major axis is called a **latus rectum** of the ellipse. The length of the latus rectum is $\frac{2b^2}{a}$.

EXAMPLE 26 Find the equation of an ellipse with its centre at the origin, major axis on the x-axis and passing through the points (6, 4) and (8, 3).

SOLUTION The standard form of the equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Substituting $x = 6, y = 4$ gives: $\frac{6^2}{a^2} + \frac{4^2}{b^2} = 1$ [1]

Substituting $x = 8, y = 3$ gives: $\frac{8^2}{a^2} + \frac{3^2}{b^2} = 1$ [2]

Multiplying [1] by 3^2 gives: $\frac{324}{a^2} + \frac{144}{b^2} = 9$ [3]

Multiplying [2] by 4^2 gives: $\frac{1024}{a^2} + \frac{144}{b^2} = 16$ [4]

$$\left(\frac{1024}{a^2} + \frac{144}{b^2}\right) - \left(\frac{324}{a^2} + \frac{144}{b^2}\right) = 16 - 9 \qquad [4] - [3]$$

$$\Rightarrow \frac{700}{a^2} = 7, \qquad a^2 = 100$$

Substituting $a^2 = 100$ into [1] gives:

$$\frac{36}{100} + \frac{16}{b^2} = 1$$

$$\Rightarrow \quad \frac{16}{b^2} = 1 - \frac{36}{100}$$

$$\frac{16}{b^2} = \frac{64}{100}$$

$$\Rightarrow \quad b^2 = 16 \times \frac{100}{64}$$

$$b^2 = 25$$

Hence, the equation of the ellipse is $\frac{x^2}{100} + \frac{y^2}{25} = 1$.

Note

(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is the equation of an ellipse with centre (0, 0), foci at $(-c, 0)$ and $(c, 0)$ and vertices at $(-a, 0)$ and $(a, 0)$. The major axis is the x-axis. $c^2 = a^2 - b^2$.

(ii) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ is the equation of an ellipse with centre (0, 0), foci at $(0, -c)$ and $(0, c)$ and vertices at $(0, -a)$ and $(0, a)$. The major axis is the y-axis. $c^2 = a^2 - b^2$.

(iii) When drawing the graph of the ellipse, we can find the coordinates when $x = 0$ and $y = 0$, and draw the ellipse using these points.

EXAMPLE 27 Sketch the graph of each of these.

(a) $\frac{x^2}{4} + \frac{y^2}{9} = 1$

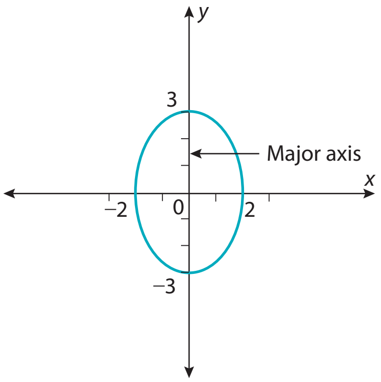
(b) $\frac{x^2}{9} + \frac{y^2}{4} = 1$

SOLUTION

(a) When $x = 0$, $\frac{y^2}{9} = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$

When $y = 0$, $\frac{x^2}{4} = 1, \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

We can draw the ellipse passing through the points (0, 3), (0, −3), (2, 0) and (−2, 0).

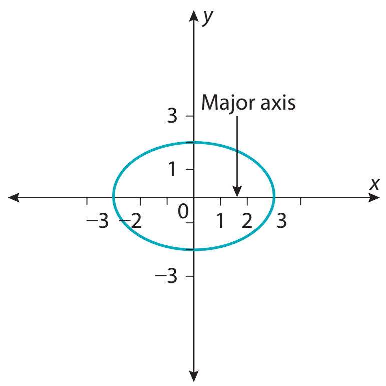


The y-axis is the major axis and the vertices are at (0, 3) and (0, −3).

(b) When $x = 0, \frac{y^2}{4} = 1 \Rightarrow y^2 = 4 \Rightarrow y = \pm 2$

When $y = 0, \frac{x^2}{9} = 1, \Rightarrow x^2 = 9 \Rightarrow x = \pm 3$

We can draw the ellipse passing through the points (0, 2), (0, −2), (3, 0) and (−3, 0).



The x-axis is the major axis and the vertices are at (−3, 0) and (3, 0).

Equation of an ellipse with centre (h, k)

If the centre of the ellipse is at (h, k) and the major axis is parallel to the x-axis, the standard form of the equation is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

The vertices will be at $(h \pm a, k)$ and the foci at $(h \pm c, k)$ where $c^2 = a^2 - b^2$.

If the major axis is parallel to the y-axis and the centre is (h, k), the equation is

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

The vertices will be at $(h, k \pm a)$ and the foci at $(h, k \pm c)$ where $c^2 = a^2 - b^2$.

We can rearrange the equation to get the general form of the equation of an ellipse as:

$Ax^2 + By^2 + Dx + Ey + F = 0$ where A and B are of the same sign and $A = B$.

EXAMPLE 28 Given that the equation of an ellipse is $4x^2 + 9y^2 - 48x + 72y + 144 = 0$, find its centre and vertices and sketch the graph of the ellipse.

SOLUTION $4x^2 + 9y^2 - 48x + 72y + 144 = 0$

Arranging the xs and ys together and taking the constant on the right-hand side gives:

$4x^2 - 48x + 9y^2 + 72y = -144$

Completing the square of the two quadratics gives:

$4(x^2 - 12x) + 9(y^2 + 8y) = -144$

$4(x-6)^2 - (4)(36) + 9(y+4)^2 - (9)(16) = -144$

$4(x-6)^2 + 9(y+4)^2 = 144$

$$4\frac{(x-6)^2}{144} + \frac{9(y+4)^2}{144} = 1$$

$$\frac{(x-6)^2}{36} + \frac{(y+4)^2}{16} = 1$$

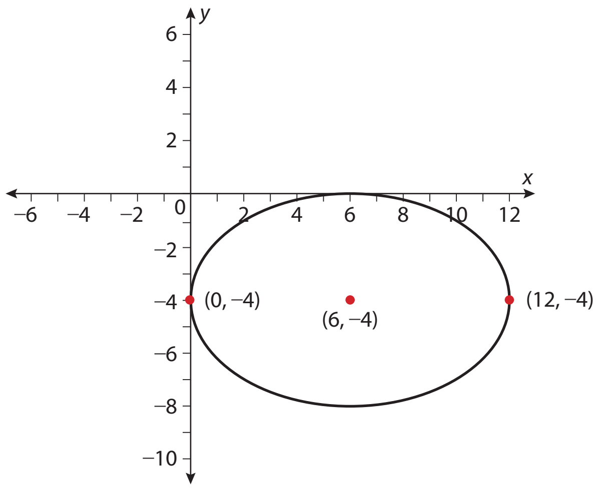
This is of the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ with $h = 6$ and $k = -4$.

Hence, the ellipse has its centre at $(6, -4)$.

Since $36 > 16$, the major axis is the x-axis.

The vertices are at $(h \pm a, k)$, which gives: $(6 + 6, -4)$ and $(6 - 6, -4)$.

The vertices are at $(12, -4)$ and $(0, -4)$.



EXAMPLE 29 Identify the centre and vertices of the ellipse with equation $3x^2 + 6y^2 - 12x + 24y + 6 = 0$.

SOLUTION We rewrite the equation by completing the square as follows.

Let us divide by 3 and take the constant to the right-hand side:

$x^2 + 2y^2 - 4x + 8y = -2$

$x^2 - 4x + 2y^2 + 8y = -2$ (Rearranging terms)

$(x-2)^2 - 4 + 2(y^2 + 4y) = -2$

$(x-2)^2 - 4 + 2(y+2)^2 - 8 = -2$

$(x-2)^2 + 2(y+2)^2 = 10$

$\frac{(x-2)^2}{10} + \frac{(y+2)^2}{5} = 1$ (Dividing by 10)

Comparing with the general form of an ellipse, we have an ellipse centre $(2, -2)$. The major axis is parallel to the x-axis since $a^2 = 10$ and $b^2 = 5$.

The vertices of this ellipse are at $(h \pm a, k)$.

Substituting $h = 2$, $k = -2$ and $a = \sqrt{10}$ gives $(2 + \sqrt{10}, -2)$ and $(2 - \sqrt{10}, -2)$.

The vertices are at $(2 + \sqrt{10}, -2)$ and $(2 - \sqrt{10}, -2)$.

Hence, $3x^2 + 6y^2 - 12x + 24y + 6 = 0$ is the equation of an ellipse with centre $(2, -2)$ and vertices $(2 + \sqrt{10}, -2)$ and $(2 - \sqrt{10}, -2)$.

EXAMPLE 30 Find the points of intersection of the line $y - 2x + 2 = 0$ and the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

SOLUTION Using the equation of the line $y - 2x + 2 = 0$ and writing y in terms of x, we have:

$y = 2x - 2$ [1]

The equation of the ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

Substituting [1] into the equation of the ellipse gives:

$$\frac{x^2}{9} + \frac{(2x-2)^2}{4} = 1$$

$$\frac{x^2}{9} + \frac{4x^2 - 8x + 4}{4} = 1 \qquad \text{(Expanding the brackets)}$$

$$\frac{x^2}{9} + x^2 - 2x + 1 = 1 \qquad \text{(Simplifying)}$$

$$x^2 + 9x^2 - 18x + 9 = 9 \qquad \text{(Multiplying both sides by 9)}$$

$$x^2 + 9x^2 - 18x = 0$$

$$10x^2 - 18x = 0$$

$$\Rightarrow 2x(5x - 9) = 0$$

Hence, $x = 0, x = \frac{9}{5}$.

When $x = 0, y = 2(0) - 2 = -2$

When $x = \frac{9}{5}, y = 2\left(\frac{9}{5}\right) - 2 = \frac{8}{5}$

Therefore, the points of intersections are $(0, -2)$ and $\left(\frac{9}{5}, \frac{8}{5}\right)$.

Focus–directrix property of an ellipse

The directrix of an ellipse is the line $x = \frac{a^2}{c}$. The distance from the point P on the ellipse to the focus F is always in a constant ratio e to its distance from the directrix, where $0 < e < 1$, The constant ratio e is called the eccentricity of the ellipse.

For a parabola, $e = 1$ and for a hyperbola, $e > 1$.

Parametric equations of ellipses

The Cartesian equation of an ellipse with centre (h, k) is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$. The parametric equations of this ellipse are $x = h + a\cos t$ and $y = k + b\sin t$.

PROOF

Using $x = h + a\cos t$ and making $\cos t$ the subject of the formula gives:

$$x - h = a\cos t$$

$$\Rightarrow \cos t = \frac{x-h}{a}$$

Squaring both sides gives:

$$\cos^2 t = \left(\frac{x-h}{a}\right)^2 \qquad [1]$$

Using $y = k + b \sin t$ and making $\sin t$ the subject of the formula gives:

$$y - k = b \sin t$$
$$\Rightarrow \sin t = \frac{y-k}{b}$$

Squaring both sides gives:

$$\sin^2 t = \left(\frac{y-k}{b}\right)^2 \qquad [2]$$

Adding [1] and [2] gives:

$$\cos^2 t + \sin^2 t = \left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2$$

Since $\cos^2 t + \sin^2 t = 1$:

$\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, which is the Cartesian equation of the curve.

Note

(i) An ellipse centred at the origin has parametric equations $x = h + a \cos t$ and $y = k + b \sin t$.

(ii) $x = h + a \cos wt$ and $y = k + b \sin wt$ also represent the parametric equations of an ellipse with centre (h, k). Can you identify the difference between the two equations and the ellipses generated by each equation?

EXAMPLE 31 Find the Cartesian equation of the curve represented in parametric form by $x = 2 + 3 \cos t$ and $y = 4 + 5 \sin t$. Identify the curve.

SOLUTION Making $\cos t$ the subject of the formula for x gives:

$$\cos t = \frac{x-2}{3}$$

Squaring both sides gives:

$$\cos^2 t = \left(\frac{x-2}{3}\right)^2 \qquad [1]$$

Making $\sin t$ the subject of the formula gives:

$$\sin t = \frac{y-4}{5}$$

Squaring both sides gives:

$$\sin^2 t = \left(\frac{y-4}{5}\right)^2 \qquad [2]$$

Adding [1] and [2] gives:

$$\sin^2 t + \cos^2 t = \left(\frac{x-2}{3}\right)^2 + \left(\frac{y-4}{5}\right)^2$$

$$\Rightarrow \left(\frac{x-2}{3}\right)^2 + \left(\frac{y-4}{5}\right)^2 = 1$$

The equation is $\dfrac{(x-2)^2}{3^2} + \dfrac{(y-4)^2}{5^2} = 1$.

This is the equation of an ellipse with centre $(2, 4)$.

EXAMPLE 32 The equation of an ellipse is given by $16x^2 + 9y^2 - 96x - 36y + 36 = 0$. Find the values of p, q, r and l such that the parametric equations of the ellipse are $x = p + q\cos\theta$ and $y = r + l\sin\theta$.

SOLUTION

$16x^2 + 9y^2 - 96x - 36y + 36 = 0$

First, we rearrange the equation and complete the square.

$16x^2 - 96x + 9y^2 - 36y + 36 = 0$

$\Rightarrow 16(x^2 - 6x) + 9(y^2 - 4y) + 36 = 0$

$\Rightarrow 16(x - 3)^2 - (16 \times 3^2) + 9(y - 2)^2 - (9 \times 2^2) + 36 = 0$

Therefore, $16(x - 3)^2 + 9(y - 2)^2 = 144$

Dividing by 144 gives:

$$\frac{(x-3)^2}{9} + \frac{(y-2)^2}{16} = 1$$

$$\Rightarrow \frac{(x-3)^2}{3^2} + \frac{(y-2)^2}{4^2} = 1$$

The parametric form of an ellipse is $x = p + q\cos\theta$ and $y = r + l\sin\theta$ where (p, q) is the centre.

$x = 3 + 3\cos\theta$ and $y = 2 + 4\sin\theta$.

Therefore $p = 3$, $q = 3$, $r = 2$, $l = 4$.

Equations of tangents and normals to an ellipse

Equation of a tangent to an ellipse

The tangent at a point P$(a\cos t, b\sin t)$ to the ellipse has gradient $\frac{-b\cos t}{a\sin t}$.

The equation of the tangent at P is:

$y - y_P = m(x - x_P)$

$$y - b\sin t = \frac{-b\cos t}{a\sin t} \times (x - a\cos t)$$

$\Rightarrow ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$

$\Rightarrow ay\sin t + bx\cos t = ab\sin^2 t + ab\cos^2 t$

$\Rightarrow ay\sin t + bx\cos t = ab$ (Since $\sin^2 t + \cos^2 t = 1$)

$\Rightarrow \frac{y}{b}\sin t + \frac{x}{a}\cos t = 1$ (Dividing both sides by ab)

The equation of the tangent to the ellipse at P is $\frac{y}{b}\sin t + \frac{x}{a}\cos t = 1$.

Equation of a normal to an ellipse

The equation of the normal at P$(a\cos t, b\sin t)$ to the ellipse has gradient $\frac{a\sin t}{b\cos t}$.

(Since the gradient of a normal at a point $\times$ gradient of tangent at that point $= -1$.)

The equation of the normal at P is:

$y - y_P = m(x - x_P)$

$y - b\sin t = \dfrac{a\sin t}{b\cos t} \times (x - a\cos t)$

$\Rightarrow by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$

The equation of the normal is $by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$.

EXAMPLE 33 Find the equation of the tangent to the ellipse $2y^2 + x^2 = 3$ at the point (1, 1).

SOLUTION Writing the equation in the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we have $\dfrac{x^2}{3} + \dfrac{2y^2}{3} = 1$.

$\dfrac{x^2}{(\sqrt{3})^2} + \dfrac{y^2}{\left(\sqrt{\frac{3}{2}}\right)^2} = 1 \qquad \left(\text{Since } \dfrac{2y^2}{3} = \dfrac{y^2}{\frac{3}{2}}\right)$

Therefore, $a = \sqrt{3}$, $b = \sqrt{\dfrac{3}{2}}$.

The gradient of the tangent at point $P(a\cos t, b\sin t)$ is $\dfrac{-b\cos t}{a\sin t}$.

At the point (1, 1), we have $a\cos t = 1$ and $b\sin t = 1$.

$a\cos t = 1$

$\Rightarrow \sqrt{3}\cos t = 1$

$\Rightarrow \cos t = \dfrac{1}{\sqrt{3}}$

$b\sin t = 1$

$\Rightarrow \sqrt{\dfrac{3}{2}}\sin t = 1$

$\Rightarrow \sin t = \dfrac{1}{\sqrt{\frac{3}{2}}}$

Now, the gradient of the tangent at (1, 1) is $\dfrac{-b\cos t}{a\sin t} = \dfrac{-\sqrt{\frac{3}{2}} \times \sqrt{\frac{1}{3}}}{\sqrt{3} \times \dfrac{1}{\sqrt{\frac{3}{2}}}} = \dfrac{-1}{2}$

The equation of the tangent at (1, 1) is:

$y - y_1 = m(x - x_1)$

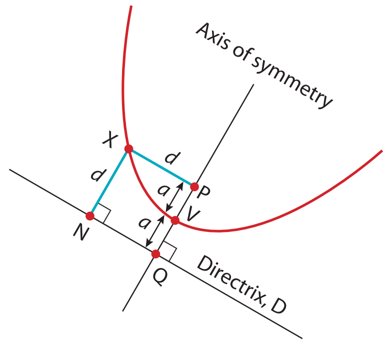
$y - 1 = \dfrac{-1}{2}(x - 1)$

$\Rightarrow 2y - 2 = -x + 1, \qquad 2y + x = 3.$

Hence, the equation of the tangent at (1, 1) is $2y + x = 3$.

Parabolas

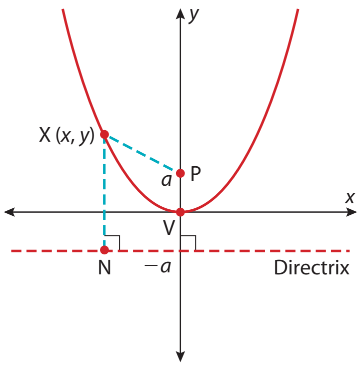
A parabola is the set of all points in a plane, which are the same distance from a fixed point P as they are from a fixed line l. The fixed point P is called the **focus** of the parabola, and the fixed line l is called the **directrix** of the parabola.



The diagram shows a parabola.

- The point V is called the vertex of the parabola.
- The point X is on the parabola and $|XP| = |XN|$.
- The line through the focus P and the perpendicular to the directrix is the axis of the symmetry of the parabola.
- The distance from the vertex to the directrix is the same as the distance from the vertex to the focus that is $|VP| = |VQ|$.

Equation of a parabola



Since the distance between a point on the parabola and the focus is equal to the distance between the point and the directrix:

$|XP| = |XN|$

$\Rightarrow \sqrt{x^2 + (y - a)^2} = y + a$ (Using Pythagoras' theorem $|XP| = \sqrt{x^2 + (y - a)^2}$)

$\Rightarrow x^2 + (y - a)^2 = (y + a)^2$ (Squaring both sides)

$\Rightarrow x^2 + y^2 - 2ay + a^2 = y^2 + 2ay + a^2$

Hence, $x^2 = 4ay$.

$x^2 = 4ay$ is the equation of a parabola with centre (0, 0) and the parabola is upwards.

The equation of the parabola with centre (0, 0) and opening to the right is $y^2 = 4ax$.

The parabolas with equation $y = ax^2 + bx + c$, $a \neq 0$ are all parabolas with the axis of symmetry parallel to the y-axis.

The equation $x = ay^2 + by + c$, $a \neq 0$, represents a parabola whose axis of symmetry is parallel to the x-axis.

The equation of a parabola can be written in two different forms, either vertex form or conics form.

The vertex form of a parabola is $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$ where (h, k) is the vertex of the parabola.

The conics form of the equation of the parabola with vertex (h, k) is $4p(y - k) = (x - h)^2$ or $4p(x - h) = (y - k)^2$ where p represents the distance from the vertex to the focus, which is also the distance from the vertex to the directrix. ($2p$ is the distance from the focus to the directrix.)

EXAMPLE 34 State the vertex and the focus of the parabola having the equation $(y - 4)^2 = 6(x - 7)$.

SOLUTION The equation is of the form $4p(x - h) = (y - k)^2$.

Comparing with $(y - 4)^2 = 6(x - 7)$ gives $4p = 6$, $h = 7$ and $k = 4$.

$4p = 6 \Rightarrow p = \frac{6}{4} = \frac{3}{2}$

The vertex is $(7, 4)$.

Because the equation involves y^2 and p is positive, the parabola is sideways and opens to the right.

Since the focus is inside the parabola, it is $\frac{3}{2}$ units to the right of the vertex.

Therefore, the focus has coordinates $\left(7 + \frac{3}{2}, 4\right) = \left(\frac{17}{2}, 4\right)$.

Parametric equations of parabolas

The parametric equations of a parabola are $x = at^2$, $y = 2at$, $t \in \mathbb{R}$, $a > 0$.

From $y = 2at$, we get $t = \frac{y}{2a}$.

Substituting into x gives:

$$x = a\left(\frac{y}{2a}\right)^2$$

$$x = \frac{y^2}{4a}$$

Hence, $y^2 = 4ax$ which is the Cartesian equation of a parabola with centre $(0, 0)$ and opening to the right.

Equations of tangents and normals to a parabola

Equations of a tangent to a parabola

Let $P(at^2, 2at)$ be any point on the parabola with equation $y^2 = 4ax$.

From calculus, we have the gradient of the tangent to the parabola at P is $\frac{1}{t}$.

The equation of the tangent to the parabola at P is:

$$y - 2at = \frac{x - at^2}{t}$$

Therefore, $ty - 2at^2 = x - at^2$.

Hence, $ty = x + at^2$ is the equation of the tangent to the parabola.

Equations of a normal to a parabola

The gradient of the normal at P is $-t$.

The equation of the normal is:

$y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3$

Hence, $y + tx = 2at + at^3$ is the equation of the normal at P.

EXAMPLE 35 Find an equation of the tangent and of the normal to the curve $y^2 = 4x$ at the point (1, 2).

SOLUTION We use the parametric equations for a parameter $x = at^2$ and $y = 2at$.

Comparing with $y^2 = 4ax$, $a = 1$.

At $P(at^2, 2at)$, $a = 1$, $t = 1$, gives the point (1, 2).

The gradient of the tangent $= \frac{1}{t} = 1$

The equation of the tangent at (1, 2) is

$y - 2 = 1(x - 1)$

$\Rightarrow y = x + 1$

The gradient of the normal is -1 and the equation of the normal at (1, 2) is

$y - 2 = -1(x - 1)$

$\Rightarrow y = -x + 3$

Hence, the equations of the tangent and normal at (1, 2) are $y = x + 1$ and $y = -x + 3$ respectively.

EXERCISE 10B

1 Find the Cartesian equation of the curve $x = t^2 + t + 1$, $y = t^2 - 2$.

2 Find the Cartesian equation of the curve $x = t + \frac{1}{t}$, $y = t - \frac{1}{t}$.

3 Find the Cartesian equation of the curve $x = 4t^2 - 2$, $y = 3t^2 + 5$.

4 Find the Cartesian equation of the curve $x = \frac{t + 2}{4}$, $y = \frac{4t^2 - 3}{t}$.

5 Find the Cartesian equation of the curve $y = 3\cos t$, $x = 4\sin^2 t$.

6 Find the equation in Cartesian form of the curve $x = 3\sin t + 2$, $y = 2\cos t - 1$.

7 The parametric equation of a curve is $x = 5\sin t$, $y = \tan t - 1$. Find the Cartesian equation.

8 The parametric equations of a curve are $x = 4\sec t - 1$, $y = 3\tan t + 7$. Find the Cartesian equation of the curve.

9 The parametric equations of a curve are $x = 6\operatorname{cosec}\theta$, $y = 2\cos\theta$. Find the Cartesian equation of the curve.

10 Show that the Cartesian equation of the curve $x = 3\cos\theta + 2$, $y = 5\sin\theta + 2$ represents the equation of an ellipse. Find the centre and length of the major axis.

11 The equation of an ellipse is $\frac{x^2}{9} + \frac{y^2}{25} = 1$.

(a) Find the x- and y-intercept of the graph of the equation.

(b) Find the coordinates of the foci.

(c) Find the length of the major axis and the minor axis.

(d) Sketch the graph of the equation.

12 Identify the centre of the following ellipses.

(a) $x^2 + 4y^2 + 4x - 8y + 4 = 0$

(b) $9x^2 + 4y^2 - 18x + 16y - 11 = 0$

13 The equation of an ellipse is $\frac{(x-1)^2}{4} + \frac{(y+3)^2}{9} = 1$.

(a) Find the coordinates of the centre of the ellipse.

(b) Find the coordinates of the foci.

(c) Find the parametric equations of the ellipse.

(d) Sketch of the graph of the ellipse.

14 Find the Cartesian equation of the parabola given in parametric form by each of the following equations.

(a) $x = t^2$, $y = 4t$

(b) $x = 6t^2$, $y = 12t$

(c) $x = t - 1$, $y = t^2 + 1$

15 Find the equation of the tangent and normal to the curve $y^2 = 16x$ at the point $(1, 4)$.

SUMMARY

Coordinate geometry

Circles

A circle is the locus of a point which moves in a plane so that it is equidistant from a fixed point.

Equation of a circle centre (0, 0) and radius r:

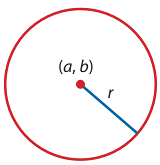
$$x^2 + y^2 = r^2$$

Equation of a circle centre (a, b) and radius r:

$$(x - a)^2 + (y - b)^2 = r^2$$

General equation of a circle:

$Ax^2 + By^2 + Cx + Dy + E = 0$, $A = B$



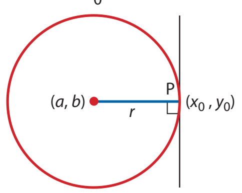
Parametric equation of a circle centre (a, b) and radius r:

$x = a + r\cos\theta \quad y = b + r\sin\theta$

or $\quad x = a + r\sin\theta \quad y = b + r\cos\theta$

Gradient of the normal at

$$P = \frac{y_0 - b}{x_0 - a}$$



Since the tangent is perpendicular to the radius, the gradient of the tangent of P is $-\left(\frac{y_0 - b}{x_0 - a}\right)$

Given three points on a circle we can find its equation by using the general form $(x - a)^2 + (y - b)^2 = r^2$. We form three equations and solve them simultaneously to find a, b and r.

Or, if the circle passes through P, Q and R, the centre of the circle is the point of intersection of the perpendicular bisectors of PQ and QR. The radius can be found using the centre and any of the points P, Q and R.

Ellipses

An ellispse is the locus of points, the sum of whose distance from two fixed points is constant. The two fixed points are the **foci** of the ellipse.

Equation of an ellipse with centre (0, 0) and foci at $(c, 0)$ and $(-c, 0)$ is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of an ellipse with foci $(0, c)$ and $(0, -c)$ is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of an ellipse with centre (h, k) and major axis parallel to the x-axis is:

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Foci are at $(h + c, k)$ and $(h - c, k)$.

Equation of an ellipse with centre (h, k) and major axis parallel to the y-axis is:

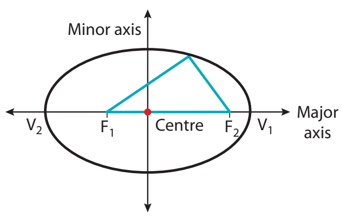
$$\frac{(x - h)^2}{b^2} + \frac{(y - k)^2}{a^2} = 1$$

Foci are at $(h, k + c)$ and $(h, k - c)$.

General equation of an ellipse:

$Ax^2 + B^2 + Cx + Dy + E = 0$, $A \neq B$

Graph of an ellipse centre C, foci F_1 and F_2



Parametric equations of an ellipse centre (h, k) and radius r:

$x = h + a\cos t \quad y = k + b\sin t$

Gradient of the tangent at $P(a\cos t, b\sin t)$ is

$$\frac{-b\cos t}{a\sin t}$$

and the gradient of the normal is

$$\frac{a\sin t}{b\cos t}$$

Equation of the tangent can be found by using the gradient and point.

Parabolas

A parabola is a set of all points in a plane, which are at the same distance from a fixed point P as they are from a fixed line l. P is called the focus and l is called the directrix.

Equation of a parabola with centre (0, 0) and opening upwards is:

$$x^2 = 4ay$$

Equation of a parabola with centre (0, 0) and opening to the right is:

$$y^2 = 4ax$$

Equation of a parabola with axis of symmetry parallel to the y-axis is:

$$y = ax^2 + bx + c \quad a \neq 0$$

Equation of a parabola with axis of symmetry parallel to the x-axis is:

$$x = ay^2 + by + c \quad a \neq 0$$

Equation of a parabola in vertex form:

$y = a(x - h)^2 + k$

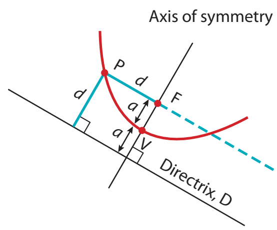
or $\quad y = a(y - k)^2 + h$

Equation of a parabola in conics form:

$4p(y - k) = (x - h)^2$

or $\quad 4p(x - h) = (y - k)^2$

where p is the distance from the vertex to the focus (also the distance from the vertex to the directrix).
($2p$ is the distance from the focus to the directrix).



Parametric equations of a parabola are:

$x = at^2 \quad y = 2at \quad t \in \mathbb{R}, \quad a > 0$

Gradient of the tangent to a parabola at $P(at^2, 2at)$ is $\frac{1}{t}$

Gradient of the normal to a parabola at $P(at^2, 2at)$ is $-t$.

Checklist

Can you do these?

- Recognise the Cartesian equation of a circle, ellipse and parabola.
- Identify the centre and radius of a circle.
- Find the equation of a circle given its centre and radius.
- Find the equation of a tangent to a circle, ellipse and parabola.
- Find the equation of a normal to a circle, ellipse and parabola.
- Find the point of intersection of a curve and a straight line.
- Find the points of intersection of two curves.
- Find the Cartesian equation of a circle, ellipse or parabola given its parametric equations.
- Find the parametric equations of a circle, ellipse or parabola given its Cartesian equation.
- Find the foci of an ellipse.
- Find the length of the major axis and the length of the minor axis of an ellipse.
- Draw the graph of an ellipse.
- Draw the graph of a parabola.

Review Exercise 10

1 Prove that the line $3x + 4y = 25$ is a tangent to the circle $x^2 + y^2 = 25$.

2 Find the equation of the diameter $x^2 + y^2 - 2x + 4y - 1 = 0$ which passes through the point $(3, 1)$.

3 Show that the circle with centre $(4, -1)$ and radius $\sqrt{2}$ units touches the line $x + y = 1$.

4 Given that $(5, 1)$, $(4, 6)$ and $(2, -2)$ are three points on a circle, find the equation of the circle.

5 Find the equations of the circles having radius $\sqrt{13}$ and tangent $2x - 3y + 1 = 0$ to the circle at $(1, 1)$.

6 Show that the Cartesian equation of the curve $x = 4\cos t - 3$, $y = 4\sin t + 4$ represents a circle. Identify the centre and radius of the circle.

7 Find the centre and radius of the circle C_1 represented by the equation $x^2 + y^2 - 6x + 8y = 5$. Find also the length of the line joining the centres of the circle C_1 and C_2 where C_2 is the circle $4x^2 + 4y^2 - 12x + 16y = 12$.

8 Show that the circles $x^2 + y^2 + 6x - 2y - 54 = 0$ and $x^2 + y^2 - 22x - 8y - 12 = 0$ intersect.

9 A curve is given parametrically by $x = \cos t - 1$ and $y = \cos 2t$.

(a) Calculate the distance between the points with parameters 0 and $\frac{\pi}{3}$.

(b) Find the Cartesian equation of the curve.

10 Find the equation of a circle, given that (0, 6) and (8, −8) are the end points of a diameter.

11 The end points of the diameter of a circle are (−3, 2) and (5, −6). Find the centre and radius of the circle. Write down the equation of the circle.

12 Find an equation of the line containing the centres of the two circles $x^2 + y^2 - 2x - 4y + 3 = 0$ and $2x^2 + 2y^2 + 4x + 6y + 9 = 0$.

13 Given a line $y - 7x = 2$, and a circle $x^2 + y^2 + 8x + 2y - 8 = 0$, determine whether the line and the circle touch, intersect at two points or never meet. If they intersect, find the point(s) of intersection.

14 Find the points of intersections of the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4x - 4y + 4 = 0$.

15 Given that the circle with equation $x^2 + y^2 - 2x - 4y = 0$ and the circle with equation $x^2 + y^2 - 2y - 2 = 0$ intersect at two points, find the points of intersection.

16 What are the points of intersection of the circle of radius 1, centre the origin, and the circle of radius 2, centre (0, 3)?

17 Show that the circle with centre (0, 0) and radius 1 does not intersect the circle with centre (3, 1) and radius 1.

18 The centre of a circle is (−2, 3) and a point on the circumference is (−5, −1).

(a) Find the equation of the line joining the two points.

(b) Show that the radius of the circle is 5 units.

(c) Write down the equation the circle.

(d) Determine the equation of the tangent to the circle at the points (−5, −1).

(e) Find the points of intersection (if any exist) of the circle above with the circle $x^2 + y^2 + 6x - 7y - 10 = 0$.

19 Show that the line with equation $y = 5 + \frac{3}{4}x$ is a tangent to the parabola $y^2 = 15x$.

20 Given the equation $4x^2 + 9y^2 = 36$, complete the following.

(a) (i) Find the x and y intercepts of the graph of the equation.

(ii) Find the coordinates of the foci.

(iii) Find the length of the major axis. Hence, sketch the graph of the equation.

(b) Write down the parametric equations of the curve.

21 The equation of a line l is $x + y = 5$ and an ellipse E is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(a) Find the coordinates of the points of intersection of the line and ellipsis.

(b) Write the equation of E in the form $x = p\cos\theta$, $y = p\sin\theta$.

22 Show that the lines with equation $y = x + 1$ is a tangent to the curve $\frac{x^2}{3} - \frac{y^2}{2} = 1$

23 The equation of an ellipse is $4x^2 + 9y^2 + 32x + 36y + 64 = 0$.

(a) Write the equation in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

(b) Write down the centre of the ellipse.

(c) Find the point of intersection of the ellipse with the line $y = x + 2$.

(d) Write the equation of the ellipse in parametric form.

24 Find the equation of the normal to the parabola with equation $y^2 = 4x$ at the point $(t^2, 2t)$. At the points P and Q on the parabola, $t = 2$ and $t = \frac{1}{2}$. The normal at P and the normal at Q meet at C. Find the coordinates of C.

Chapter 11

Vectors in Three Dimensions ($\mathbb{R}^3$)

At the end of this chapter you should be able to:

- Add two vectors
- Subtract two vectors
- Multiply a vector by a scalar quantity
- Find the length of a vector
- Identify a position vector
- Derive and use displacement vectors
- Derive and use unit vectors
- Express a vector in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $xi + yj + zk$
- Find the scalar product of two vectors
- Find the angle between two vectors
- Find the equation of a line given a point on the line and a vector parallel to the line
- Find the equation of a line given two points on the line
- Determine whether two lines are parallel, intersect, or skewed
- Identify a vector normal to a plane
- Find the equation of a plane given a point on the plane and a vector perpendicular to the plane

KEYWORDS/TERMS

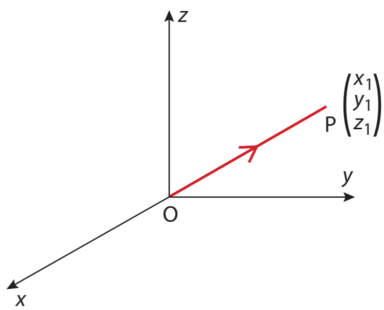
vectors • position vector • displacement vector • unit vector • scalar product • dot product • direction vector • parallel • perpendicular • skew • intersect • normal vector • plane

Vectors in 3D

Vectors have numerous physical and geometric applications since they have both magnitude and direction. Forces, acceleration, velocity and displacement are all vector quantities. Computer graphics and 3D game programming makes extensive use of vectors.

Colour information is displayed on most monitors. These make use of the three wavelengths for red, green and blue referred to as RGB colour. Other colours are derived from these three basic colours. For example, purple can be derived from a combination of red and blue. We can express colours as triplets of red, green and blue whose values will range from 0 to 1. We can write $C = \begin{pmatrix} C_r \\ C_g \\ C_b \end{pmatrix}$. Using this vector, we can produce different colour schemes by adding or multiplying by a scalar. Colour multiplication is also called modulation.

A vector quantity has both magnitude and direction while a scalar quantity has magnitude only. A vector can be written as a letter in bold typeface **p**. A vector starting at O and ending at P can be written as $\overrightarrow{OP}$. $\mathbb{R}^3$(read as 'r three') is the set of all ordered triples of real numbers. Let O be the origin and O_x, O_y, O_z be perpendicular axes.

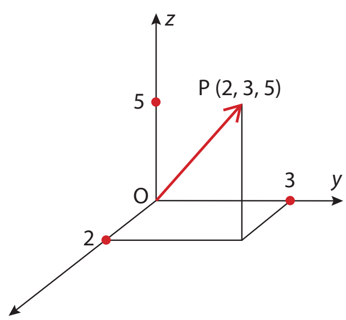


If $P(x_1, y_1, z_1)$ is a point in three dimensions, then the position vector of P, that is $\overrightarrow{OP}$, is as shown.
To get from O to P, we move x_1 units along the x-axis, y_1 units along the y-axis and z_1 units up the z-axis.

The position vector $\overrightarrow{OP}$ has components $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$.

Plotting a point in three dimensions

To plot a point in three dimensions, you need to move parallel to the axes in the x- and y-direction as shown. Then from the point of intersection, you move up the z-axis. The point (2, 3, 5) is plotted by moving 2 units along the x-axis, 3 units parallel to the y-axis and then 5 units upwards.

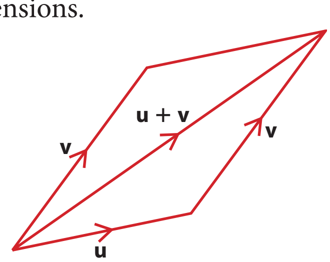


Algebra of vectors

We carry out vector addition, subtraction and scalar multiplication in three dimensions in the same way that we carry them out in two dimensions.

Addition of vectors

The result of adding two vectors is the diagonal of the parallelogram shown in the diagram below.



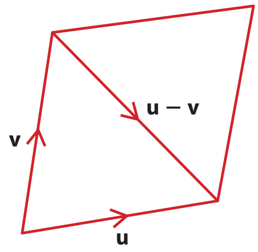
We can also add vectors by adding their corresponding components as shown below.

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

Subtraction of vectors

We can also subtract vectors by subtracting the corresponding components.



$$\mathbf{u} - \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} - \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix}$$

Multiplication by a scalar

We multiply each component by a scalar, λ.

$$\lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix} \text{ where } \lambda \in \mathbb{R}$$

EXAMPLE 1 Given that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$, find the following.

(a) $\overrightarrow{OA} + \overrightarrow{OB}$ **(b)** $\overrightarrow{OB} - \overrightarrow{OA}$

(c) $2\overrightarrow{OA} + 3\overrightarrow{OB}$

SOLUTION

(a) $\overrightarrow{OA} + \overrightarrow{OB} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 - 1 \\ 1 + 2 \\ 3 + 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix}$

(b) $\overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

(c) $2\overrightarrow{OA} - 3\overrightarrow{OB} = 2\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix} + \begin{pmatrix} -3 \\ 6 \\ 12 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 18 \end{pmatrix}$

Equality of vectors

Recall that two vectors are equal if and only if the magnitude and direction of the vectors are equal. Also two vectors **u** and **v** are equal if and only if the corresponding components are equal.

Therefore, $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ iff $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$.

EXAMPLE 2 **u** and **v** are equal vectors. Given that $\mathbf{u} = \begin{pmatrix} 2 \\ 1 \\ x - 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} y \\ x - 2 \\ 2 \end{pmatrix}$, find x and y.

SOLUTION Since $\begin{pmatrix} 2 \\ 1 \\ x - 1 \end{pmatrix} = \begin{pmatrix} y \\ x - 2 \\ 2 \end{pmatrix}$, equating corresponding components gives:

$$2 = y$$
$$1 = x - 2$$
$$x - 1 = 2$$
$$\Rightarrow \quad x = 3$$

Hence, $x = 3$, $y = 2$.

Magnitude of a vector

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$.

The modulus or magnitude (length) of $\mathbf{u}$ is $|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$.

EXAMPLE 3 Find the magnitude of $\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

SOLUTION

$$\left|\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}\right| = \sqrt{(1)^2 + (2)^2 + (-2)^2}$$
$$= \sqrt{1 + 4 + 4}$$
$$= \sqrt{9}$$
$$= 3$$

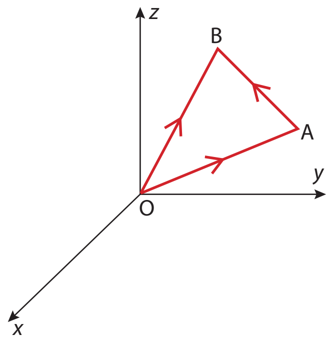
EXAMPLE 4 Given that $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ z - 1 \end{pmatrix}$. Find the values of z given that $|\overrightarrow{OA}| = 5$ units.

SOLUTION

$$\overrightarrow{OA} = \sqrt{3^2 + 2^2 + (z - 1)^2}$$
$$\Rightarrow \sqrt{9 + 4 + (z - 1)^2} = 5$$
$$13 + (z - 1)^2 = 25 \qquad \text{(Squaring both sides)}$$
$$(z - 1)^2 = 12$$
$$z - 1 = \pm\sqrt{12}$$
$$z = 1 \pm \sqrt{12}$$
$$z = 1 \pm 2\sqrt{3}$$

Hence, $z = 1 + 2\sqrt{3}, 1 - 2\sqrt{3}$.

Displacement vectors



The displacement vector $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$$= -\overrightarrow{OA} + \overrightarrow{OB}$$
$$= \overrightarrow{OB} - \overrightarrow{OA}$$

EXAMPLE 5 Given that $\overrightarrow{OA} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$, find the following.

(a) $\overrightarrow{AB}$ **(b)** $|\overrightarrow{AB}|$

SOLUTION

(a) $$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$$

(b) $$|\overrightarrow{AB}| = \sqrt{(2)^2 + (1)^2 + (-5)^2}$$
$$= \sqrt{4 + 1 + 25}$$
$$= \sqrt{30}$$

Unit vectors

A unit vector is a vector whose magnitude is 1 unit.

For example, $\left|\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right| = \sqrt{1^2 + 0^2 + 0^2} = 1$

Since the magnitude of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is 1, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is a unit vector.

EXAMPLE 6 Is $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ a unit vector?

SOLUTION To decide whether the vector is a unit vector, we find the magnitude of the vector. If the magnitude of the vector is 1 unit, then vector is a unit vector.

$$\left|\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}\right| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = 1$$

Since $\left|\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}\right| = 1$, $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ is a unit vector.

EXAMPLE 7 Show that $\frac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector, where $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

SOLUTION Since $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $= |\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$.

Therefore, $\dfrac{\mathbf{v}}{|\mathbf{v}|} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2+z^2}} \\ \dfrac{y}{\sqrt{x^2+y^2+z^2}} \\ \dfrac{z}{\sqrt{x^2+y^2+z^2}} \end{pmatrix}$

$$\left|\frac{\mathbf{v}}{|\mathbf{v}|}\right| = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)^2 + \left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)^2}$$

$$= \sqrt{\frac{x^2}{x^2+y^2+z^2} + \frac{y^2}{x^2+y^2+z^2} + \frac{z^2}{x^2+y^2+z^2}}$$

$$= \sqrt{\frac{x^2+y^2+z^2}{x^2+y^2+z^2}}$$

$$= \sqrt{1} = 1$$

Since $\left|\dfrac{\mathbf{v}}{|\mathbf{v}|}\right| = 1$, $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector.

Note

We have just shown that any vector divided by its length is a unit vector. $\hat{\mathbf{v}}$ is the notation used for a unit vector.

EXAMPLE 8 Find a unit vector in the direction of $\overrightarrow{PQ}$ where $\overrightarrow{OP} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$.

SOLUTION First we find $\overrightarrow{PQ}$:

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$$

$$= \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

Now we find the length of $\overrightarrow{PQ}$:

$$|\overrightarrow{PQ}| = \sqrt{(1)^2 + (0)^2 + (2)^2} = \sqrt{5}$$

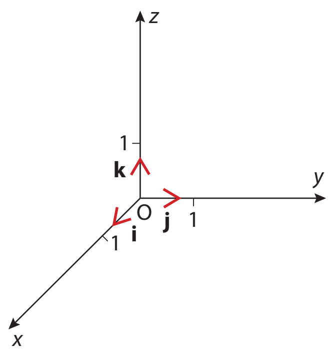
A unit vector in the direction of $\overrightarrow{PQ}$ is:

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}{\sqrt{5}} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

Special unit vectors

The vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are unit vectors in the direction of the x-axis, y-axis and z-axis respectively. We use the letters **i**, **j** and **k** to represent them.

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



We can represent any vector as a sum or difference of the three unit vectors as follows.

Let $\overrightarrow{OA} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$
$$= a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ can be written as $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

EXAMPLE 9 Write the following vectors in terms of **i**, **j** and **k**.

(a) $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ **(b)** $\begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix}$ **(c)** $\begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix}$

SOLUTION The x-component is the coefficient of **i**, the y-component is the coefficient of **j** and the z-component is the coefficient of **k**.

(a) $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 2i + j + 4k$

(b) $\begin{pmatrix} -2 \\ 3 \\ 3 \end{pmatrix} = -2i + 3j + 3k$

(c) $\begin{pmatrix} 5 \\ 4 \\ 5 \end{pmatrix} = 5i + 4j + 5k$

Scalar product or dot product

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

The dot product or scalar product of **u** and **v** is given by:

$$\mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$
$$= u_1v_1 + u_2v_2 + u_3v_3$$

We find the product of the corresponding components and then sum.

EXAMPLE 10 Given that $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$, find $\mathbf{u} \cdot \mathbf{v}$.

SOLUTION

$$\mathbf{u} \cdot \mathbf{v} = (\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$$
$$= 2\mathbf{i} \cdot \mathbf{i} - 3\mathbf{j} \cdot \mathbf{j} + 5\mathbf{k} \cdot \mathbf{k}$$
$$= 2 - 3 + 5$$
$$= 4$$

Note

$\mathbf{i} \cdot \mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1$ and similarly $\mathbf{j} \cdot \mathbf{j} = 1$ and $\mathbf{k} \cdot \mathbf{k} = 1$.

EXAMPLE 11 Find the value of a for which the scalar product of $\begin{pmatrix}2\\1\\a\end{pmatrix}$ and $\begin{pmatrix}-1\\a-3\\2\end{pmatrix}$ is 10.

SOLUTION

$$\begin{pmatrix}2\\1\\a\end{pmatrix}\cdot\begin{pmatrix}-1\\a-3\\2\end{pmatrix} = 10$$

$\Rightarrow (2)(-1) + (1)(a-3) + (2)(a) = 10$

$-2 + a - 3 + 2a = 10$

$3a = 15 \Rightarrow a = 5$

Hence, $a = 5$.

Properties of the scalar product

If **u**, **v** and **w** are vectors, then:

(i) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$ — commutative property

(ii) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$ — distributive property

(iii) $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$ — (a vector dot itself is the square of the magnitude of the vector)

(iv) $\mathbf{0}\cdot\mathbf{v} = 0$

Angle between two vectors

Let $\mathbf{u} = \begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$.

If the angle between **u** and **v** is θ then

$|\mathbf{u}|\,|\mathbf{v}|\cos\theta = \mathbf{u}\cdot\mathbf{v}$

Or $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$

EXAMPLE 12 Find the angle between the vectors $\begin{pmatrix}2\\1\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}$.

SOLUTION Let θ be the angle between the two vectors.

$$\text{Now } \cos\theta = \frac{\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\left|\begin{pmatrix}2\\1\\-1\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\right|} \qquad \left(\text{Using } \cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right)$$

$$= \frac{2+1+0}{\sqrt{2^2+1^2+(-1)^2}\sqrt{1^2+1^2+0^2}}$$

$$= \frac{3}{\sqrt{6}\sqrt{2}}$$

$$\theta = \cos\left(\frac{3}{\sqrt{6}\sqrt{2}}\right)^{-1}$$

$$= 30°$$

Hence, the angle between the two vectors is 30°.

EXAMPLE 13 Let $\overrightarrow{OA} = \begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}-2\\1\\2\end{pmatrix}$.

Find the following

(a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$

(b) The angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

SOLUTION

(a) $\overrightarrow{OA} \cdot \overrightarrow{OB} = \begin{pmatrix}1\\3\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\2\end{pmatrix} = (1)(-2) + (3)(1) + (2)(2) = 5$

(b) Let θ be the angle between the two vectors.

Since $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}$

$$\cos\theta = \frac{\begin{pmatrix}1\\3\\2\end{pmatrix} \cdot \begin{pmatrix}-2\\1\\2\end{pmatrix}}{\left|\begin{pmatrix}1\\3\\2\end{pmatrix}\right|\left|\begin{pmatrix}-2\\1\\2\end{pmatrix}\right|}$$

$$= \frac{5}{\sqrt{1^2 + 3^2 + 2^2}\sqrt{(-2)^2 + (1)^2 + 2^2}}$$

$$= \frac{5}{\sqrt{14}\sqrt{9}}$$

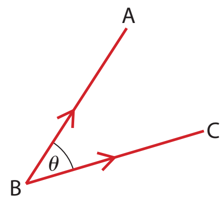
$$= \frac{5}{3\sqrt{14}}$$

Hence, $\theta = 63.55°$.

EXAMPLE 14 Find the angle ABC where $\overrightarrow{OA} = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix}1\\4\\1\end{pmatrix}$.

SOLUTION

Angle ABC is between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$. (When finding the angle between two vectors, both vectors must be in the same direction.)



Since we are finding angle ABC, this is the angle between $\overrightarrow{BA}$ and $\overrightarrow{BC}$ or the angle between $\overrightarrow{AB}$ and $\overrightarrow{CB}$.

Now $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB}$

$$= \begin{pmatrix}2\\1\\3\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}1\\4\\1\end{pmatrix} - \begin{pmatrix}1\\-1\\2\end{pmatrix} = \begin{pmatrix}0\\5\\-1\end{pmatrix}$$

Now $\overrightarrow{BA} \cdot \overrightarrow{BC} = \begin{pmatrix}1\\2\\1\end{pmatrix} \cdot \begin{pmatrix}0\\5\\-1\end{pmatrix} = 0 + 10 - 1 = 9$

$$|\overrightarrow{BA}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$$

$$|\overrightarrow{BC}| = \sqrt{0^2 + 5^2 + (-1)^2} = \sqrt{26}$$

$$\cos A\hat{B}C = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{9}{\sqrt{6}\sqrt{26}}$$

$$A\hat{B}C = \cos^{-1}\left(\frac{9}{\sqrt{6}\sqrt{26}}\right)$$
$$= 43.9°$$

Hence, angle ABC is 43.9°

EXAMPLE 15 Given the vectors $\overrightarrow{OP} = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix}-1\\1\\-1\end{pmatrix}$, find the angle between the two vectors.

SOLUTION Let θ be the angle between $\overrightarrow{OP}$ and $\overrightarrow{OQ}$.

$$\cos\theta = \frac{\begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\-1\end{pmatrix}}{\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\left|\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right|}$$

$$= \frac{-1+2-1}{\sqrt{1^2+2^2+1^2}\sqrt{(-1)^2+(1)^2+(-1)^2}}$$

$$= \frac{0}{\sqrt{6}\sqrt{3}} = 0$$

$$\Rightarrow \theta = \cos^{-1}(0) = 90°$$

Hence, the angle between the two vectors is 90°.

Perpendicular and parallel vectors

Perpendicular vectors

Any two vectors **u** and **v** are perpendicular if and only if the scalar product of the two vectors is 0, that is $\mathbf{u}\cdot\mathbf{v} = 0$.

EXAMPLE 16 Prove that $\begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$ are perpendicular vectors.

SOLUTION To prove that two vectors are perpendicular, we need to prove that their scalar product is 0.

$$\begin{pmatrix}2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\0\\1\end{pmatrix} = -4+0+4$$
$$= 0$$

Since $\begin{pmatrix}2\\1\\4\end{pmatrix}\cdot\begin{pmatrix}-2\\0\\1\end{pmatrix} = 0 \Rightarrow \begin{pmatrix}2\\1\\4\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\1\end{pmatrix}$ are perpendicular vectors.

EXAMPLE 17 The vectors $2\mathbf{i}+\mathbf{j}+a\mathbf{k}$ and $(a-2)\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ are perpendicular vectors. Find the value of a.

SOLUTION Since $\begin{pmatrix}2\\1\\a\end{pmatrix}$ and $\begin{pmatrix}a-2\\3\\4\end{pmatrix}$ are perpendicular:

$$\begin{pmatrix}2\\1\\a\end{pmatrix}\cdot\begin{pmatrix}a-2\\3\\4\end{pmatrix} = 0$$

$\Rightarrow 2a - 4 + 3 + 4a = 0$

$\Rightarrow 6a = 1$

$\Rightarrow a = \frac{1}{6}$

Hence, $a = \frac{1}{6}$.

Parallel vectors

Two vectors are parallel if and only if one is a scalar multiple of the other.

The vectors $\begin{pmatrix}2\\1\\2\end{pmatrix}$ and $\begin{pmatrix}4\\2\\4\end{pmatrix}$ are parallel since $\begin{pmatrix}4\\2\\4\end{pmatrix} = 2\begin{pmatrix}2\\1\\2\end{pmatrix}$, that is $\begin{pmatrix}4\\2\\4\end{pmatrix}$ is a scalar multiple of $\begin{pmatrix}2\\1\\2\end{pmatrix}$.

EXAMPLE 18

Relative to an origin O, the position vectors of the points P, Q, R and S are given by $\overrightarrow{OP} = \begin{pmatrix}4\\-2\\3\end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix}2\\1\\2\end{pmatrix}, \overrightarrow{OR} = \begin{pmatrix}1\\q\\1\end{pmatrix}, \overrightarrow{OS} = \begin{pmatrix}3\\p\\1\end{pmatrix}$ where p and q are constants. Find the following.

(a) A unit vector in the direction of $\overrightarrow{PQ}$

(b) The value of q for which angle POR is 90°

(c) The value of p for which the length of $\overrightarrow{PS}$ is $\sqrt{21}$

SOLUTION

(a) First we find $\overrightarrow{PQ}$.

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}2\\1\\2\end{pmatrix} - \begin{pmatrix}4\\-2\\3\end{pmatrix} = \begin{pmatrix}-2\\3\\-1\end{pmatrix}$$

We next need the length of $\overrightarrow{PQ}$.

$$|\overrightarrow{PQ}| = \sqrt{(-2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

A unit vector in the direction of $\overrightarrow{PQ}$ is $\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{1}{\sqrt{14}}\begin{pmatrix}-2\\3\\-1\end{pmatrix}$.

(b) Since angle POR is 90°, $\overrightarrow{OP}$ is perpendicular to $\overrightarrow{OR}$ and hence, $\overrightarrow{OP} \cdot \overrightarrow{OR} = 0$

Therefore, $\begin{pmatrix}4\\-2\\3\end{pmatrix} \cdot \begin{pmatrix}1\\q\\1\end{pmatrix} = 0 \Rightarrow 4 - 2q + 3 = 0 \Rightarrow 2q = 7, q = \frac{7}{2}$.

(c) $\overrightarrow{PS} = \overrightarrow{OS} - \overrightarrow{OP} = \begin{pmatrix}3\\p\\1\end{pmatrix} - \begin{pmatrix}4\\-2\\3\end{pmatrix} = \begin{pmatrix}-1\\p+2\\-2\end{pmatrix}$

$|\overrightarrow{PS}| = \sqrt{1 + (p+2)^2 + 4} = \sqrt{21}$

Therefore, $5 + (p + 2)^2 = 21$.

$\Rightarrow (p + 2)^2 = 16$

$\Rightarrow p + 2 = \pm 4$

Hence, $p = -2 + 4 = 2, p = -4 - 2 = -6$.

EXAMPLE 19

Relative to an origin O, the position vectors of the points P and Q are given by $\overrightarrow{OP} = \begin{pmatrix}1\\-2\\1\end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix}1\\1\\3\end{pmatrix}$.

(a) Find angle QOP.

(b) The point R is such that $\overrightarrow{OR} = 2\mathbf{i} + 5b\mathbf{j}$ where b is a constant. The lengths of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are equal. Find the value(s) of b.

SOLUTION

(a) The angle QOP is the angle between the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$. Let θ be angle QOP.

$$\text{Now } \cos\theta = \frac{\begin{pmatrix}1\\-2\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\1\\3\end{pmatrix}\right|} = \frac{1-2+3}{\sqrt{6}\,\sqrt{11}} = \frac{2}{\sqrt{66}}$$

Therefore, $\theta = \cos^{-1}\left(\frac{2}{\sqrt{66}}\right) = 75.7°$.

(b) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}1\\1\\3\end{pmatrix} - \begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}0\\3\\2\end{pmatrix}$

$|\overrightarrow{PQ}| = \sqrt{9+4} = \sqrt{13}$

Now, $\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = \begin{pmatrix}2\\5b\\0\end{pmatrix} - \begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\5b+2\\-1\end{pmatrix}$

$|\overrightarrow{PR}| = \sqrt{(1)^2 + (5b+2)^2 + (-1)^2}$

$= \sqrt{13}$, since $|\overrightarrow{PQ}| = |\overrightarrow{PR}|$

Therefore, $2 + (5b+2)^2 = 13$.

$\Rightarrow (5b+2)^2 = 11$

$\Rightarrow 5b + 2 = \pm\sqrt{11}$

Hence, $b = \frac{-2 \pm \sqrt{11}}{5}$.

EXAMPLE 20 Relative to an origin O, the position vectors of A and B are given by $\overrightarrow{OA} = \begin{pmatrix}4\\-2\\1\end{pmatrix}, \overrightarrow{OB} = \begin{pmatrix}-1\\2\\5\end{pmatrix}$.

(a) Find the position vector of the midpoint M of A and B.

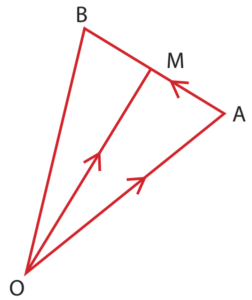
(b) Given that C is the point such that $\overrightarrow{AC} = 3\overrightarrow{AB}$, find the position vector of C.

SOLUTION

(a) Using coordinate geometry,

$$\overrightarrow{OM} = \begin{pmatrix}\frac{4-1}{2}\\ \frac{-2+2}{2}\\ \frac{1+5}{2}\end{pmatrix} = \begin{pmatrix}\frac{3}{2}\\0\\3\end{pmatrix}$$

Alternative method:



$\overrightarrow{OM} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \begin{pmatrix}4\\-2\\1\end{pmatrix} + \frac{1}{2}\left[\begin{pmatrix}-1\\2\\5\end{pmatrix} - \begin{pmatrix}4\\-2\\1\end{pmatrix}\right]$

$= \begin{pmatrix}4\\-2\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-5\\4\\4\end{pmatrix}$

$= \begin{pmatrix}\frac{3}{2}\\0\\3\end{pmatrix}$

(b) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}-1\\2\\5\end{pmatrix} - \begin{pmatrix}4\\-2\\1\end{pmatrix} = \begin{pmatrix}-5\\4\\4\end{pmatrix}$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \overrightarrow{OC} - \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$$

Since $\overrightarrow{AC} = 3\overrightarrow{AB}$:

$$\overrightarrow{OC} - \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix}$$

$$\Rightarrow \overrightarrow{OC} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -5 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -11 \\ 10 \\ 13 \end{pmatrix}$$

EXERCISE 11A

1 Find the scalar products of the following pairs of vectors.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ -4 \end{pmatrix}$

2 Find the length of the following vectors, referred to an origin O.

(a) $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

(b) $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

(c) $\overrightarrow{OC} = \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix}$

3 Are the vectors in each of the following pairs perpendicular to each other?

(a) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$

(d) $\begin{pmatrix} 4 \\ 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

4 Referred to the origin O, the position vectors of A, B and C are given respectively by $\overrightarrow{OA} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\overrightarrow{OC} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$. Find a unit vector parallel to each of these.

(a) $\overrightarrow{AB}$

(b) $\overrightarrow{AC}$

(c) $\overrightarrow{BC}$

5 Given that $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ p \\ 2p-1 \end{pmatrix}$, find the following.

(a) The angle between the directions of **a** and **b**

(b) The value of p for which **a** and **c** are perpendicular

6 The points A, B and C have position vectors $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$, $3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} + \mathbf{j} - \mathbf{k}$ respectively, relative to an origin O. Find angle ABC.

7 Relative to an origin O, the position vectors of the points A and B are given by $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Given that C is the point such that $\overrightarrow{AC} = 2\overrightarrow{AB}$, find the unit vector in the direction of $\overrightarrow{OC}$.

8 Relative to the origin O, the position vectors or the points P, Q, R and S are given by $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$, $\overrightarrow{OR} = \begin{pmatrix} -2 \\ 1 \\ a \end{pmatrix}$ and $\overrightarrow{OS} = \begin{pmatrix} -1 \\ 3 \\ b \end{pmatrix}$, where a and b are constants, find the following.

(a) The unit vector in the direction of $\overrightarrow{PQ}$

(b) The value of a for which the POR is 90°

(c) The values of b for which the length of $\overrightarrow{PS}$ is 5 units

9 Relative to an origin O, the position vectors of A and B are given by $\overrightarrow{OA} = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 1 \\ p \end{pmatrix}$, where p is a constant.

(a) Find angle AOB when $p = 1$

(b) Find the values of p for which the length of AB is 8 units

10 Referred to the origin O, the position vectors of P and Q are given respectively by $OP = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $OQ = 5\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$.

(a) Find angle POQ.

(b) Find the position vector of the point A on OQ such that PA is perpendicular to OQ.

Equation of a line

A line is defined in space by either one of the following.

(a) A point on the line and a vector parallel to the line (direction of the line)

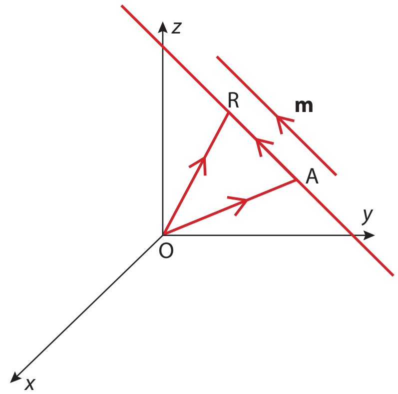
(b) Two points on the line

Finding the equation of a line given a point on a line and the direction of the line

Let **a** be a fixed point on the line and **m** be a vector parallel to the line. The equation of the line is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}$ where $\lambda \in \mathbb{R}$.

The vector **r** represents any point on the line $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

PROOF



Let R be a point on the line with position vector **r** and A be a known point on the line with position **a**. Since **m** is a vector parallel to the line:

$\overrightarrow{AR}$ is parallel to **m**

$\Rightarrow \overrightarrow{AR} = \lambda\mathbf{m}, \lambda \in \mathbb{R}$. i.e. $\overrightarrow{AR}$ is a scalar multiple of **m**.

Since $\overrightarrow{AR} = \overrightarrow{OR} - \overrightarrow{OA} = \mathbf{r} - \mathbf{a}$:

$\Rightarrow \overrightarrow{OR} - \overrightarrow{OA} = \lambda\mathbf{m}$

$\Rightarrow \mathbf{r} - \mathbf{a} = \lambda\mathbf{m}$

Therefore, $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \lambda \in \mathbb{R}$.

> **Note**
>
> A vector **parallel** to a line is called the **direction vector** of the line.

EXAMPLE 21 Find the equation of the line parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and passing through the point $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

SOLUTION: Using $\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \lambda \in \mathbb{R}$, where **m** is a vector parallel to the line, i.e. $\mathbf{m} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, the equation of the line is $\mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$.

EXAMPLE 22 Find the value of p for which the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$ has the same direction as $\begin{pmatrix} -2 \\ 3 \\ p-1 \end{pmatrix}$.

SOLUTION Since the line is in the same direction as $\begin{pmatrix} -2 \\ 3 \\ p-1 \end{pmatrix}$ we have $\begin{pmatrix} -2 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ p-1 \end{pmatrix}$.

Therefore, $p - 1 = 2$.

Hence, $p = 3$.

Finding the equation of a line given two points on the line

Let P and Q be two points on the line with position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ respectively. The equation of the line passing through P and Q is:

$\mathbf{r} = \overrightarrow{OP} + \lambda(\overrightarrow{OQ} - \overrightarrow{OP}), \lambda \in \mathbb{R}$.

PROOF

Can you prove this result?

EXAMPLE 23 Find the equation of the line passing through the points $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and $\begin{pmatrix}4\\2\\4\end{pmatrix}$.

SOLUTION Using $\mathbf{r} = \overrightarrow{OP} + \lambda(\overrightarrow{OQ} - \overrightarrow{OP}), \lambda \in \mathbb{R}$, where $\overrightarrow{OP} = \begin{pmatrix}2\\1\\3\end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix}4\\2\\4\end{pmatrix}$:

$$\overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix}4\\2\\4\end{pmatrix} - \begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix}$$

Therefore, the equation of the line is $\mathbf{r} = \begin{pmatrix}2\\1\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\1\end{pmatrix}, \lambda \in \mathbb{R}$.

EXAMPLE 24 Relative to an origin O, a line contains the points $2\mathbf{i} + \mathbf{j}$ and $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$. Find the equation of this line.

SOLUTION Let $\mathbf{p} = \mathbf{i} + \mathbf{j}$ and $\mathbf{q} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

$$\mathbf{q} - \mathbf{p} = (3\mathbf{i} - 2\mathbf{j} + \mathbf{k}) - (2\mathbf{i} + \mathbf{j})$$
$$= \mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

The equation of the line is $\mathbf{r} = 2\mathbf{i} + \mathbf{j} + \lambda(\mathbf{i} - 3\mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$.

EXAMPLE 25 Identify two points on the line with equation $\mathbf{r} = \begin{pmatrix}1\\2\\5\end{pmatrix} + t\begin{pmatrix}0\\2\\4\end{pmatrix}, t \in \mathbb{R}$.

SOLUTION To identify a point on the line we assign a value to t.

When $t = 0$, $\mathbf{r} = \begin{pmatrix}1\\2\\5\end{pmatrix} + 0\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix}$

When $t = 1$, $\mathbf{r} = \begin{pmatrix}1\\2\\5\end{pmatrix} + (1)\begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}1\\2\\5\end{pmatrix} + \begin{pmatrix}0\\2\\4\end{pmatrix} = \begin{pmatrix}1\\4\\9\end{pmatrix}$

Hence, two points on the line are: $\begin{pmatrix}1\\2\\5\end{pmatrix}$ and $\begin{pmatrix}1\\4\\9\end{pmatrix}$.

EXAMPLE 26 Is the point $\begin{pmatrix}2\\1\\2\end{pmatrix}$ on the line with equation $\mathbf{r} = \begin{pmatrix}-1\\2\\0\end{pmatrix} + t\begin{pmatrix}2\\3\\2\end{pmatrix}, t \in \mathbb{R}$?

SOLUTION If we can find a value of t which gives the point $\begin{pmatrix}2\\1\\2\end{pmatrix}$ on the line, then the point is on the line. To do this, we can form three equations using the components and solve one of them. Then we check if all the equations are satisfied using this value of t.

If the point is on the line, then we have a value of t for which this is true:

$$\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}-1\\2\\0\end{pmatrix} + t\begin{pmatrix}2\\3\\2\end{pmatrix} = \begin{pmatrix}-1 + 2t\\2 + 3t\\0 + 2t\end{pmatrix}$$

Equating corresponding components gives:

$2 = -1 + 2t \Rightarrow t = \frac{3}{2}$

$1 = 2 + 3t \Rightarrow t = -\frac{1}{3}$

$2 = 2t \Rightarrow t = 1$

All the values of t are different. There is no value of t giving the point. Hence, the point is not on the line.

EXAMPLE 27 Is the point $\begin{pmatrix}4\\4\\3\end{pmatrix}$ on the line with equation $\mathbf{r} = \begin{pmatrix}3\\2\\2\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix}, s \in \mathbb{R}$?

SOLUTION Let $\begin{pmatrix}4\\4\\3\end{pmatrix} = \begin{pmatrix}3\\2\\2\end{pmatrix} + s\begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}3+s\\2+2s\\2+s\end{pmatrix}$.

Equating the corresponding components gives:

$4 = 3 + s \Rightarrow s = 1$

$4 = 2 + 2s \Rightarrow s = 1$

$3 = 2 + s \Rightarrow s = 1$

When $s = 1$, we get the point $\begin{pmatrix}4\\4\\3\end{pmatrix}$ on the line.

Therefore, the point $\begin{pmatrix}4\\4\\3\end{pmatrix}$ is on the line.

Note

In this case all three equations gave the same value for s.

Vector equation of a line

The equation of a line in the form $\mathbf{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\0\end{pmatrix}, \lambda \in \mathbb{R}$ is the vector equation of the line.

In this form, $\begin{pmatrix}1\\2\\1\end{pmatrix}$ is a fixed point on the line and $\begin{pmatrix}-1\\2\\0\end{pmatrix}$ is a vector parallel to the line.

Parametric equation of a line

Let $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ be a fixed point on a line and $x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ be a vector parallel to the line. The equation of the line is

$\mathbf{r} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \lambda\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$, where $\lambda \in \mathbb{R}$.

Since r represents any point on the line we have $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$

Therefore, $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \lambda\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}, \lambda \in \mathbb{R}$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \begin{pmatrix}\lambda x_0\\\lambda y_0\\\lambda z_0\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a+\lambda x_0\\b+\lambda y_0\\c+\lambda z_0\end{pmatrix}$$

Equating corresponding components gives:

$$\left.\begin{aligned}x &= a + \lambda x_0\\ y &= b + \lambda y_0\\ z &= c + \lambda z_0\end{aligned}\right\}\lambda \in \mathbb{R}$$

These are the parametric equations of the line. λ is called the parameter in the equation.

Note

We have an equation for x in terms of λ, one for y in terms of λ and an equation for z in terms of λ.

EXAMPLE 28 The equation of a line is given as $\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}, \lambda \in \mathbb{R}$. Write down the parametric equations of the line.

SOLUTION

$$\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}, \lambda \in \mathbb{R}$$

Replacing $\mathbf{r}$ by $\begin{pmatrix}x\\y\\z\end{pmatrix}$ gives:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\2\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \begin{pmatrix}-\lambda\\2\lambda\\2\lambda\end{pmatrix}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1-\lambda\\2+2\lambda\\4+2\lambda\end{pmatrix}$$

Equating corresponding components:

$$\left.\begin{aligned}x &= 1-\lambda\\ y &= 2+2\lambda\\ z &= 4+2\lambda\end{aligned}\right\}\lambda \in \mathbb{R}$$

Hence, the parametric equations of the line are:

$$x = 1-\lambda,\ y = 2+2\lambda,\ z = 4+2\lambda,\ \lambda \in \mathbb{R}$$

Cartesian equation of a line

Let the vector equation of a line be:

$$\mathbf{r} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \lambda\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}, \lambda \in \mathbb{R}$$

If we replace $\mathbf{r}$ by $\begin{pmatrix}x\\y\\z\end{pmatrix}$ we have:

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}a\\b\\c\end{pmatrix} + \lambda\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}$$

$x = a + \lambda x_0$ [1]

$y = b + \lambda y_0$ [2]

$z = c + \lambda z_0$ [3]

From [1]: $x - a = \lambda x_0 \Rightarrow \lambda = \dfrac{x-a}{x_0}$

From [2]: $y - b = \lambda y_0 \Rightarrow \lambda = \dfrac{y-b}{y_0}$

From [3]: $z - c = \lambda z_0 \Rightarrow \lambda = \dfrac{z-c}{z_0}$

Equating equations for λ

$$\frac{x-a}{x_0} = \frac{y-b}{y_0} = \frac{z-c}{z_0}$$

Note

The denominator of each is a component of the direction vector, and the numerator contains the components of the fixed point on the line.

$\dfrac{x-a}{x_0} = \dfrac{y-b}{y_0} = \dfrac{z-c}{z_0}$, $\begin{pmatrix}a\\b\\c\end{pmatrix}$: fixed point

EXAMPLE 29 Write the following equation in Cartesian form:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \lambda \in \mathbb{Z}$$

SOLUTION: Replacing **r** by $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ gives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 + 2\lambda \\ 4 + 3\lambda \\ -1 + 5\lambda \end{pmatrix}$$

Equating components gives:

$x = 2 + 2\lambda$ [1]

$y = 4 + 3\lambda$ [2]

$z = -1 + 5\lambda$ [3]

From [1]: $x = 2 + 2\lambda$

$$\lambda = \frac{x - 2}{2}$$

From [2]: $y - 4 = 3\lambda$

$$\lambda = \frac{y - 4}{3}$$

From [3]: $z = -1 + 5\lambda$

$z + 1 = 5\lambda$

$$\lambda = \frac{z + 1}{5}$$

Therefore, $\frac{x - 2}{2} = \frac{y - 4}{3} = \frac{z + 1}{5}$, which is the Cartesian equation of the line.

Equation of a line

The different forms of the equation of a line are:

Vector form:

$$r = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \lambda \in \mathbb{R}$$

Parametric form:

$$\left.\begin{aligned} x &= a + \lambda x_0 \\ y &= b + \lambda y_0 \\ z &= c + \lambda z_0 \end{aligned}\right\} \lambda \in \mathbb{R}$$

Cartesian form:

$$\frac{x - a}{x_0} = \frac{y - b}{y_0} = \frac{z - c}{z_0}$$

EXAMPLE 30 The Cartesian equation of a line is given by:

$$\frac{x-2}{3}=\frac{y+5}{7}=\frac{2z-4}{5}$$

Write this equation in vector form and parametric form.

SOLUTION We can write the equation in the form:

$$\frac{x-a}{x_0}=\frac{y-b}{y_0}=\frac{z-c}{z_0}$$

$$\Rightarrow \frac{x-2}{3}=\frac{y-(-5)}{7}=\frac{2(z-2)}{5}$$

$$\Rightarrow \frac{x-2}{3}=\frac{y-(-5)}{7}=\frac{(z-2)}{\frac{5}{2}}$$

The vector equation is:

$$\mathbf{r}=\begin{pmatrix}2\\-5\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\7\\\frac{5}{2}\end{pmatrix},\ \lambda\in\mathbb{R}$$

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$$\mathbf{r}=\begin{pmatrix}2\\-5\\2\end{pmatrix}+\begin{pmatrix}3\lambda\\7\lambda\\\frac{5}{2}\lambda\end{pmatrix}$$

$$=\begin{pmatrix}2+3\lambda\\-5+7\lambda\\2+\frac{5}{2}\lambda\end{pmatrix}$$

$$\left.\begin{aligned}x&=2+3\lambda\\y&=-5+7\lambda\\z&=2+\tfrac{5}{2}\lambda\end{aligned}\right\}\lambda\in\mathbb{R}$$

The parametric equations are:

$x=2+3\lambda$
$y=-5+7\lambda$
$z=2+\frac{5}{2}\lambda$
where $\lambda\in\mathbb{R}$.

Finding the angle between two lines, given the equations of the lines

The angle between two lines is the angle between their direction vectors. Let θ be the angle between the lines with equations $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{c}+\mu\mathbf{d}$, where $\lambda,\mu\in\mathbb{R}$. Since the directions of the lines are **b** and **d** respectively, the angle between the two lines is:

$$\cos\theta=\frac{\mathbf{b}\cdot\mathbf{d}}{|\mathbf{b}|\,|\mathbf{d}|}$$

Note

(i) Two lines are parallel if and only if their direction vectors are scalar multiples of each other.
(ii) Two lines are perpendicular if and only if the scalar product of their direction vectors is zero.

EXAMPLE 31 Find the angle between the lines $\mathbf{r} = \begin{pmatrix}1\\0\\1\end{pmatrix} + t\begin{pmatrix}-1\\1\\2\end{pmatrix}, t \in \mathbb{R}$ and $\mathbf{r} = \begin{pmatrix}1\\4\\2\end{pmatrix} + s\begin{pmatrix}1\\4\\1\end{pmatrix}, s \in \mathbb{R}$.

SOLUTION The angle between two lines is the angle between their direction vectors. Therefore, we first identify the direction vectors of the line.

The direction vectors are $\begin{pmatrix}-1\\1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\4\\1\end{pmatrix}$, let θ be the angle between the lines.

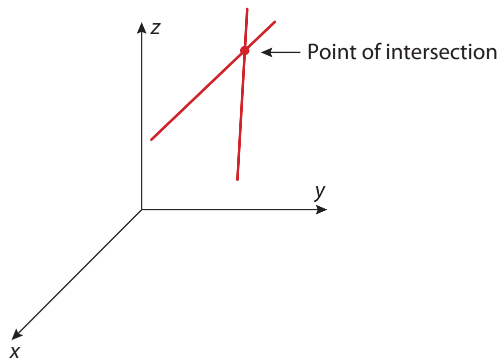
$$\cos\theta = \frac{\begin{pmatrix}-1\\1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\4\\1\end{pmatrix}}{\left|\begin{pmatrix}-1\\1\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\4\\1\end{pmatrix}\right|} = \frac{-1+4+2}{\sqrt{6}\,\sqrt{18}} = \frac{5}{6\sqrt{3}}$$

Hence, the angle between the two lines is 61.2°.

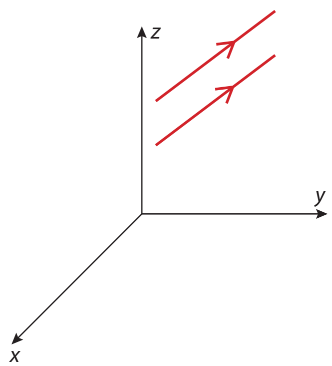
Skew lines

Skew lines are lines that are not parallel and do not intersect. If two lines are on the same plane, they are either parallel to each other or they intersect each other. Hence, skew lines will be on different planes.

These lines intersect.

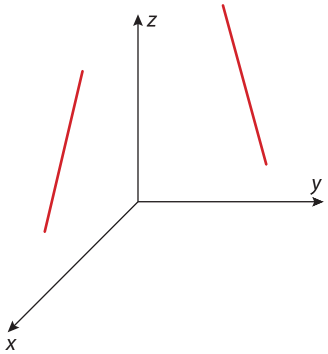


These lines are parallel lines.



These lines are skew lines.

Skew lines will not intersect even though they are not parallel.



EXAMPLE 32 Decide whether the lines $\mathbf{r} = \begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\2\end{pmatrix}$, where $\lambda, \mu \in \mathbb{R}$, are skew.

SOLUTION Skew lines are not parallel and do not intersect. Therefore, we need to decide:

- whether the lines are parallel
- whether the lines intersect

Since $\begin{pmatrix}1\\3\\2\end{pmatrix}$ is not a scalar multiple of $\begin{pmatrix}-1\\2\\4\end{pmatrix}$, then $\begin{pmatrix}1\\3\\2\end{pmatrix}$ is not parallel to $\begin{pmatrix}-1\\2\\4\end{pmatrix}$. Therefore, the lines are not parallel.

Let us decide whether the lines intersect. If they intersect, there will be a point that satisfies the equations of both lines for particular values of λ and μ.

Equating the equations of the lines gives:

$$\begin{pmatrix}1\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\4\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\2\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}1-\lambda\\2+2\lambda\\1+4\lambda\end{pmatrix} = \begin{pmatrix}2+\mu\\1+3\mu\\2\mu\end{pmatrix}$$

$\Rightarrow 1 - \lambda = 2 + \mu$ [1]

$2 + 2\lambda = 1 + 3\mu$ [2]

$1 + 4\lambda = 2\mu$ [3]

Now we solve equations [1] and [2] simultaneously:

$1 - \lambda = 2 + \mu$

$1 + \lambda = \frac{1}{2} + \frac{3}{2}\mu$

Adding gives: $2 = \frac{5}{2} + \frac{5}{2}\mu$

$\Rightarrow -\frac{1}{2} = \frac{5}{2}\mu$

$\Rightarrow \mu = -5$

Substituting $\mu = -5$ into [1] gives:

$\lambda = 4$

Equations [1] and [2] are satisfied by $\mu = -5$ and $\lambda = 4$.

Substituting $\lambda = 4$ and $\mu = -5$ into equation [3] gives:

$1 + 4(4) = 2(-5)$

$\Rightarrow 17 = -10$, which is inconsistent.

Since all three equations are not satisfied by $\lambda = 4$ and $\mu = -5$, the lines do not intersect.

Since the two lines are not parallel and do not intersect, the lines are skew.

EXAMPLE 33

(a) Find the equation of the line passing through the points $\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\begin{pmatrix}-1\\0\\1\end{pmatrix}$.

(b) Find also the equation of the line passing through $\begin{pmatrix}2\\1\\0\end{pmatrix}$ and parallel to $\begin{pmatrix}1\\2\\4\end{pmatrix}$.

(c) Do the lines from **(a)** and **(b)** intersect?

SOLUTION

(a) The equation of the line is

$$\mathbf{r} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\left[\begin{pmatrix}-1\\0\\1\end{pmatrix} - \begin{pmatrix}1\\2\\2\end{pmatrix}\right], \lambda \in \mathbb{R}, \text{ using } \mathbf{r} = \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}), \lambda \in \mathbb{R}$$

$$\mathbf{r} = \begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-2\\-1\end{pmatrix}, \lambda \in \mathbb{R}$$

(b) The equation of the line is

$$\mathbf{r} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\2\\4\end{pmatrix}, \mu \in \mathbb{R}, \text{ using } \mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \lambda \in \mathbb{R}$$

(c) Equating components gives:

$$\begin{pmatrix}1\\2\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-2\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\2\\4\end{pmatrix}$$

$1 - 2\lambda = 2 + \mu$ [1]

$2 - 2\lambda = 1 + 2\mu$ [2]

$2 - \lambda = 4\mu$ [3]

We solve equations [1] and [2]:

$1 = -1 + \mu$ [2] − [1]

$\Rightarrow \mu = 2$

Substituting into [1] gives:

$1 - 2\lambda = 2 + 2$

$-2\lambda = 3$

$\lambda = -\frac{3}{2}$

Substituting $\mu = 2$ and $\lambda = -\frac{3}{2}$ into [3] gives:

$2 + \frac{3}{2} = 4(2)$, which is inconsistent.

Since all three equations are not satisfied by $\mu = 2$ and $\lambda = -\frac{3}{2}$, the lines do not intersect.

EXAMPLE 34 The vector equation of a straight line l is $\mathbf{r} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} + 2\mathbf{k}),\ t \in \mathbb{R}$. All position vectors are with respect to the origin O.

(a) The point A$(-2, a, b)$ is on l. Find the values of a and b.

(b) The point B on l is given by $t = 4$. Find the coordinates of B.

(c) Find the coordinates of C on l, such that $\overrightarrow{OC}$ is perpendicular to l.

SOLUTION

(a) Equating gives:

$$\begin{pmatrix} -2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -4 + t \\ 3 + t \\ 1 + 2t \end{pmatrix}$$

Therefore, $-2 = -4 + t \Rightarrow t = 2$

$a = 3 + t \Rightarrow a = 3 + 2 = 5$

$b = 1 + 2t = 1 + 2(2) = 5$

Hence $a = 5$, $b = 5$.

(b) When $t = 4$, $\begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 7 \\ 9 \end{pmatrix}$

The coordinates of B are $(0, 7, 9)$.

(c) Since C is on l: $\overrightarrow{OC} = \begin{pmatrix} -4 + t \\ 3 + t \\ 1 + 2t \end{pmatrix}$.

We need to find t such that $\overrightarrow{OC}$ is perpendicular to l.

Since $\overrightarrow{OC}$ and l are perpendicular:

$\overrightarrow{OC} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$, $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is the direction of l.

$$\Rightarrow \begin{pmatrix} -4 + t \\ 3 + t \\ 1 + 2t \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$$

$\Rightarrow -4 + t + 3 + t + 2 + 4t = 0$

$\Rightarrow 6t + 1 = 0, \qquad t = -\frac{1}{6}$

Hence, $\overrightarrow{OC} = \begin{pmatrix} -4 - \frac{1}{6} \\ 3 - \frac{1}{6} \\ 1 - \frac{2}{6} \end{pmatrix} = \begin{pmatrix} -\frac{25}{6} \\ \frac{17}{6} \\ \frac{2}{3} \end{pmatrix}$.

The coordinates of C are $\left(-\frac{25}{6}, \frac{17}{6}, \frac{2}{3}\right)$.

Equation of a plane

Note

A vector perpendicular to a plane is called a **normal vector** to the plane.

A plane can be specified by either of the following.

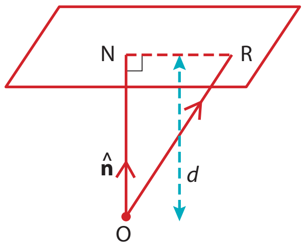
(i) A unit vector perpendicular to the plane and the distance from the origin to the plane.

(ii) A point on the plane and a vector perpendicular to the plane.

Equation of a plane, given the distance from the origin to the plane and a unit vector perpendicular to the plane

Given a unit vector $\hat{\mathbf{n}}$ perpendicular to a plane, and the distance d from the origin to the plane, the equation of the plane is $\mathbf{r} \cdot \hat{\mathbf{n}} = d$

PROOF



Let $\overrightarrow{OR} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{r} = \overrightarrow{OR}$.

Since $\overrightarrow{OR}$ is on the plane and $\overrightarrow{ON}$ is perpendicular to the plane, we have:

$\overrightarrow{NR} \cdot \overrightarrow{ON} = 0$

Since d is the distance from O to N and $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane:

$\overrightarrow{ON} = d\hat{\mathbf{n}}$

$\overrightarrow{NR} = \overrightarrow{OR} - \overrightarrow{ON}$

$= \mathbf{r} - d\hat{\mathbf{n}}$ (Since $\overrightarrow{ON} = d\hat{\mathbf{n}}$)

$\overrightarrow{NR} \cdot \overrightarrow{ON} = 0$

$\Rightarrow (\mathbf{r} - d\hat{\mathbf{n}}) \cdot d\hat{\mathbf{n}} = 0$

$\Rightarrow \mathbf{r} \cdot d\hat{\mathbf{n}} - d^2\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 0$

$\Rightarrow \mathbf{r} \cdot d\hat{\mathbf{n}} = d^2\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}$

$\Rightarrow d\,(\mathbf{r} \cdot \hat{\mathbf{n}}) = d^2$ (Since $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$)

$\Rightarrow \mathbf{r} \cdot \hat{\mathbf{n}} = d$ (Dividing both sides by d)

Q.E.D.

EXAMPLE 35 Given that a unit vector perpendicular to a plane is $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix}$ and the distance from the origin to the plane is 2 units, find the equation of the plane.

SOLUTION Using $\mathbf{r} \cdot \hat{\mathbf{n}} = d$ gives:

$\mathbf{r} \cdot \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} = 2$

$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 6$ (Multiply throughout by 3)

The equation of the plane is $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 6$.

EXAMPLE 36 The equation of a plane is given by $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5$. Find the distance from the origin to the plane.

SOLUTION We will convert $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 5$ to $\mathbf{r} \cdot \hat{\mathbf{n}} = d$.

We first find the magnitude of $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$:

$$\left|\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right| = \sqrt{2^2 + (1)^2 + (-1)^2} = \sqrt{6}$$

$$\Rightarrow \mathbf{r} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \frac{5}{\sqrt{6}}$$

Hence, the distance from the origin to the plane is $\frac{5}{\sqrt{6}}$.

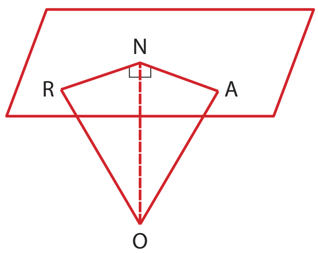
Equation of a plane, given a point on the plane and a normal to the plane

Let $\mathbf{a}$ be a point on a plane and $\mathbf{n}$ a vector perpendicular to the plane.

The equation of the plane is

$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$

PROOF



Note

Any vector perpendicular to a plane is perpendicular to every vector on the plane.

Since $\overrightarrow{NR}$ and $\overrightarrow{NA}$ are on the plane and $\overrightarrow{ON}$ is perpendicular to the plane, $\overrightarrow{NR}$ and $\overrightarrow{NA}$ are perpendicular to $\overrightarrow{ON}$.

$\Rightarrow \overrightarrow{NR} \cdot \overrightarrow{ON} = 0$

$\Rightarrow \overrightarrow{NR} = \overrightarrow{OR} - \overrightarrow{ON}$

Therefore, $\left(\overrightarrow{OR} - \overrightarrow{ON}\right) \cdot \overrightarrow{ON} = 0$

Similarly, $\overrightarrow{NA} = \overrightarrow{OA} - \overrightarrow{ON}$

$\Rightarrow \left(\overrightarrow{OA} - \overrightarrow{ON}\right) \cdot \overrightarrow{ON} = 0$

$\Rightarrow \left(\overrightarrow{OR} - \overrightarrow{ON}\right) \cdot \overrightarrow{ON} = \left(\overrightarrow{OA} - \overrightarrow{ON}\right) \cdot \overrightarrow{ON} = 0$

$\Rightarrow \overrightarrow{OR} \cdot \overrightarrow{ON} = \overrightarrow{OA} \cdot \overrightarrow{ON}$

Therefore, $\overrightarrow{OR} \cdot \overrightarrow{ON} = \overrightarrow{OA} \cdot \overrightarrow{ON}$.

Hence, $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ is the equation of a plane perpendicular to the vector $\mathbf{n}$ and containing the point $\mathbf{a}$.

EXAMPLE 37 Find the equation of the plane passing through $\begin{pmatrix}1\\2\\4\end{pmatrix}$ and perpendicular to $\begin{pmatrix}-1\\2\\1\end{pmatrix}$. Find also the distance of the plane from the origin.

SOLUTION Using $\mathbf{r}\cdot\mathbf{n} = \mathbf{a}\cdot\mathbf{n}$ gives:

$$\mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

$$\Rightarrow \mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = -1 + 4 + 4$$

$$\Rightarrow \mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = 7$$

Hence, the equation of the plane is $\mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix} = 7$.

The magnitude of $\begin{pmatrix}-1\\2\\1\end{pmatrix}$ is $\sqrt{1+4+1} = \sqrt{6}$

Therefore, $\mathbf{r}\cdot\begin{pmatrix}\frac{-1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\\\frac{1}{\sqrt{6}}\end{pmatrix} = \frac{7}{\sqrt{6}}$.

Hence, the distance from the origin to the plane is $\frac{7}{\sqrt{6}}$ units.

EXAMPLE 38 Find the equation of the plane passing through $\begin{pmatrix}1\\4\\1\end{pmatrix}$ and perpendicular to the line passing through the points $\begin{pmatrix}4\\2\\0\end{pmatrix}$ and $\begin{pmatrix}3\\2\\1\end{pmatrix}$.

SOLUTION A vector parallel to the normal to the plane is the direction of the required line. Since the line is perpendicular to the plane, equation of the line is given by:

$$\mathbf{r} = \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda\left[\begin{pmatrix}3\\2\\1\end{pmatrix} - \begin{pmatrix}4\\2\\0\end{pmatrix}\right]$$

$$\Rightarrow \mathbf{r} = \begin{pmatrix}4\\2\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}$$

A normal to the plane is $\mathbf{n} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$.

The equation of the plane is

$$\mathbf{r}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\4\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$\Rightarrow \mathbf{r}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = -1 + 1$$

$$\Rightarrow \mathbf{r}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = 0$$

Cartesian equation of a plane

The vector equation of a plane is

$\mathbf{r} \cdot \mathbf{n} = D$

If $\mathbf{n} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, then

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} A \\ B \\ C \end{pmatrix} = D$

$\Rightarrow Ax + By + Cz = D$

$Ax + By + Cz = D$ is the Cartesian equation of the plane with the normal vector being $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$, i.e. the coefficient of x, y and z respectively.

EXAMPLE 39 Find the equation of the plane passing through $\begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}$ and perpendicular to $\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$ in vector form and Cartesian form.

SOLUTION The equation of the plane is given by:

$\mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}$

$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 2 + 20 - 3$

$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 19$

The vector equation of the plane is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 19$.

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} = 19 \Rightarrow 2x + 4y + 3z = 19$

The Cartesian equation of the plane is $2x + 4y + 3z = 19$.

EXAMPLE 40 Find the Cartesian equation for each of the following planes.

(a) The plane π_1 through A(4, 2, −1) and parallel to the plane $2x - y + 3z = 4$

(b) The plane π_2 through B(−1, 2, −3) and perpendicular to the line l_1:
$\frac{x-2}{3} = \frac{y-1}{2} = \frac{z-1}{1}$

SOLUTION Since the two planes are parallel, they have the same normal vectors.

(a) The normal to $2x - y + 3z = 4$ is $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

Therefore, the normal to π_1 is $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

Using $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$ gives:

$$\mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 8 - 2 - 3 = 3$$

Substituting $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ gives:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = 3$$

The Cartesian equation of the plane is $2x - y + 3z = 3$.

(b) Since π_2 is perpendicular to l_1, the direction vector of l_1 is perpendicular to π_2.

The direction vector of l_1 is $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. The equation of π_2 is given by:

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{r} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = -3 + 4 - 3 = -2$$

Hence the Cartesian equation is $3x + 2y + z = -2$.

EXAMPLE 41 The equation of a line is $x = 2 + t$, $y = 1 - 3t$, $z = 4 + t$.

(a) Find the value of t which satisfies the equation of the plane π: $x + 2y + z = 12$. Hence, find the point of intersection of the line and the plane.

(b) Find also the acute angle between the normal to the plane and the line. Hence, deduce the angle between the line and the plane.

SOLUTION

(a) Substituting $x = 2 + t$, $y = 1 - 3t$, $z = 4 + t$ into $x + 2y + z = 12$, gives:

$$\begin{aligned} 2 + t + 2(1 - 3t) + 4 + t &= 12 \\ 8 - 4t &= 12 \\ t &= -1 \end{aligned}$$

Substituting $t = -1$ gives:

$x = 2 - 1 = 1$

$y = 1 - 3(-1) = 4$

$z = 4 + (-1) = 3$

Hence, the point of intersection is $x = 1$, $y = 4$, $z = 3$ or $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$.

(b) The direction vector of the line is $\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ and the normal to the plane is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. Let the angle be θ.

$$\cos\theta = \frac{\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\right|} = \frac{-4}{\sqrt{11}\sqrt{6}}$$

Therefore, $\theta = 180° - 119.5° = 60.5°$.

Hence, the angle between the line and the plane is $(180° - 90° - 60.5°) = 29.5°$.

Note

A diagram will assist in identifying the angle between the line and the plane.

EXERCISE 11B

1 Find the equation of each line given the direction vector and a fixed point on the line.

(a) $\begin{pmatrix}1\\2\\4\end{pmatrix}$ is the point and the direction is $\begin{pmatrix}2\\1\\3\end{pmatrix}$

(b) $\begin{pmatrix}1\\-1\\4\end{pmatrix}$ is the point and $\begin{pmatrix}3\\2\\-4\end{pmatrix}$ is the direction

(c) $\begin{pmatrix}4\\1\\4\end{pmatrix}$ is the point and $\begin{pmatrix}-3\\0\\2\end{pmatrix}$ is the direction

2 Find the equation of the lines passing through the following points.

(a) $\begin{pmatrix}2\\1\\1\end{pmatrix}$ and $\begin{pmatrix}4\\1\\3\end{pmatrix}$

(b) $\begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\begin{pmatrix}4\\2\\4\end{pmatrix}$

(c) $\begin{pmatrix}4\\4\\5\end{pmatrix}$ and $\begin{pmatrix}6\\2\\7\end{pmatrix}$

3 Find the vector equation and Cartesian equation of the plane passing through the point A and perpendicular to the vector B with position vectors to an origin O.

(a) $\overrightarrow{OA} = \begin{pmatrix}2\\4\\-2\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}1\\4\\1\end{pmatrix}$

(b) $\overrightarrow{OA} = \begin{pmatrix}2\\2\\5\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}4\\2\\2\end{pmatrix}$

(c) $\overrightarrow{OA} = \begin{pmatrix}6\\7\\3\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}-1\\0\\-4\end{pmatrix}$

4 A line passes through the point $\begin{pmatrix}2\\1\\3\end{pmatrix}$ and is parallel to $\begin{pmatrix}-1\\2\\1\end{pmatrix}$. Find the equation of l in vector form, parametric form and Cartesian form.

5 The Cartesian equation of a line is $\frac{x-2}{4} = \frac{y-3}{5} = \frac{2-z}{3}$. Write the equation in vector form and parametric form.

6 Given the equation of the line $x = 2 + \lambda$, $y = 3 + 4\lambda$, $z = 2 + 2\lambda$, $\lambda \in \mathbb{R}$, write the equation in vector form.

7 The line L passes through the point A$(2, -3, 4)$ and is parallel to $4\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$. Find the following.

(a) The vector equation of L

(b) The parametric equations of L

(c) The Cartesian equation of L

8 The equations of the lines l_1, l_2 and l_3 are given as:

l_1: $r = (2i - 5j - k) + s(-i - 2j - k)$

l_2: $x = 4 - 3t$, $y = 1 + 5t$, $z = 2 - t$

l_3: $\dfrac{x+2}{3} = \dfrac{y-1}{4} = \dfrac{z+2}{-1}$

(a) Find a vector that is parallel to each line.

(b) Find a point on each of the lines.

9 Relative to an origin O, points C and D have position vectors $\begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \\ q \end{pmatrix}$ respectively, where p and q are constants.

(a) The line through C and D has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 4 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\lambda \in \mathbb{R}$. Find the values of p and q.

(b) Find the position vector of the point A on the line CD such that $\overrightarrow{OA}$ is perpendicular to $\overrightarrow{CD}$.

10 The equation of a straight line l is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

(a) Find the position vector of P on l such that $\overrightarrow{OP}$ is perpendicular to l and O is the origin of position vectors.

(b) A point Q is on l such that the length of OQ is 4 units. Find the exact possible values of λ for which the length of OQ is 4 units.

11 Find the point of intersection of the lines $\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -2 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -9 \\ 36 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$, $s, t \in \mathbb{R}$.

12 With respect to an origin O, points A and B have position vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$. Find the equation of the plane passing through A and perpendicular to B. Write the equation in both vector form and Cartesian form.

13 Relative to an origin O, the position vectors of A and B are $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$ respectively. Find the equation of the plane passing through B and perpendicular to A. Find also the distance from the origin to the plane.

14 The points P, Q and R have position vectors $2\mathbf{i} + \mathbf{j} + \mathbf{k}$, $-32\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ respectively, with respect to an origin O.

(a) Find an equation of the line PQ.

(b) Find the equation of a plane passing through R and perpendicular to the line PQ.

15 Find the angle between the lines l_1 and l_2 where the equation of l_1 is $\dfrac{x-1}{2} = \dfrac{y+2}{3} = \dfrac{2z-4}{4}$ and the equation of l_2 is $x = 2 + 3t$, $y = 1 - t$, $z = 3 + 4t$. Are the lines perpendicular?

16 The vector equation of a straight line l is $-\mathbf{i} + \mathbf{j} + t(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $t \in \mathbb{R}$. All position vectors are with respect to the origin O.

(a) The point A(p, q, 3) is on l. Find the values of p and q.

(b) The point B on l is given by $t = 2$. Find the coordinates of B.

(c) Find the coordinates of C on l, such that $\overrightarrow{OC}$ is perpendicular to l.

SUMMARY

Vectors in three dimensions ($\mathbb{R}^3$)

Vectors

Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$$\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

$$\mathbf{u} - \mathbf{v} = \begin{pmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \end{pmatrix}$$

$$\lambda \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{pmatrix}, \ \lambda \in \mathbb{R}$$

$\mathbf{u} = \mathbf{v}$ iff $u_1 = v_1, u_2 = v_2, u_3 = v_3$

$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$

$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

A unit vector is a vector with magnitude 1 unit.

A unit vector in the direction of $\mathbf{u}$ is $\dfrac{\mathbf{u}}{|\mathbf{u}|}$.

$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

$\mathbf{u}$ and $\mathbf{v}$ are perpendicular iff $\mathbf{u} \cdot \mathbf{v} = 0$.

$\mathbf{u}$ and $\mathbf{v}$ are parallel iff one is a scalar multiple of the other.

$$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Lines

Let $\mathbf{a}$ be a point on a line and $\mathbf{m}$ be the direction vector. The equation of the line is

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{m}, \ \lambda \in \mathbb{R}$$

Let $\mathbf{p}$ and $\mathbf{q}$ be two points on a line. The equation of the line is

$$\mathbf{r} = \mathbf{p} + \lambda(\mathbf{q} - \mathbf{p}), \quad \lambda \in \mathbb{R}$$

Vector equation of a line:

$$\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}, \lambda \in \mathbb{R}$$

Cartesian equation of a line:

$$\frac{x-a}{x_0} = \frac{y-b}{y_0} = \frac{z-c}{z_0}$$

Parametric equations of a line:

$$\left.\begin{aligned} x &= a + \lambda x_0 \\ y &= b + \lambda y_0 \\ z &= c + \lambda z_0 \end{aligned}\right\} \lambda \in \mathbb{R}$$

Skew lines are lines that are not parallel and do not intersect.

The angle between two lines is the angle between their direction vectors.

Two lines are perpendicular iff the scalar product of their direction vectors is 0.

Planes

Let $\hat{\mathbf{n}}$ be a unit vector perpendicular to a plane and d be the distance from the origin to the plane. The equation of the plane is:

$$\mathbf{r} \cdot \hat{\mathbf{n}} = d$$

Let $\mathbf{a}$ be a point on a plane and $\mathbf{n}$ a vector perpendicular to the plane. The equation of the plane is:

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Vector equation of a plane:

$$\mathbf{r} \, . \begin{pmatrix} A \\ B \\ C \end{pmatrix} \cdot D$$

Cartesian equation of a plane:

$$Ax + By + Cz = D$$

where $\begin{pmatrix} A \\ B \\ C \end{pmatrix}$ is a vector perpendicular to the plane.

Checklist

Can you do these?

- Add two vectors.
- Subtract two vectors.
- Multiply a vector by a scalar quantity.
- Find the length of a vector.
- Identify a position vector.
- Derive and use displacement vectors.
- Derive and use unit vectors.
- Express a vector in the form $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.
- Find the scalar product of two vectors.
- Find the angle between two vectors.
- Find the equation of a line given a point on the line and a vector parallel to the line.
- Find the equation of a line given two points on the line.
- Find the vector equation of a line.
- Find the parametric equations of a line.
- Find the Cartesian equation of a line.
- Determine whether two lines are parallel, intersect, or skewed.
- Identify a vector normal to a plane.
- Find the equation of a plane given a point on the plane and a vector perpendicular to the plane.
- Write the equation of a plane in Cartesian form and in vector form.

Review Exercise 11

1 With reference to an origin O, the position vectors of A, B and C are $2\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, $-2\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $\mathbf{i} + \mathbf{j} + \mathbf{k}$ respectively. Find angle ABC.

2 The equation on a plane π is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 6$. Find the distance from the origin to the plane.

3 Are these two lines skew?

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R} \qquad \frac{x+4}{1} = \frac{y-2}{3} = \frac{z+1}{-2}$$

4 The Cartesian equation of a line l_1 is $\frac{x-2}{3} = \frac{y-4}{2} = \frac{z-1}{2}$ and the equation of a line l_2 is $\mathbf{r} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$, where $t \in \mathbb{R}$. Are l_1 and l_2 skew lines?

5 Find the angle between the normal to the planes with the equations $x + 2y + z = 4$ and $2x - y - z = 1$.

6 Referred to the origin, the points A and B have position vectors $\overrightarrow{OA} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\overrightarrow{OB} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$. The point C on AB is such that $AC : CB = p : 1 - p$.

Show that $\overrightarrow{OC} = (2 - p)\mathbf{i} + (1 - 3p)\mathbf{j} + (-1 + 4p)\mathbf{k}$.

Hence, find the value of p for which $\overrightarrow{OC} = \overrightarrow{AB}$.

Find also the values of p for which angles AOB and COB are equal.

7 Decide whether the following lines intersect.

$$\mathbf{r} = \begin{pmatrix} 4 \\ 9 \\ 12 \end{pmatrix} + t\begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -3 \\ -15 \\ -19 \end{pmatrix} + s\begin{pmatrix} 2 \\ 8 \\ 8 \end{pmatrix},\ t, s \in \mathbb{R}$$

8 The equation of a line l is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and the point A with respect to an origin O has position vector $3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$. The point N is on the line l and is such that $\overrightarrow{AN}$ is perpendicular to l. Find the position vector of N and the magnitude of $\overrightarrow{AN}$.

9 Are the lines l_1 and l_2 skew?

$$l_1: \frac{x-1}{2} = \frac{y+1}{1} = \frac{z}{-2}$$

$$l_2: \left.\begin{aligned} x &= 1 - 3\mu \\ y &= 2 \\ z &= 2 + 4\mu \end{aligned}\right\}\ \mu \in \mathbb{R}$$

10 Referred to an origin O, the point A is (1, 3, 1), B is (−2, 1, 1) and C is a variable point such that $\overrightarrow{OC} = \overrightarrow{OA} + \lambda\overrightarrow{AB}$. Find the following.

(a) $\overrightarrow{OA} \cdot \overrightarrow{OB}$

(b) Angles AOB

(c) The values of λ for which $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$

(d) The value of λ for which $|\overrightarrow{OC}| = |\overrightarrow{AC}|$

11 With respect to an origin O, the points A, B and C have position vectors $5\mathbf{i} - \mathbf{j} - 3\mathbf{k}$, $-4\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $5\mathbf{i} - 2\mathbf{j} + 11\mathbf{k}$ respectively. Find the following.

(a) The angle between the vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$

(b) A vector equation for the line BC

(c) The equation of a plane π passing through $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and normal to $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

(d) The angle between the normal to π and the direction of the line BC

Hence find the angle between the line BC and the plane π.

12 Decide whether the following lines are skew.

$$\mathbf{r} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix},\ t, s \in \mathbb{R}$$

13 The points A and B have position vectors $3\mathbf{i} + 4\mathbf{j} - \mathbf{k}$ and $5\mathbf{i} + 7\mathbf{j} + 6\mathbf{k}$ respectively.

(a) Find the equation of the plane passing through A and perpendicular to AB.

(b) The point C has position vector $6\mathbf{i} + 23\mathbf{j} + 8\mathbf{k}$. Find the parametric equations of the line AC.

(c) Find the position vector of the point D where the line AC and the plane in **(a)** meet.

14 The equation of a plane π is $2x - 3y + z = 6$ and the equation of a line l is

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix},\ t \in \mathbb{R}.$$

(a) Write the equation of l in parametric form.

(b) Find the value of t for which the line l and the plane π meet. Hence, find the point of intersection of the line and the plane.

(c) Find the acute angle between the line l and the normal to the plane π.

15 With respect to an origin O, the point A has position vector $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. The line l has equation $\mathbf{r} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix},\ s \in \mathbb{R}$. The point B on l is such that AB is perpendicular to l.

(a) Find the position vector of B.

(b) Find the length of AB.

Module 2 Tests

Module 2 Test 1

1 (a) Find the general solution of the equation $\sin 2x + \sin 3x + \sin 4x = 0$. [8]

(b) Given that $A + B + C = 180°$, prove that $\sin A + \sin B + \sin C = 4\cos\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}$. [8]

(c) (i) Express $2\sin\theta + \cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

(ii) Find the general solution of the equation $2\sin\theta + \cos\theta = 2$. [4]

(iii) Write down the maximum value of $\dfrac{1}{4 + 2\sin\theta + \cos\theta}$. [2]

2 (a) Find the equation of the circle which passes through the points P(2, 3), Q(4, −1) and R(3, −1). [9]

(b) Show that the Cartesian equation of the curve represented by parametric equations $x = 1 + 4\cos\theta$, $y = 4\sin\theta - 2$ is a circle with centre $(1, -2)$ and radius 4 units. [8]

(c) The parametric equations of a curve are $x = 2 + \cos t$ and $y = 3 + 2\sin t$. Find the equation of the curve and describe the curve. [8]

3 (a) With respect to the origin O, the position vectors of A, B and C are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\overrightarrow{OB} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\overrightarrow{OC} = p\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, where p is a constant.

(i) Find angle AOB. [4]

(ii) Find the value of p for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$. [4]

(b) The straight line $y + x = 11$ cuts the circle $x^2 + y^2 - 8y = 9$ at the points A and B.

(i) Calculate the mid-point of AB.

(ii) Calculate the equation of the perpendicular bisector of AB. [11]

(c) The equation of an ellipse is $25x^2 + 9y^2 = 225$. Find the centre of the ellipse and the length of the major axis. Write the equation in parametric form.

4 (a) The equation of a line l is given by $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$.

Write this equation in parametric form. [4]

(b) The equation of a plane is $\mathbf{r} \cdot \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 8$. Write this equation in Cartesian form. [3]

(c) Find the value of t for which the line in (a) meets the plane in (b). (Solve the two equations simultaneously.) [4]

(d) Hence, find the point of intersection of the line in (a) and the plane in (b). [2]

5 (a) Find the general solution of the equation $\cos x + \cos 3x + \cos 5x + \cos 7x = 0$. [9]

(b) Find the Cartesian equation of the curve $x = 1 + \tan t$, $y = 2 + \cos t$. [5]

Module 2 Test 2

1 (a) The circle C has equation $x^2 + y^2 - 4x - 9y - 12 = 0$.

(i) Find the radius and the coordinates of the centre of the circle. [3]

(ii) Find the equation of the tangent at the point (6, 0) on C. [4]

(iii) Calculate the exact value of the x-coordinates of the points of intersection of C with the straight line $y = x + 8$. [7]

(b) Are the lines with equations l_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + t\begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$ and l_2: $x = 2 + 3s$, $y = 1 - s$, $z = s$, $t, s \in \mathbb{R}$ skew?

2 (a) Find the value of $\tan 22.5°$ in surd form without using a calculator or tables. [8]

(b) Given that A and B are acute angles such that $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, find, without the use of a calculator, the exact values of

(i) $\sin(A + B)$ [3]

(ii) $\cos(A - B)$ [3]

(c) Prove that $\dfrac{\sin\theta - \sin 2\theta + \sin 3\theta}{\cos\theta - \cos 2\theta + \cos 3\theta} = \tan 2\theta$. [5]

3 (a) Find the Cartesian equation of the curve $x = 4 + 2\tan t$, $y = \sec t + 3$. [6]

(b) Two planes π_1 and π_2 have equations $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 4$ and $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 6$. Write the equations in Cartesian form. Find the angles between the normal to the planes.

4 (a) (i) Find the equation of the normal to the parabola $y^2 = 16x$ at the point $(t^2, 4t)$.

(ii) At the points P and Q on the parabola, $t = 3$ and $t = \frac{1}{3}$. Find the intersection of the normals at P and Q.

(b) (i) Prove that $\cos\theta + 2\cos 3\theta + \cos 5\theta = 4\cos^2\theta\cos 3\theta$. [6]

(ii) Hence, find the general solution of the equation $\cos\theta + 2\cos 3\theta + \cos 5\theta = 0$. [4]

(c) Given that the circle $x^2 + y^2 - 7x + 2y + a = 0$ passes through the point (7, 1), find the value of a. Show that the equation of the diameter of this circle which passes through (7, 1) is $7y = 4x - 21$. Find the coordinates of the other end of the diameter. [9]

3

Calculus I

CHAPTER 12

Limits and Continuity

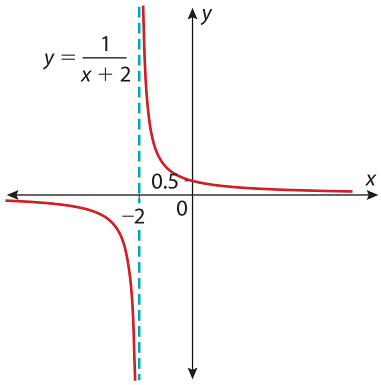
At the end of this chapter you should be able to:

- Use graphs to determine the continuity or discontinuity of a function
- Describe the behaviour of a function as x gets close to some given fixed number
- Use the limit notation $\lim_{x \to a} f(x) = L$, $f(x) \to L$ as $x \to a$
- Use simple limit theorems
- Use limit theorems for specific cases
- Show and use $\lim_{x \to 0} \frac{\sin x}{x} = 1$, by a geometric approach
- Solve problems involving limits
- Identify the region over which a function is continuous
- Identify the points where a function is discontinuous and describe its discontinuity
- Use the concept of left-handed and right-handed limits
- Use the concept of continuity on a closed interval

KEYWORDS/TERMS

limits • continuity • discontinuity • left hand limits • right hand limits • jump discontinuity • point discontinuity • infinite discontinuity • removable discontinuity • non-removable discontinuity • existence

Limits



Limits are important in the development of calculus and in other branches of mathematical analysis such as series. The gradient function is defined in terms of limits and will be used in a later chapter to find the gradients of different functions. Let us look at what we mean by a limit.

Let $f(x) = \frac{1}{x+2}$.

Let us see what happens to $f(x)$ as x approaches 2.
Look at the tables and the graph.

x	$f(x) = \frac{1}{x+2}$
1.9	0.25641026
1.99	0.25062657
1.999	0.25006251
1.9999	0.25000625
1.99999	0.25000063
1.999999	0.25000006

x	$f(x) = \frac{1}{x+2}$
2.1	0.24390244
2.01	0.24937656
2.001	0.24993752
2.0001	0.24999375
2.00001	0.24999938
2.000001	0.24999994

Notice that as x gets closer and closer to the value 2, $f(x)$ gets closer to $\frac{1}{4}$. This value is the limit of $f(x)$ as x approaches 2.

A **limit** of a function $f(x)$ is the value the function approaches at a given x-value. The function may not actually reach the value for the given x. For $f(x) = \frac{1}{x+2}$, we say that 'the limit, as x approaches 2, of $f(x)$ equals $\frac{1}{4}$, and we write $\lim\limits_{x \to 2} \frac{1}{x+2} = \frac{1}{4}$.

Let us look at what happens to the function $f(x) = \frac{x^2 + 3x + 2}{x+1}$ as x approaches -1.

x	$\frac{x^2 + 3x + 2}{x+1}$	$x + 2$
-4	$\frac{(-4)^2 + 3(-4) + 2}{-4+1} = -2$	$-4 + 2 = -2$
-3	$\frac{(-3)^2 + 3(-3) + 2}{-3+1} = -1$	$-3 + 2 = -1$
-2	$\frac{(-2)^2 + 3(-2) + 2}{-2+1} = 0$	$-2 + 2 = 0$
-1	$\frac{(-1)^2 + 3(-1) + 2}{-1+1} = \frac{0}{0}$	$-1 + 2 = 1$
0	$\frac{(0)^2 + 3(0) + 2}{0+1} = 2$	$0 + 2 = 2$
1	$\frac{(1)^2 + 3(1) + 2}{1+1} = 3$	$1 + 2 = 3$
2	$\frac{(2)^2 + 3(2) + 2}{2+1} = 4$	$2 + 2 = 4$
3	$\frac{(3)^2 + 3(3) + 2}{3+1} = 5$	$3 + 2 = 5$
4	$\frac{(4)^2 + 3(4) + 2}{4+1} = 6$	$4 + 2 = 6$

When we substitute $x = -1$ directly into $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$, we get $f(-1) = \dfrac{(-1)^2 + 3(-1) + 2}{(-1) + 1} = \dfrac{0}{0}$. Let us investigate this a little further.

Note that for the two functions $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$ and $f(x) = x + 2$ all the values correspond except at $x = -1$. At $x = -1$, the function $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$ is undefined and for the function $f(x) = x + 2$ the value of the function is 1. When drawing the graph of $y = \dfrac{x^2 + 3x + 2}{x + 1}$, we can draw $y = x + 2$ instead and leave an unshaded circle at the point $(-1, 1)$ indicating that this point is not part of the graph. We say that there is a 'hole' in the graph at the point $(-1, 1)$. Does this function have a limit at $x = -1$ and if the function does have a limit, how can we find it?

From the table below, we see that as $x \to -1$, $f(x)$ aproaches 1. The limit of $f(x)$ as x approaches -1 is 1. As indicated above, the graph of $f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$ is equivalent to $f(x) = x + 2$ with the point $(-1, 1)$ excluded from this graph. Since $\lim\limits_{x \to -1} f(x) = 1$, the limit exists but the function will not actually reach this value.

Think of someone driving towards a wall to park their car. Assuming that they do not crash, they will drive as close as possible, without touching the wall. They have reached their limit.

Let us investigate this by completing the table below.

x	$f(x) = \dfrac{x^2 + 3x + 2}{x + 1}$
-0.9	1.1
-0.99	1.01
-0.999	1.001
-0.9999	1.0001
-0.99999	1.00001
-0.999999	1.000001

x	$f\left(x = \dfrac{x^2 + 3x + 2}{x + 1}\right)$
-1.1	0.9
-1.01	0.99
-1.001	0.999
-1.0001	0.9999
-1.00001	0.99999
-1.000001	0.999999

As $x \to -1$, $f(x)$ approaches 1. The limit of $f(x)$ as x approaches -1 is 1. We can graph this function in the following way:

$$f(x) = \frac{x^2 + 3x + 2}{x + 1} = \frac{(x + 1)(x + 2)}{(x + 1)} = x + 2$$

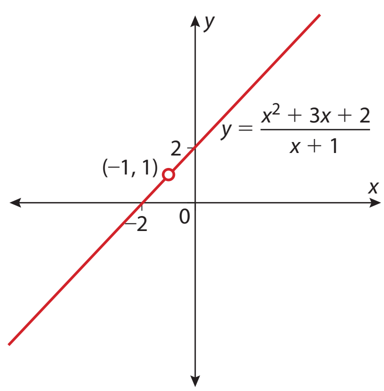
The graph has a hole at $(-1, 1)$ even though $\lim\limits_{x \to -1} f(x) = 1$. In this case, the limit exists but the function will not actually reach this limit.

Note

An indeterminate form of a limit is a form in which we do not have enough information to evaluate its limit. $\frac{0}{0}$ is considered indeterminate since it can be 1, ∞ or 0.

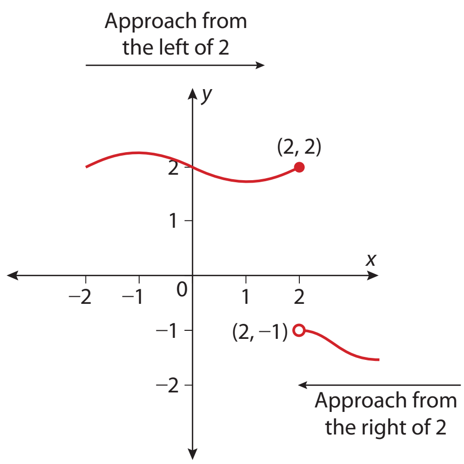
The following table gives some determinate and indeterminate forms.

Indeterminate	Determinate
$\frac{0}{0}$	$\infty + \infty \to \infty$
$\frac{\infty}{\infty}$	$-\infty - \infty \to -\infty$
$\infty \times -\infty$	$(0^+)^{\infty} \to 0$
$0 \times \infty$	$(0^+)^{-\infty} \to 0$
0^0	$\frac{0}{\infty} \to 0$
∞^0	
1^{∞}	



Left-hand limits and right-hand limits

Some functions will approach to different values of y at a given value of x: one y-value as we come from the left and a different y-value as we come from the right. These y-values are called left-hand limits and right-hand limits. Let us look at how this works.



As x approaches 2 from the left, the function approaches the y-value of 2 and as x approaches 2 from the right the function approaches the y-value of -1. Therefore, we write $\lim\limits_{x \to 2^-} f(x) = 2$ and $\lim\limits_{x \to 2^+} f(x) = -1$.

The notation used for left-hand limits is $\lim\limits_{x \to a^-} f(x)$. This is read as 'the limit of $f(x)$ as x approaches a from the left' and $\lim\limits_{x \to a^+} f(x)$ is used for the right hand limit and is read as 'the limit of $f(x)$ as x approaches a from the right'.

The existence of a limit

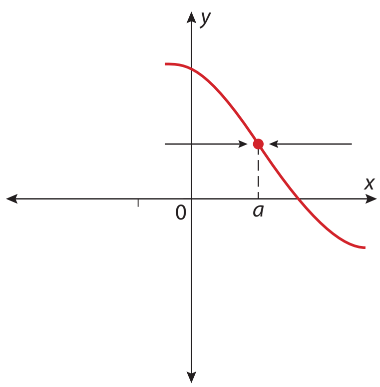
For a limit to exist for a function $f(x)$ at some value $x = a$, then the following must be true.

(i) The left-hand limit must exist at $x = a$.

(ii) The right-hand limit must exist at $x = a$.

(iii) The left-hand limit and the right-hand limit at $x = a$ must be equal.

This means that if $\lim\limits_{x \to a} f(x)$ exists, then both the left-hand limit and the right-hand limit must exist and $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} f(x)$.



There are situations where a limit does not exist.

(i) A limit of a function $f(x)$ does not exist if the left-hand limit and right-hand limits of $f(x)$ are not equal.

(ii) A limit of a function $f(x)$ does not exist if $f(x)$ oscillates infinitely.

(iii) A limit of a function $f(x)$ does not exist if $f(x)$ increases or decreases infinitely.

Limit laws

Suppose that the limits $\lim\limits_{x \to b} f(x)$ and $\lim\limits_{x \to b} g(x)$ exist.

Law 1

$\lim\limits_{x \to b} c = c$, where c is a constant.

For example: $\lim\limits_{x \to b} 5 = 5$

Law 2

$\lim_{x \to b} x = b$

For example: $\lim_{x \to 2} x = 2$

Law 3

$\lim_{x \to b} x^n = b^n$, where b is a positive integer and in the domain for x.

For example: $\lim_{x \to 3} x^2 = 3^2$
$= 9$

Law 4

$\lim_{x \to b} (f(x) + g(x)) = \lim_{x \to b} f(x) + \lim_{x \to b} g(x)$

This means that the limit of a sum is equal to the sum of the limits.

For example: $\lim_{x \to b} (x^2 + x - 3) = \lim_{x \to b} x^2 + \lim_{x \to b} x - \lim_{x \to b} 3$
$= b^2 + b - 3$

Law 5

$\lim_{x \to b} (cf(x)) = c \lim_{x \to b} f(x)$, where c is a constant.

For example: $\lim_{x \to b} 4x^2 = 4 \lim_{x \to b} x^2 = 4b^2$

Law 6

$\lim_{x \to b} f(x)g(x) = \lim_{x \to b} f(x) \times \lim_{x \to b} g(x)$

This means that the limit of a product of two functions is equal to the product of the limits of the functions.

For example: $\lim_{x \to b} (x + 2)(2x + 3)^2 = \lim_{x \to b} (x + 2) \times \lim_{x \to b} (2x + 3)^2$
$= (b + 2)(2b + 3)^2$

Law 7

$\lim_{x \to b} \frac{f(x)}{g(x)} = \frac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)}$, provided that $\lim_{x \to b} g(x) \neq 0$.

For example: $\lim_{x \to b} \frac{3x^2 + 2}{x^3 + 1} = \frac{\lim_{x \to b} 3x^2 + 2}{\lim_{x \to b} x^3 + 1}$
$= \frac{3b^2 + 2}{b^3 + 1}$

Law 8

$\lim_{x \to b} (f(x))^n = \left(\lim_{x \to b} f(x)\right)^n$

For example: $\lim_{x \to b} (4x + 3)^5 = \left(\lim_{x \to b} (4x + 3)\right)^5$
$= (4b + 3)^5$

Law 9

$\lim_{x \to b} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to b} f(x)}$

EXAMPLE 1 Using the limit laws, evaluate the following limits.

(a) $\lim_{x \to 2} (3x^2 + 4x - 2)$

(b) $\lim_{x \to 2} (x + 2)(3x - 2)^2$

(c) $\lim_{x \to 3} \left(\dfrac{x^2 + 1}{3x - 4}\right)$

SOLUTION

(a)
$$\begin{aligned}\lim_{x \to 2} (3x^2 + 4x - 2) &= 3\lim_{x \to 2} x^2 + 4\lim_{x \to 2} x - \lim_{x \to 2} 2\\ &= 3(2)^2 + 4(2) - 2 \qquad \text{(Substitute } x = 2\text{)}\\ &= 12 + 8 - 2\\ &= 18\end{aligned}$$

(b)
$$\begin{aligned}\lim_{x \to 2} (x + 2)(3x - 2)^2 &= \lim_{x \to 2} (x + 2) \times \lim_{x \to 2} (3x - 2)^2\\ &= (2 + 2) \times (3(2) - 2)^2 \qquad \text{(Substitute } x = 2\text{)}\\ &= 4 \times 16\\ &= 64\end{aligned}$$

(c)
$$\begin{aligned}\lim_{x \to 3} \left(\frac{x^2 + 1}{3x - 4}\right) &= \frac{\lim_{x \to 3} (x^2 + 1)}{\lim_{x \to 3} (3x - 4)}\\ &= \frac{(3^2 + 1)}{(3(3) - 4)} \qquad \text{(Substitute } x = 3\text{)}\\ &= \frac{10}{5}\\ &= 2\end{aligned}$$

Evaluating limits

Most limits can be found by using one of three methods: direct substitution (as we did in Example 1), factorising or the conjugate method. When finding limits you can try one method at a time until one works (of course you can try direct substitution first). Let us look at the different methods of finding limits, starting with the easiest, which is direct substitution.

> **Note**
>
> If you substitute into a function like $\lim_{x \to 2} \frac{4}{x - 2}$ and you get $\frac{4}{0}$ (any number other than zero divided by zero), then the limit does not exist.

Direct substitution

To find $\lim_{x \to a} f(x)$ by direct substitution, we substitute $x = a$ into $f(x)$ and we arrive at the value where $\lim_{x \to a} f(x) = f(a)$.

Direct substitution is the first method one should attempt when evaluating a limit. This method works for functions that are continuous everywhere or continuous over its domain. For these functions, the limit is also equal to the function value at that particular point. Direct substitution works for any type of functions, including piecewise functions, unless the function is not continuous at the point.

If this method does not work, one of the other methods described below should be used to evaluate the limit.

EXAMPLE 2 Find $\lim_{x \to 1}(x^2 + 4x + 1)$.

SOLUTION We can find this limit by substituting $x = 1$ into the function $x^2 + 4x + 1$.

Hence, $\lim_{x \to 1}(x^2 + 4x + 1) = 1^2 + 4(1) + 1$
$= 6$

EXAMPLE 3 Find $\lim_{x \to 0} \dfrac{x^2 - 3x}{x^3 + 4x + 5}$.

SOLUTION

$$\lim_{x \to 0} \frac{x^2 - 3x}{x^3 + 4x + 5} = \frac{\lim_{x \to 0}(x^2 - 3x)}{\lim_{x \to 0}(x^3 + 4x + 5)}$$

$$= \frac{0^2 - 3(0)}{0^3 + 4(0) + 5} \quad \text{(Substitute } x = 0\text{)}$$

$$= \frac{0}{5}$$

$$= 0$$

EXAMPLE 4 Find $\lim_{t \to -1}(3t^2 + 1)^4(t + 2)^3$.

SOLUTION

$$\lim_{t \to -1}(3t^2 + 1)^4(t + 2)^3 = (3(-1)^2 + 1)^4 \times (-1 + 2)^3 \quad \text{(Substitute } t = -1\text{)}$$

$$= (4^4)(1)^3$$

$$= 256$$

EXAMPLE 5 Evaluate $\lim_{t \to 2}\sqrt{t^3 + 3t^2 + 5}$.

SOLUTION

$$\lim_{t \to 2}\sqrt{t^3 + 3t^2 + 5} = \sqrt{\lim_{t \to 2}(t^3 + 3t^2 + 5)}$$

$$= \sqrt{2^3 + 3(2)^2 + 5} \quad \text{(Substitute } t = 2\text{)}$$

$$= \sqrt{8 + 12 + 5}$$

$$= \sqrt{25}$$

$$= 5$$

Try these 12.1 In questions **a** to **j**, evaluate the following limits.

(a) $\lim_{x \to 2}(4x^2 + 6x + 1)$ **(b)** $\lim_{x \to 1}\left(\dfrac{x^2 + 1}{x - 4}\right)$

(c) $\lim_{t \to 0}\sqrt{\dfrac{t + 1}{t + 9}}$ **(d)** $\lim_{t \to 4}(3t + 1)^3$

(e) $\lim_{t \to 4}(t + 1)^2(4t - 2)$ **(f)** $\lim_{x \to 0}(3x - 4)(2x^2 + 7)$

(g) $\lim_{x \to -2}(4x^3 - 3x^2 + 5)$ **(h)** $\lim_{x \to -1}\dfrac{x + 1}{2x + 3}$

(i) $\lim_{x \to 0}\sqrt{\dfrac{x^2 + 2x + 1}{x + 2}}$ **(j)** $\lim_{x \to 2}\sqrt{\dfrac{(x + 1)^3}{(4x - 2)}}$

Factorising method

Let us look at $\lim_{x \to 2} f(x)$ where $f(x) = \frac{x^2 - 4}{x - 2}$.

Using direct substitution gives:

$$f(2) = \frac{2^2 - 4}{2 - 2} = \frac{4 - 4}{2 - 2} = \frac{0}{0}$$

This value is considered indeterminate. To evaluate this limit we need to look at another method and in this case factorising will work. After factorising, we cancel and then substitute directly to evaluate the limit.

$$\text{Now, } f(x) = \frac{x^2 - 4}{x - 2} = \frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}}$$
$$= x + 2$$

$$\text{Therefore, } \lim_{x \to 2} f(x) = \lim_{x \to 2} (x + 2)$$
$$= (2 + 2)$$
$$= 4$$

$$\text{Therefore, } \lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2}\right) = 4$$

EXAMPLE 6 Find $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2}$.

SOLUTION

$$\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2} = \lim_{x \to -2} \frac{(x + 2)(x + 3)}{x + 2} \quad \text{(Factorise the numerator and cancel)}$$
$$= \lim_{x \to -2} (x + 3)$$
$$= -2 + 3 \quad \text{(Substitute } x = -2\text{)}$$
$$= 1$$

EXAMPLE 7 Evaluate $\lim_{x \to -2} \frac{x^3 + 8}{x + 2}$.

SOLUTION

Factorise $x^3 + 8$, recalling $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$:

$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$

$$\text{Therefore, } \lim_{x \to -2} \frac{x^3 + 8}{x + 2} = \lim_{x \to -2} \frac{\cancel{(x + 2)}(x^2 - 2x + 4)}{\cancel{x + 2}}$$
$$= \lim_{x \to -2} (x^2 - 2x + 4)$$
$$= (-2)^2 - 2(-2) + 4 \quad \text{(Substitute } x = -2\text{)}$$
$$= 4 + 4 + 4$$
$$= 12$$

Try these 12.2 Find the limits of

(a) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(b) $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

(c) $\lim_{x \to 0} \frac{x^2 + x}{x}$

Conjugate method

This method for evaluating limits is useful when the limits contain radicals. We learned to conjugate radicals in Module 1 (a revision of radicals may be useful at this stage).

EXAMPLE 8 Evaluate $\lim\limits_{x \to 25} \dfrac{x - 25}{\sqrt{x} - 5}$.

SOLUTION Multiply the numerator and denominator by $\sqrt{x} + 5$:

$$\lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5} = \lim_{x \to 25} \frac{x - 25}{\sqrt{x} - 5} \times \frac{\sqrt{x} + 5}{\sqrt{x} + 5}$$

$$= \lim_{x \to 25} \frac{\cancel{(x - 25)}(\sqrt{x} + 5)}{\cancel{x - 25}}$$

$$= \lim_{x \to 25} (\sqrt{x} + 5) = \sqrt{25} + 5$$

$$= 5 + 5$$

$$= 10$$

EXAMPLE 9 Evaluate $\lim\limits_{x \to 1} \dfrac{\sqrt{x + 8} - 3}{x - 1}$.

SOLUTION Multiplying numerator and denominator by $\sqrt{x + 8} + 3$, we get:

$$\lim_{x \to 1} \frac{\sqrt{x + 8} - 3}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x + 8} - 3}{x - 1} \times \frac{\sqrt{x + 8} + 3}{\sqrt{x + 8} + 3}$$

$$= \lim_{x \to 1} \frac{(x + 8) - 9}{(x - 1)(\sqrt{x + 8} + 3)}$$

$$= \lim_{x \to 1} \frac{\cancel{x - 1}}{\cancel{(x - 1)}(\sqrt{x + 8} + 3)}$$

$$= \lim_{x \to 1} \left(\frac{1}{\sqrt{x + 8} + 3}\right)$$

$$= \frac{1}{\sqrt{9} + 3} \qquad \text{(Substitute } x = 1\text{)}$$

$$= \frac{1}{3 + 3} = \frac{1}{6}$$

Note

In this case, do not expand the denominator. The numerator
$= (\sqrt{x + 8} - 3)(\sqrt{x + 8} + 3)$
$= (\sqrt{x + 8})^2 - 3^2$
$= (x + 8) - 9$

EXAMPLE 10 Evaluate $\lim\limits_{x \to 6} \dfrac{x - 6}{\sqrt{x - 6}}$.

SOLUTION Multiplying the numerator and denominator by $\sqrt{x - 6}$, we get:

$$\lim_{x \to 6} \frac{x - 6}{\sqrt{x - 6}} = \lim_{x \to 6} \frac{x - 6}{\sqrt{x - 6}} \times \frac{\sqrt{x - 6}}{\sqrt{x - 6}}$$

$$= \lim_{x \to 6} \frac{(x - 6)\sqrt{x - 6}}{x - 6}$$

$$= \lim_{x \to 6} \sqrt{x - 6}$$

$$= \sqrt{6 - 6}$$

$$= \sqrt{0}$$

$$= 0$$

EXAMPLE 11 Find the value of $\lim\limits_{x \to 14} \dfrac{x - 14}{4 - \sqrt{x + 2}}$.

SOLUTION

$$\lim_{x \to 14} \frac{x - 14}{4 - \sqrt{x + 2}} = \lim_{x \to 14} \frac{x - 14}{4 - \sqrt{x + 2}} \times \frac{4 + \sqrt{x + 2}}{4 + \sqrt{x + 2}}$$

$$= \lim_{x \to 14} \frac{(x - 14)(4 + \sqrt{x + 2})}{16 - (x + 2)}$$

$$= \lim_{x \to 14} \frac{(x - 14)(4 + \sqrt{x + 2})}{14 - x}$$

$$= \lim_{x \to 14} \frac{(x - 14)(4 + \sqrt{x + 2})}{14 - x}$$

$$= \lim_{x \to 14} -(4 + \sqrt{x + 2}) = -(4 + \sqrt{14 + 2})$$

$$= -(4 + 4) = -8$$

Note

$$\frac{x-14}{14-x} = \frac{x-14}{-(x-14)}$$
$$= -1$$

Try these 12.3 Evaluate the following limits.

(a) $\lim\limits_{x \to 25} \dfrac{5 - \sqrt{x}}{25 - x}$

(b) $\lim\limits_{x \to 19} \dfrac{x - 19}{5 - \sqrt{x + 6}}$

(c) $\lim\limits_{x \to 5} \dfrac{x - 5}{\sqrt{x - 5}}$

Tending to infinity

A limit does not exist if a function increases or decreases infinitely. When we substitute our x-value into $f(x)$ and we get $\frac{0}{0}$, there may be a hole in the graph. If we get a fraction of the form $\frac{a}{0}$ where $a \neq 0$, this indicates that there is a vertical asymptote and this indicates that the function is increasing or decreasing without bound.

EXAMPLE 12 For what value(s) of x does no limit exist for $f(x) = \dfrac{x^2 + 4x + 3}{2x^2 + 3x + 1}$?

SOLUTION Factorising the expression we have:

$$f(x) = \frac{(x + 1)(x + 3)}{(x + 1)(2x + 1)}$$

$$f(x) = \frac{(x + 3)}{(2x + 1)}$$

When $x = -1$, $f(-1) = \dfrac{-1+3}{2(-1) + 1} = \dfrac{2}{-1} = -2$

Therefore, the function has a hole at $x = -1$ and the limit is 1.

When $x = -\frac{1}{2}$, $f\left(-\frac{1}{2}\right) = \dfrac{-\frac{1}{2} + 3}{2\left(-\frac{1}{2}\right) + 1} = \dfrac{2\frac{1}{2}}{0}$

$\lim\limits_{x \to -\frac{1}{2}} f(x)$ does not exist since $f(x)$ will either increase or decrease infinitely.

Therefore, the limit does not exist when $x = -\frac{1}{2}$.

EXAMPLE 13 Determine the x-values at which the function $f(x) = \dfrac{x^2 + x + 1}{x^2 + 5x + 6}$ is undefined. Decide whether the limit of $f(x)$ exists at these values of x.

SOLUTION Factorising the denominator gives:

$$f(x) = \frac{x^2 + x + 1}{(x + 2)(x + 3)}$$

When the denominator is zero:

$(x + 2)(x + 3) = 0$

$\Rightarrow x = -2, -3$

$\therefore f(x)$ is undefined when $x = -2, x = -3$.

Let us find out if the limit exists when $x = -2$:

$$\begin{aligned}\lim_{x \to -2} f(x) &= \lim_{x \to -2} \frac{x^2 + x + 1}{(x + 2)(x + 3)} \\ &= \frac{(-2)^2 + (-2) + 1}{(-2 + 2)(-2 + 3)} \qquad \text{(Substitute } x = -2\text{)} \\ &= \frac{4 - 2 + 1}{0} \\ &= \frac{3}{0}\end{aligned}$$

Therefore, the limit does not exist.

Let us find out if the limit exists when $x = -3$:

$$\begin{aligned}\lim_{x \to -2} f(x) &= \lim_{x \to -2} \frac{x^2 + x + 1}{(x + 2)(x + 3)} \\ &= \frac{(-3)^2 + (-3) + 1}{(-3 + 2)(-3 + 3)} \qquad \text{(Substitute } x = -3\text{)} \\ &= \frac{9 - 3 + 1}{0} \\ &= \frac{7}{0}\end{aligned}$$

Therefore, the limit does not exist.

Hence, $f(x)$ is undefined at $x = -2, x = -3$ and the limit does not exist at these two values.

Limits at infinity

When finding $\lim\limits_{x \to \infty} f(x)$, where $f(x)$ is a rational function, we can divide the numerator and denominator by the highest term in x and evaluate the limits.

EXAMPLE 14 Find $\lim\limits_{x \to \infty} \dfrac{2x^2 + x + 1}{x^2 + 2x + 3}$.

SOLUTION Since we are dealing with infinite limits, we can divide the numerator and denominator by x^2 (highest power).

$$\lim_{x \to \infty} \frac{2x^2 + x + 1}{x^2 + 2x + 3} = \lim_{x \to \infty} \frac{\frac{2x^2}{x^2} + \frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} + \frac{2x}{x^2} + \frac{3}{x^2}}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x} + \frac{1}{x^2}}{1 + \frac{2}{x} + \frac{3}{x^2}}$$

$$= \frac{2 + 0 + 0}{1 + 0 + 0} \qquad \left(\text{As } x \to \infty, \frac{1}{x}, \frac{1}{x^2}, \frac{2}{x} \text{ and } \frac{3}{x^2} \text{ all tend to } 0.\right)$$

$$= 2$$

Hence, $\lim\limits_{x \to \infty} \dfrac{2x^2 + x + 1}{x^2 + 2x + 3} = 2.$

EXAMPLE 15 Evaluate $\lim\limits_{x \to \infty} \dfrac{x^2 + 6}{3x^2 - 4x + 1}$.

SOLUTION Since we are dealing with infinite limits, we can divide the numerator and denominator by the term with the highest power of x, which is x^2.

$$\lim_{x \to \infty} \frac{x^2 + 6}{3x^2 - 4x + 1} = \lim_{x \to \infty} \frac{\frac{x^2}{x^2} + \frac{6}{x^2}}{\frac{3x^2}{x^2} - \frac{4x}{x^2} + \frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 + \frac{6}{x^2}}{3 - \frac{4}{x} + \frac{1}{x^2}}$$

$$= \frac{1 + 0}{3 - 0 + 0} \qquad \left(\text{Since } \frac{6}{x^2}, \frac{4}{x}, \frac{1}{x^2} \to 0 \textit{ as } x \to \infty.\right)$$

$$= \frac{1}{3}$$

EXAMPLE 16 Evaluate $\lim\limits_{x \to \infty} \dfrac{3x^2 + x + 3}{x^3 + 8x + 9}$.

SOLUTION Since we are dealing with infinite limits, we can divide the numerator and denominator by the term with the highest power of x, which is x^3.

$$\lim_{x \to \infty} \frac{3x^2 + x + 3}{x^3 + 8x + 9} = \lim_{x \to \infty} \frac{\frac{3x^2}{x^3} + \frac{x}{x^3} + \frac{3}{x^3}}{\frac{x^3}{x^3} + \frac{8x}{x^3} + \frac{9}{x^3}}$$

$$= \lim_{x \to \infty} \frac{\frac{3}{x} + \frac{1}{x^2} + \frac{3}{x^3}}{1 + \frac{8}{x^2} + \frac{9}{x^3}}$$

$$= \frac{0 + 0 + 0}{1 + 0 + 0} \qquad \left(\text{Since } \frac{3}{x}, \frac{1}{x^2}, \frac{3}{x^2}, \frac{8}{x^2} \text{ and } \frac{9}{x^3} \to 0 \text{ as } x \to \infty.\right)$$

$$= \frac{0}{1}$$

$$= 0$$

Try these 12.4

Evaluate the following limits.

(a) $\lim\limits_{x \to \infty} \dfrac{x^2 + 6x + 9}{2x^2 + 7x + 8}$

(b) $\lim\limits_{x \to \infty} \dfrac{x^3 + 7x^2 + 2}{3x^3 + 6x + 1}$

EXERCISE 12A

Where possible, evaluate the following limits.

1 $\lim_{x \to 2} (x^2 + 4x + 5)$

2 $\lim_{x \to 0} 3^x$

3 $\lim_{x \to 0} \frac{3x + 2}{4x - 1}$

4 $\lim_{x \to 2} \frac{3x^2 + 5x + 2}{x - 4}$

5 $\lim_{x \to -1} \frac{4x^2 - 3x + 2}{x^2 + x + 2}$

6 $\lim_{x \to 1} (4x + 3)^2$

7 $\lim_{x \to 5} \frac{(x - 5)(x + 2)}{x - 5}$

8 $\lim_{x \to 3} \frac{(x - 3)(2x + 1)}{x - 3}$

9 $\lim_{x \to 0} \frac{3x^3 - 4x^2}{x^2}$

10 $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x + 2}$

11 $\lim_{x \to -5} \frac{x^2 + 9x + 20}{x + 5}$

12 $\lim_{x \to -2} \frac{x^4 - 16}{x + 2}$

13 $\lim_{x \to 5} \frac{\sqrt{x + 11} - 4}{x - 5}$

14 $\lim_{x \to -2} \frac{x + 2}{\sqrt{x + 6} - 2}$

15 $\lim_{x \to 16} \frac{x - 16}{\sqrt{x} - 4}$

16 $\lim_{x \to \infty} \frac{3x + 1}{2x - 1}$

17 $\lim_{x \to \infty} \frac{6x - 5}{4x - 1}$

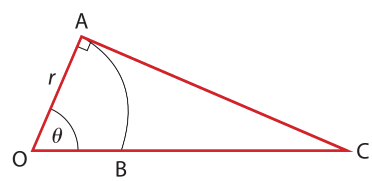
18 $\lim_{x \to \infty} \frac{x^2 + 3x + 1}{x^2 - 2x - 1}$

19 $\lim_{x \to \infty} \frac{x^3 - 6x + 1}{2x^2 - 1}$

Special limits

1 $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

PROOF



Triangle ABC is a right-angled triangle.

$\tan \theta = \frac{AC}{r}$

$\Rightarrow AC = r \tan \theta$

$$\text{Area of triangle OAC} = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2}r \times (r \tan \theta)$$
$$= \frac{1}{2}r^2 \tan \theta$$

Area of sector OAB $= \frac{1}{2}r^2 \theta$

Area of triangle OAB $= \frac{1}{2}r^2 \sin \theta$

Note

OAB is the sector of a circle with center O and radius *r* units.

Area of triangle OAB < area of the sector OAB < area of triangle OAC

$\Rightarrow \frac{1}{2}r^2 \sin \theta < \frac{1}{2}r^2 \theta < \frac{1}{2}r^2 \tan \theta$

Dividing by $\frac{1}{2}r^2$, we get:

$\sin \theta < \theta < \tan \theta$

$\sin \theta < \theta < \frac{\sin \theta}{\cos \theta}$

Dividing by $\sin\theta$, we get:

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\frac{\sin\theta}{\cos\theta}}{\sin\theta}$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

As $\theta \to 0$, $\cos\theta \to 1$

Therefore, $\frac{1}{\cos\theta} \to 1$

Therefore, as $\theta \to 0$, $\frac{\theta}{\sin\theta} \to 1$

$$\Rightarrow \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1$$

Since $\frac{\theta}{\sin\theta} = \frac{1}{\frac{\sin\theta}{\theta}}$,

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = \lim_{\theta\to 0}\frac{1}{\frac{\sin\theta}{\theta}}$$

$$= \frac{1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}$$

$$\Rightarrow 1 = \frac{1}{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}$$

Hence, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

2 $\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = 0$

PROOF

$$\lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\frac{\cos\theta - 1}{\theta} \times \frac{\cos\theta + 1}{\cos\theta + 1}$$

$$= \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)}$$

$$= \lim_{\theta\to 0}\frac{-(1 - \cos^2\theta)}{\theta(\cos\theta + 1)}$$

$$= \lim_{\theta\to 0}\frac{-\sin^2\theta}{\theta(\cos\theta + 1)} \qquad (\text{Since } 1 - \cos^2\theta = \sin^2\theta)$$

$$= \lim_{\theta\to 0} -\frac{\sin\theta}{\theta} \times \frac{\sin\theta}{\cos\theta + 1}$$

$$= \lim_{\theta\to 0} -\frac{\sin\theta}{\theta} \times \lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1}$$

$$= -1 \times \left(\frac{0}{1+1}\right) = 0$$

3 $\lim_{x\to\infty}\left(1 + \frac{1}{x}\right)^x = e$

EXAMPLE 17 Evaluate $\lim_{\theta \to 0} \frac{\sin 6\theta}{\theta}$.

SOLUTION Multiply numerator and denominator by 6:

$$\lim_{\theta \to 0} \frac{6 \times \sin 6\theta}{6 \times \theta} = \lim_{\theta \to 0} 6\left(\frac{\sin 6\theta}{6\theta}\right)$$

$$= \lim_{\theta \to 0} 6 \times \lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta}$$

We can now use $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$, since $\lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta}$ is of the same form as $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$. We have:

$$\lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta} = 1$$

Alternatively, let $u = 6\theta$

$$\lim_{\theta \to 0} \left(\frac{\sin 6\theta}{6\theta}\right) = \lim_{\theta \to 0} \left(\frac{\sin u}{u}\right)$$

$$= 1$$

Therefore, $\lim_{\theta \to 0} \frac{\sin 6\theta}{\theta} = \lim_{\theta \to 0} 6 \times \lim_{\theta \to 0} \frac{\sin 6\theta}{6\theta}$

$$= 6 \times 1$$

$$= 6$$

EXAMPLE 18 Evaluate $\lim_{\theta \to 0} \frac{\cos 4\theta - 1}{2\theta}$.

SOLUTION Multiply numerator and denominator by 2:

$$\lim_{\theta \to 0} \frac{\cos 4\theta - 1}{2\theta} \times \frac{2}{2} = \lim_{\theta \to 0} 2\left(\frac{\cos 4\theta - 1}{4\theta}\right)$$

$$= \lim_{\theta \to 0} 2 \times \lim_{\theta \to 0} \frac{\cos 4\theta - 1}{4\theta}$$

$$= 2 \times 0 \qquad \left(\text{Since } \lim_{\theta \to 0} 2 = 2 \text{ and } \lim_{\theta \to 0} \frac{\cos 4\theta - 1}{4\theta} = 0\right)$$

$$= 0$$

EXAMPLE 19 Find the value of $\lim_{\theta \to 0} \frac{\sin 7\theta}{\sin 4\theta}$.

SOLUTION Since we know that $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1$, we can introduce this form by dividing numerator and denominator by θ:

$$\lim_{\theta \to 0} \frac{\frac{\sin 7\theta}{\theta}}{\frac{\sin 4\theta}{\theta}} = \frac{\lim_{\theta \to 0} \frac{\sin 7\theta}{\theta}}{\lim_{\theta \to 0} \frac{\sin 4\theta}{\theta}}$$

Now, $\frac{\sin 7\theta}{\theta} = \frac{7\sin 7\theta}{7\theta}$.

Therefore, $\lim_{\theta \to 0} \frac{\sin 7\theta}{\theta} = \lim_{\theta \to 0} \frac{7\sin 7\theta}{7\theta}$

$$= \lim_{\theta \to 0} 7\left(\frac{\sin 7\theta}{7\theta}\right)$$

$$= \lim_{\theta \to 0} 7 \times \lim_{\theta \to 0} \frac{\sin 7\theta}{7\theta}$$

$$= 7 \times 1$$

$$= 7$$

Similarly, $\frac{\sin 4\theta}{\theta} = \frac{4\sin 4\theta}{4\theta}$.

Therefore, $\lim\limits_{\theta \to 0} \frac{\sin 4\theta}{\theta} = \lim\limits_{\theta \to 0} \frac{4\sin 4\theta}{4\theta}$

$$= \lim_{\theta \to 0} 4\left(\frac{\sin 4\theta}{4\theta}\right)$$

$$= \lim_{\theta \to 0} 4 \times \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta}$$

$$= 4 \times 1$$

$$= 4$$

Hence, $\lim\limits_{\theta \to 0} \frac{\sin 7\theta}{\sin 4\theta} = \frac{7}{4}$.

EXAMPLE 20 Evaluate $\lim\limits_{\theta \to 0} \frac{\tan 4\theta}{\theta}$.

SOLUTION

$$\lim_{\theta \to 0} \frac{\tan 4\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin 4\theta}{\theta \cos 4\theta} \qquad \left(\text{Since } \tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}\right)$$

$$= \lim_{\theta \to 0} \left(\frac{\sin 4\theta}{\theta} \times \frac{1}{\cos 4\theta}\right)$$

Now, $= \lim\limits_{\theta \to 0} \frac{\sin 4\theta}{\theta} = \lim\limits_{\theta \to 0} \frac{4\sin 4\theta}{4\theta}$.

Therefore, $\lim\limits_{\theta \to 0} \frac{\sin 4\theta}{\theta} = \lim\limits_{\theta \to 0} 4\left(\frac{\sin 4\theta}{4\theta}\right)$

$$= \lim_{\theta \to 0} 4 \times \lim_{\theta \to 0} \frac{\sin 4\theta}{4\theta}$$

$$= 4 \times 1$$

$$= 4$$

Now we look at $\lim\limits_{\theta \to 0} \frac{1}{\cos 4\theta}$:

$$\lim_{\theta \to 0} \frac{1}{\cos 4\theta} = \frac{1}{\cos 4(0)}$$

$$= \frac{1}{1}$$

$$= 1$$

Hence, $\lim\limits_{\theta \to 0} \frac{\tan 4\theta}{\theta} = 4 \times 1 = 4$

EXAMPLE 21 Evaluate $\lim\limits_{x \to 0} \frac{\tan x - 6x}{\sin 3x - 5x}$.

SOLUTION

$$\lim_{x \to 0} \frac{\tan x - 6x}{\sin 3x - 5x} = \lim_{x \to 0} \left(\frac{\frac{\sin x}{\cos x} - 6x}{\sin 3x - 5x}\right) \qquad \left(\text{Since } \tan x = \frac{\sin x}{\cos x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\frac{\frac{\sin x}{\cos x} - 6x}{x}}{\frac{\sin 3x - 5x}{x}}\right) \qquad \text{(Divide numerator and denominator by } x\text{)}$$

$$= \lim_{x \to 0} \left(\frac{\left(\frac{\sin x}{x \cos x}\right) - 6}{\left(\frac{\sin 3x}{x}\right) - 5}\right)$$

$$= \frac{\lim\limits_{x\to 0}\frac{\sin x}{x\cos x} - \lim\limits_{x\to 0} 6}{\lim\limits_{x\to 0}\frac{\sin 3x}{x} - \lim\limits_{x\to 0} 5}$$

$$= \frac{\lim\limits_{x\to 0}\frac{\sin x}{x} \times \lim\limits_{x\to 0}\frac{1}{\cos x} - \lim\limits_{x\to 0} 6}{\lim\limits_{x\to 0}\frac{3\sin 3x}{3x} - \lim\limits_{x\to 0} 5}$$

$$= \frac{(1)(1) - 6}{3 - 5}$$

$$= \frac{-5}{-2} = \frac{5}{2}$$

Try these 12.5 Evaluate the following limits.

(a) $\lim\limits_{\theta\to 0}\frac{\sin 8\theta}{\theta}$

(b) $\lim\limits_{\theta\to 0}\frac{\tan 7\theta}{\theta}$

(c) $\lim\limits_{\theta\to 0}\frac{\sin 6\theta}{\sin 5\theta}$

Continuity

If $f(x)$ is continuous at $x = a$, then the following conditions must be met:

(i) $f(a)$ must be defined.

(ii) $\lim\limits_{x\to a} f(x)$ must exist.

(iii) $\lim\limits_{x\to a} f(x) = f(a)$

This means that for a function to be continuous at a point: the function must have a value at that point; the limit of the function must exist at that point; and the function must have the same value as the limit at that point.

EXAMPLE 22 Is the function $f(x) = 2x + 1$ continuous at $x = 2$?

SOLUTION $f(2) = 5$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-} (2x + 1) = 5$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+} (2x + 1) = 5$

Since, $\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^+} f(x)$, the limit of $f(x)$ exists at $x = 2$ and its value is 5.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(2x + 1)$$
$$= 5$$

Hence, $f(2) = \lim\limits_{x\to 2} f(x)$.

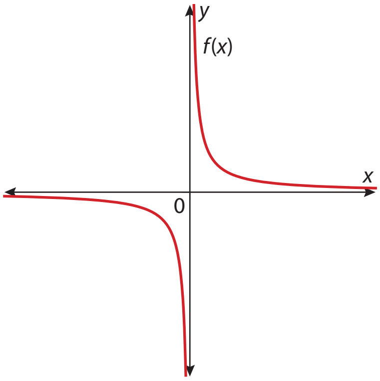
Therefore, the function is continuous at $x = 2$.

Types of discontinuity

Infinite discontinuity

Note

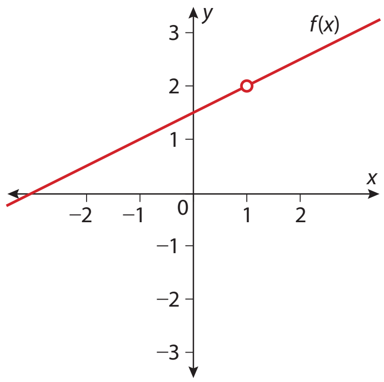
If $\lim_{x \to a} f(x) = \infty$ or $\lim_{x \to a} f(x) = -\infty$, $f(x)$ is said to demonstrate infinite discontinuity at $x = a$.



Note

If $\lim_{x \to a} f(x)$ exists but does not equal $f(x)$, $f(x)$ is said to demonstrate point discontinuity at $x = a$.

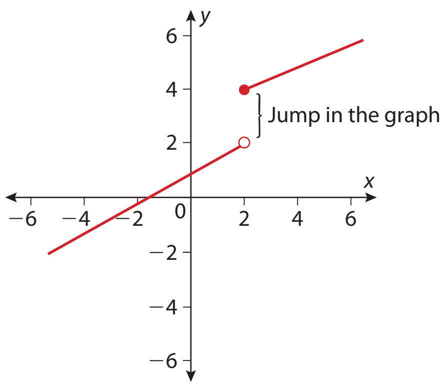
The graph of $f(x)$ demonstrates infinite discontinuity at $x = 0$.



Point discontinuity

The graph of $f(x)$ has a hole at $x = 1$. The function is discontinuous at $x = 1$ since $f(1) = 0$, while $\lim_{x \to 1} f(x) = 2$. If $f(x)$ is continuous at $x = 1$, the limit and the function values must be equal.

Jump discontinuity



Note

If $\lim_{x \to a^+} f(x) = y_1$ and $\lim_{x \to a^+} f(x) = y_2$, where $y_1 \neq y_2$, then $f(x)$ is said to exhibit jump discontinuity at $x = a$.

The graph has a break at $x = 2$. The function is discontinuous at $x = 2$.

$\lim_{x \to 2^+} f(x) = 4 \qquad \lim_{x \to 2^-} f(x) = 2$ and $4 \neq 2$.

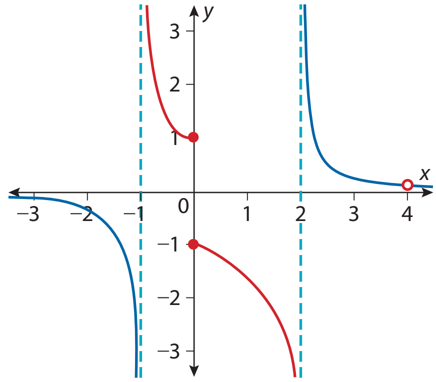
EXAMPLE 23 Using the graph of $f(x)$, identify whether the function $f(x)$ is discontinuous at the given values of x. If the function is discontinuous, identify the type of discontinuity.

(a) $x = -1$

(b) $x = 2$

(c) $x = 4$

(d) $x = 0$



SOLUTION

(a) $x = -1$, infinite discontinuity since the graph moves towards infinity at this point.

(b) $x = 2$, infinite discontinuity since the graph moves towards infinity at this point.

(c) $x = 4$, point discontinuity since there is a hole in the graph at $x = 4$.

(d) $x = 0$, jump discontinuity since there is a jump in the graph when $x = 0$.

Removable and non-removable discontinuity

Functions with point discontinuity are said to be removable discontinuous since we can redefine the function to correspond with the existing limit and remove the discontinuity from the function. Jump and infinite discontinuities cannot be removed and are classified as non-removable.

In the graph in Example 23, $x = -1$, $x = 2$ and $x = 0$ are non-removable discontinuities, while $x = 4$ is a removable discontinuity.

Can you identify why jump and infinite discontinutíes cannot be removed?

EXAMPLE **24**

Given the function $f(x)$, where $f(x) = 4x - x^2$ for $x > 2$ and $f(x) = 3x - 2$ for $x < 2$, identify the value(s) of x at which $f(x)$ is discontinuous and describe the type of discontinuity.

SOLUTION

Both parts of the function are polynomials. Therefore, they are continuous over their domains. Let us see what happens at $x = 2$.

When $f(x) = 4x - x^2$:

$f(2) = 4(2) - 2^2 = 8 - 4 = 4$

$f(x) = 3x - 2$

$f(2) = 3(2) - 2 = 6 - 2 = 4$

$\lim_{x\to 2^+} f(x) = 4$ and $\lim_{x\to 2^-} f(x) = 4$

Therefore, $\lim_{x\to 2} f(x) = 4$

But the function is not defined at $x = 2$. This function is defined for x greater than 2 or less than 2.

Therefore, the function is discontinuous at $x = 2$.

Since $\lim_{x\to 2} f(x)$ exists, $f(x)$ exhibits point discontinuity at $x = 2$.

Remember

$\left.\begin{array}{l} 4x - x^2, x > 2 \\ 3x - 2, x < 2 \end{array}\right\} f(x)$

$f(x)$ is not defined at $x = 2$.

EXAMPLE 25 Calculate the value of a that makes $f(x)$ continuous everywhere.

$$f(x) = \begin{cases} \log(2a + 4x), x < 3 \\ \log(x + 4a), x \geq 3 \end{cases}$$

SOLUTION Since the log function is continuous everywhere over its domain, we can make the function continuous by forcing the left-hand and right-hand limits to be equal at $x = 3$.

Substituting $x = 3$ and equating gives:

$\log(2a + 12) = \log(3 + 4a)$

$\therefore\ 2a + 12 = 3 + 4a$

$9 = 2a$

$a = \frac{9}{2}$

Hence, when $a = \frac{9}{2}$, $f(x)$ is continuous everywhere.

EXAMPLE 26 Given the function $f(x)$, determine the value(s) of x for which the function is discontinuous and identify the type of discontinuity.

$$f(x) = \begin{cases} 4x + 2, x > 1 \\ 8x - 4, x < 1 \end{cases}$$

SOLUTION Since both parts of the function are linear, they are continuous over their respective domains.

Let us see if the limit exists at $x = 1$.

When $f(x) = 4x + 2$,

$f(1) = 4(1) + 2 = 6$

When $f(x) = 8x - 4$,

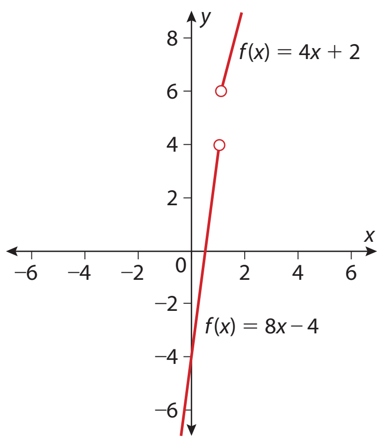
$f(1) = 8(1) - 4 = 8 - 4 = 4$

$\lim\limits_{x \to 1^+} f(x) = 6$ and $\lim\limits_{x \to 1^-} f(x) = 4$.

Therefore, $\lim\limits_{x \to 1} f(x)$ does not exist.

Hence, $f(x)$ is discontinuous at $x = 1$.

The discontinuity is jump discontinuity.



EXAMPLE 27 **(a)** Given $f(x) = \frac{x + 1}{x + 2}$, determine the value(s) of x for which $f(x)$ is discontinuous.

(b) Is the discontinuity removable?

SOLUTION $f(x) = \frac{x + 1}{x + 2}$ has an asymptote when $x = -2$.

Therefore, $\lim\limits_{x \to 2} f(x) = \infty$.

Hence, $f(x)$ is discontinuous at $x = -2$.

Since the discontinuity is an infinite discontinuity, the discontinuity is non-removable.

A polynomial function is continuous everywhere while a rational function $\frac{P(x)}{Q(x)}$ is continuous for all x, except for which $Q(x) = 0$.

EXERCISE 12B

In questions **1** to **43**, evaluate the following limits.

1 $\lim_{x \to 1} \frac{4x^3 + 5}{x - 3}$

2 $\lim_{x \to 0} (4x + 2)^5$

3 $\lim_{x \to 2} \sqrt{4x + 2}$

4 $\lim_{x \to 1} (x^2 - 2x + 1)^6$

5 $\lim_{x \to -1} (4x + 3)^3(2x + 1)^2$

6 $\lim_{x \to 0} \frac{(6x - 2)^3}{(3x + 1)^2}$

7 $\lim_{x \to -3} \frac{x^2 - 3}{2x + 3}$

8 $\lim_{x \to -\frac{1}{2}} \frac{2x^2 + 5x + 2}{2x + 1}$

9 $\lim_{x \to -2} \frac{x^5 + 32}{x + 2}$

10 $\lim_{x \to 5} \frac{x^2 + 2x - 35}{x^2 - 10x + 25}$

11 $\lim_{x \to -1} \frac{x^2 + x - 2}{x^2 - 1}$

12 $\lim_{x \to 1} \frac{2x^3 + x^2 - 2x - 1}{3x^2 - 2x - 1}$

13 $\lim_{x \to \frac{1}{3}} \frac{27x^3 - 1}{3x - 1}$

14 $\lim_{x \to 3} \frac{x^5 - 243}{x - 3}$

15 $\lim_{x \to 9} \frac{9 - x}{\sqrt{x} - 3}$

16 $\lim_{x \to 10} \frac{\sqrt{5} - \sqrt{x - 5}}{10 - x}$

17 $\lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x}$

18 $\lim_{x \to 1} \frac{\sqrt{3 + x} - \sqrt{5 - x}}{x^2 - 1}$

19 $\lim_{x \to 0} \frac{x^2}{\sqrt{x^2 + 12} - \sqrt{12}}$

20 $\lim_{\theta \to 0} \frac{\sin 5\theta}{\theta}$

21 $\lim_{\theta \to 0} \frac{\sin 3\theta}{5\theta}$

22 $\lim_{\theta \to 0} \frac{\sin 2\theta}{3\theta}$

23 $\lim_{\theta \to 0} \frac{\sin 3\theta}{\sin 5\theta}$

24 $\lim_{\theta \to 0} \frac{\sin 8\theta}{\sin 4\theta}$

25 $\lim_{\theta \to 0} \frac{\cos 2\theta - 1}{2\theta}$

26 $\lim_{\theta \to 0} \frac{1 - \cos 2\theta}{2\theta}$

27 $\lim_{\theta \to 0} \frac{\tan 2\theta}{2\theta}$

28 $\lim_{\theta \to 0} \frac{\tan 6\theta}{4\theta}$

29 $\lim_{x \to 0} \frac{x}{\sin 3x}$

30 $\lim_{x \to 0} \frac{x}{\tan x}$

31 $\lim_{x \to \infty} \frac{4x^4 + 3x^2 + x - 2}{x^4 - 1}$

32 $\lim_{x \to \infty} \frac{3x^2}{x^4 - 1}$

33 (a) Given that $\lim_{x \to 2} (3f(x)) = 10$, evaluate $\lim_{x \to 2} (f(x) + 2x^2)$.

(b) Determine the values of $x \in \mathbb{R}$ for which the function $\frac{x + 2}{x^2 + 7x + 12}$ is not continuous.

34 (a) Show that $\frac{\sqrt{x - 6} - 1}{7 - x} = \frac{-1}{\sqrt{x - 6} + 1}$. Hence, evaluate $\lim_{x \to 7} \frac{\sqrt{x - 6} - 1}{7 - x}$.

(b) Evaluate $\lim_{\theta \to 0} \frac{\tan^2 \theta}{\theta}$.

35 Evaluate $\lim_{x \to 0} \frac{5 \sin x + \cos x - 1}{4x}$.

36 Evaluate $\lim_{x \to 0} \frac{\sin^2 x + \sin x(\cos x - 1)}{x^2}$.

SUMMARY

Limits and continuity

Limits

Exists

$\lim_{x \to b} c = c$, where c is a constant.

$\lim_{x \to b} x^n = b^n$, where b is a positive integer.

$\lim_{x \to b} (f(x) + g(x)) = \lim_{x \to b} f(x) + \lim_{x \to b} g(x)$

$\lim_{x \to b} (cf(x)) = c \lim_{x \to b} f(x)$, where c is a constant.

$\lim_{x \to b} f(x)\, g(x) = \lim_{x \to b} f(x) \times \lim_{x \to b} g(x)$

$\lim_{x \to b} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to b} f(x)}{\lim_{x \to b} g(x)}$, provided that $\lim_{x \to b} g(x) \neq 0$.

$\lim_{x \to b} (f(x))^n = \left(\lim_{x \to b} f(x)\right)^n$

$\lim_{x \to b} \sqrt{f(x)} = \sqrt{\lim_{x \to b} (f(x))}$

Methods of finding limits

Direct substitution
Factorising
Rationalising

Limits tending to infinity: treat differently

Special limits

$\lim_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

$\lim_{\theta \to 0} \dfrac{\cos\theta - 1}{\theta} = 0$

Does not exist

A limit does not exist, if:

a the left-hand limit $\neq$ right-hand limit
b $f(x)$ oscillates infinitely
c $f(x)$ increases or decreases infinitely

Continuity

Continuous

$f(x)$ is continuous at $x = a$, if:

a $\lim_{x \to a} f(x) = l_1$
b $f(a) = l_2$
c $l_1 = l_2$

Discontinuous

Removable

Point

Non-removable

Jump
Infinite

Checklist

Can you do these?

- Use graphs to determine the continuity or discontinuity of a function.
- Describe the behaviour of a function as x gets close to some fixed number.
- Use simple limit theorems.
- Identify and evaluate one-sided limits.
- Distinguish between the limit of a function and the value of a function.
- Use the limit theorems for specific cases.
- Show geometrically that $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.
- Evaluate limits tending to infinity.
- Solve problems using limits.
- Identify the relationship between limits and continuity
- Identify the region over which a function is continuous.
- Identify the points where a function is discontinuous.
- Describe the discontinuity of a function.

Review Exercise 12

1 Evaluate $\lim\limits_{x \to \infty} \dfrac{4x^5 + x^2 + x - 2}{4x^3 + 5}$.

2 Find the value of $\lim\limits_{x \to \infty} \dfrac{7x^3 - 2x^2 + x}{3x^2 - 1}$.

3 Find $\lim\limits_{x \to -1} \dfrac{4x^2 - 3x + 2}{x^2 + x + 2}$.

4 Evaluate the following limits.

(a) $\lim\limits_{x \to 2} \dfrac{x^3 + x^2 + x - 14}{x^3 - x - 6}$

(b) $\lim\limits_{x \to -\frac{5}{4}} \dfrac{12x^2 + 23x + 10}{4x^2 + 13x + 10}$

5 Find $\lim\limits_{\theta \to 0} \dfrac{\sin 2\theta}{\sin \theta}$.

6 Find $\lim\limits_{\theta \to 0} \dfrac{\cos 6\theta - 1}{2\theta}$.

7 Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 3x \sin 5x}{7x^2}$.

8 Find $\lim\limits_{x \to -3} \dfrac{\sqrt{3 - 5x}}{x + 1}$.

9 (a) Find $\lim_{x \to -\frac{1}{2}} \dfrac{2x^2 + 5x + 2}{2x^2 + 9x + 4}$.

(b) Find the real values of x for which the function $f(x) = \dfrac{x^2}{x^2 + 3x + 2}$ is continuous.

10 (a) Evaluate $\lim_{\theta \to 0} \dfrac{\sin 5\theta}{\theta}$.

(b) Determine the real values of x for which the function $\dfrac{4x}{|2x - 3| - 5}$ is continuous.

11 Evaluate $\lim_{\theta \to 0} \dfrac{\tan 5\theta}{\theta}$.

12 Find the value of $\lim_{x \to 0} \dfrac{\tan 2x - 4x}{\sin 3x - 7x}$.

13 (a) Find the real values of x for which the function $f(x) = \dfrac{x}{4x^2 - 11x - 3}$ is continuous.

(b) Find $\lim_{x \to 0} \dfrac{\sin 4x}{x}$. Hence, evaluate $\lim_{x \to 0} \dfrac{\sin 4x}{\sin 5x}$.

14 (a) Show that $\dfrac{\sqrt{x + 4} - 3}{x - 5} = \dfrac{1}{\sqrt{x + 4} + 3}$. Hence, evaluate $\lim_{x \to 5} \dfrac{\sqrt{x + 4} - 3}{x - 5}$.

(b) Evaluate $\lim_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$.

(c) Find the values of x for which the function $\dfrac{2x}{|3x - 1| - 8}$ is continuous.

15 Give all the x-values for which the function $f(x) = \dfrac{9x^2 - 3x - 2}{3x^2 + 13x + 4}$ is discontinuous, and classify each instance of discontinuity.

CHAPTER 13

Differentiation 1

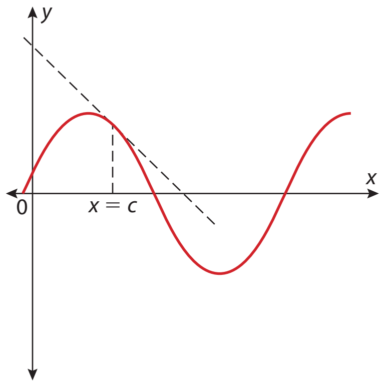
At the end of this chapter you should be able to:

- Demonstrate the concept of the derivative at $x = c$ as the gradient of the tangent at $x = c$
- Define the derivative at a point as a limit
- Use the different notations for derivative i.e. $f'(x), \frac{dy}{dx}, f^{(1)}(x)$
- Differentiate from first principles: $f(x) = x^n, f(x) = k, f(x) = \sin x$
- Know and use the derivative of x^n
- Know and use simple theorems about derivatives of $y = cf(x)$, $y = f(x) + g(x)$, where c is a constant
- Calculate derivatives of
  - polynomials
  - trigonometric functions
- Know and use the product rule and quotient rule for differentiation
- Know and use the concept of the derivative as a rate of change
- Differentiate composite functions using the chain rule
- Find the second derivatives of functions
- Differentiation of parametric equations

KEYWORDS/TERMS

scope • gradient • tangent • limit • first principles • derivative • differentiation • rate of change • product rule • quotient rule • chain rule • composite function • differentiation • first derivative • second derivative

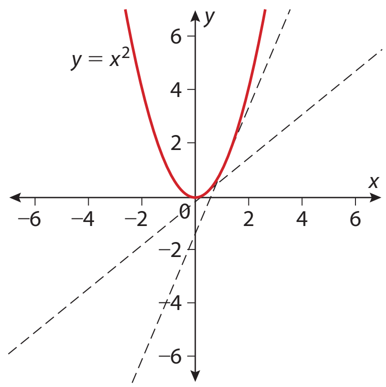
Differentiation



Differential calculus is an important tool of mathematics that has applications in disciplines such as economics, engineering, and science. Differential calculus deals with finding the rate at which one variable changes at a particular instant with respect to another. In economics for example, one of the basic rules is that individuals do the best that they can, given the scarce resources available to them. If we convert this concept to a mathematical principle, we look at the problem as one of constrained optimisation. If a consumer who has a restricted budget has utility depending on consumption of two goods, we can form a set of equations and set out to maximize utility subject to any constraints. This will make use of a technique called the method of Lagrange Multipliers, which makes use of differential calculus.

Differential calculus is the study of the properties of the derivative of a function and differentiation is the process of finding the derivative of a function. The derivative of a function is the slope or steepness of the function. The steepness of a function at any point $x = c$ is the gradient of the tangent at that point.

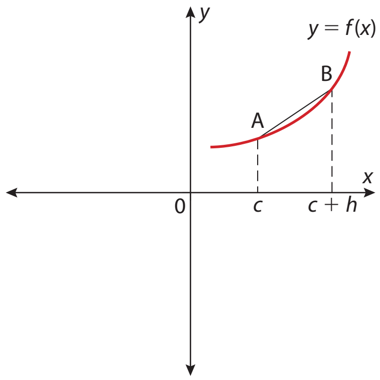
The gradient of a curve changes along the curve. For the curve $y = x^2$ the gradient is given at varying points on the table below.



x	0	$\frac{1}{2}$	1	2	3
$y = x^2$	0	$\frac{1}{4}$	1	4	9
Slope: $2x$	0	1	2	4	6

As the values of x increases, the curve gets steeper and the gradient at each point increases.

The difference quotient



Given the curve $y = f(x)$, let us calculate the gradient of the tangent line at point A, where $x = c$.

We take another point, B, on the curve, which is close to A. At B, $x = c + h$.

We draw a line through the two points. We can find the gradient of this line as:

$$\text{gradient of the line AB} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(c + h) - f(c)}{(c + h) - c}$$

$$= \frac{f(c + h) - f(c)}{h}$$

If we make h infinitely small, so small that it is close to 0, then the two points on the graph will have the same gradient. Using the limit as h tends to 0, we can calculate the slope or gradient of the tangent at A, using the gradient of the line AB. (The gradient of the chord AB will approximate to the gradient of the tangent at A.)

This limit is called the **difference quotient**, and is the definition of the derivative of a function:

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

DEFINITION

The derivative of a function $f(x)$ at any point x, is the slope of the tangent line to the function at x. It is defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

Existence of a derivative

There are three cases where the derivative of a function does not exist:

(i) There is no derivative at any type of discontinuity.

(ii) There is no derivative at a cusp on a function.

(iii) There is no derivative at a vertical tangent line.

Notation for derivatives

The first derivative of $y = f(x)$, has four different notations, each of which can be used at your convenience:

$f'(x)$

$\frac{dy}{dx}$

y'

$f^{(1)}(x)$

These all represent the first derivative of the function $y = f(x)$.

The second derivative can be represented by:

$f''(x)$

$\frac{d^2y}{dx^2}$

y''

$f^{(2)}(x)$

Similar notation can be used for the nth derivatives.

Interpretations of derivatives

The derivative is the gradient function at any point x or the gradient of the tangent to the curve at any point x. A derivative is also a rate of change and it measures how one variable changes with respect to another. $\frac{dy}{dx}$ represents the rate of change of y with respect to x. It is read as 'd y by d x'.

Finding derivatives using first principles

EXAMPLE 1 Let $f(x) = c$ where c is a constant. Find $f'(x)$, using first principles.

SOLUTION By first principles we mean 'use the definition of $f'(x)$'.

By definition, $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Since $f(x) = c$, replacing x by $x + h$, we get:

$f(x + h) = c$

$$\begin{aligned}\text{Therefore, } f'(x) &= \lim_{h \to 0} \left(\frac{c - c}{h}\right) \\ &= \lim_{h \to 0} 0 \\ &= 0\end{aligned}$$

Therefore, when $f(x) = c$, $f'(x) = 0$.

EXAMPLE 2 Let $f(x) = x$. Find $f'(x)$, using the definition of derivatives.

SOLUTION By definition, $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Since $f(x) = x$, replacing $x = x + h$, we have:

$f(x + h) = x + h$

$$\begin{aligned}\text{Therefore, } f'(x) &= \lim_{h \to 0} \frac{x + h - x}{h} \\ &= \lim_{h \to 0} \frac{h}{h} \\ &= \lim_{h \to 0} 1 \\ &= 1\end{aligned}$$

Therefore, when $f(x) = x$, $f'(x) = 1$.

EXAMPLE 3 Let $f(x) = x^2$. Use first principles to find $f'(x)$.

SOLUTION By definition, $f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Since $f(x) = x^2$:

$$\begin{aligned}f(x + h) &= (x + h)^2 \\ &= x^2 + 2hx + h^2\end{aligned}$$

$$\text{Therefore, } f'(x) = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2) - x^2}{h}$$
$$= \lim_{h \to 0} \left(\frac{2hx + h^2}{h}\right)$$
$$= \lim_{h \to 0} (2x + h)$$
$$= 2x + 0$$
$$= 2x$$

Therefore, when $f(x) = x^2$, $f'(x) = 2x$.

EXAMPLE 4 Use the definition of derivative to find $f'(x)$ where $f(x) = \sqrt{x}$.

SOLUTION By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.

Since $f(x) = \sqrt{x}$, $f(x + h) = \sqrt{x + h}$.

Substituting this into the definition, we have:

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

Now we simplify the fractional expression, using the conjugate of the numerator $(\sqrt{x + h} + \sqrt{x})$ to rationalise the numerator:

$$\frac{\sqrt{x + h} - \sqrt{x}}{h} = \frac{\sqrt{x + h} - \sqrt{x}}{h} \times \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$$
$$= \frac{(x + h) - x}{h(\sqrt{x + h} + \sqrt{x})}$$
$$= \frac{\cancel{h}}{\cancel{h}(\sqrt{x + h} + \sqrt{x})}$$
$$= \frac{1}{\sqrt{x + h} + \sqrt{x}}$$

$$\text{Therefore, } f'(x) = \lim_{h \to 0} \left(\frac{1}{\sqrt{x + h} + \sqrt{x}}\right)$$
$$= \frac{1}{\sqrt{x} + 0 + \sqrt{x}} \qquad \text{(Substitute } h = 0\text{)}$$
$$= \frac{1}{2\sqrt{x}}$$

Hence, when $f(x) = \sqrt{x}$, $f'(x) = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2}x^{-\frac{1}{2}}$.

EXAMPLE 5 Use first principles to find the derivative of $f(x) = x^3$.

SOLUTION By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

Since $f(x) = x^3$

$$f(x + h) = (x + h)^3$$
$$= (x + h)(x + h)(x + h) = (x^2 + 2hx + h^2)(x + h)$$
$$= x^3 + 3hx^2 + 3h^2x + h^3$$

$$\text{Therefore, } f'(x) = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3h^2x + h^3}{h}$$

$$= \lim_{h \to 0} (3x^2 + 3hx + h^2)$$

$$= 3x^2 + 3x(0) + 0^2 \qquad \text{(Substitute } h = 0\text{)}$$

$$= 3x^2$$

When $f(x) = x^3, f'(x) = 3x^2$.

Try these 13.1 Using first principles find the derivative of these.

(a) $f(x) = \frac{1}{x}$

(b) $f(x) = \frac{1}{x^2}$

Rule

If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. (Bring down the power and reduce the power by 1.)

General results

Function	Derivative
c, where c is a constant	0
x	1
x^2	$2x$
x^3	$3x^2$
$\sqrt{x} = x^{\frac{1}{2}}$	$\frac{1}{2}x^{-\frac{1}{2}}$

EXAMPLE 6 Given that $f(x) = x^5$, find $f'(x)$.

SOLUTION Using $f'(x) = nx^{n-1}$, when $f(x) = x^n$ with $n = 5$.

$f(x) = x^5$

$f'(x) = 5x^{5-1} = 5x^4$

EXAMPLE 7 Given that $f(x) = x^{\frac{4}{7}}$, find $f'(x)$.

SOLUTION Using $f'(x) = nx^{n-1}$, when $f(x) = x^n$.

$f(x) = x^{\frac{4}{7}}$

$f'(x) = \frac{4}{7}x^{\frac{4}{7}-1} = \frac{4}{7}x^{-\frac{3}{7}}$

EXAMPLE 8 Given that $f(x) = \frac{1}{\sqrt{x}}$, find $f'(x)$.

SOLUTION Writing $\frac{1}{\sqrt{x}}$ in index form:

$f(x) = x^{-\frac{1}{2}}$

Using $f'(x) = nx^{n-1}$, where $n = -\frac{1}{2}$, we have:

$f'(x) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}$

Try these 13.2 Find the derivatives of these.

(a) $f(x) = x^{10}$

(b) $f(x) = x^{\frac{3}{4}}$

(c) $f(x) = \frac{1}{x^4}$

(d) $f(x) = \frac{1}{x^8}$

Differentiation of $ag(x)$ where a is a constant

EXAMPLE 9 Let $f(x) = ag(x)$ where a is a constant. Show that $f'(x) = ag'(x)$.

SOLUTION By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Since $f(x) = ag(x)$

$f(x+h) = ag(x+h)$

$$\begin{aligned}
\text{Therefore, } f'(x) &= \lim_{h \to 0} \frac{ag(x+h) - ag(x)}{h} \\
&= \lim_{h \to 0} \frac{a(g(x+h) - g(x))}{h} \\
&= \lim_{h \to 0} a \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= a \times \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= ag'(x) \qquad \left(\text{Since } g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right)
\end{aligned}$$

EXAMPLE 10 Given that $f(x) = 4x$, find $f'(x)$.

SOLUTION Using $f(x) = ag(x)$, where $a = 4$, $g(x) = x$.

$g'(x) = 1$

Therefore, $f'(x) = 4 \times 1 = 4$

EXAMPLE 11 Find the derivative of $f(x) = 6x^3$.

SOLUTION

$$f'(x) = 6 \times 3x^2$$
$$= 18x^2$$

EXAMPLE 12 Find $f'(x)$ where $f(x) = -\frac{4}{x^5}$.

SOLUTION

$$f(x) = -\frac{4}{x^5} = -4x^{-5}$$
$$f'(x) = -4(-5x^{-6})$$
$$= 20x^{-6}$$
$$= \frac{20}{x^6}$$

Try these 13.3 Find the derivatives of these.

(a) $f(x) = 10x^5$

(b) $f(x) = \frac{1}{\sqrt{x}}$

(c) $f(x) = \frac{12}{x^3}$

Differentiation of sums and differences of functions

EXAMPLE 13 Given that $f(x) = g(x) + m(x)$, show that $f'(x) = g'(x) + m'(x)$.

SOLUTION

By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ [1]

Since $f(x) = g(x) + m(x)$

$f(x+h) = g(x+h) + m(x+h)$

Substituting into [1] gives:

$$f'(x) = \lim_{h \to 0} \frac{(g(x+h) + m(x+h)) - (g(x) + m(x))}{h}$$
$$= \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} + \frac{m(x+h) - m(x)}{h} \right)$$
$$= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} \frac{m(x+h) - m(x)}{h}$$
$$= g'(x) + m'(x)$$

Try these 13.4 Given that $f(x) = g(x) - m(x)$, prove that $f'(x) = g'(x) - m'(x)$.

EXAMPLE 14 Given that $y = 4x^2 + 3x$, find $\frac{dy}{dx}$.

SOLUTION Recall that $\frac{dy}{dx}$ simply means 'differentiate y with respect to x'.

Using the rules for addition:

$$\frac{dy}{dx} = 4(2x) + 3(1)$$
$$= 8x + 3$$

EXAMPLE 15 Differentiate $y = \sqrt{x} + \frac{1}{x}$ with respect to y.

SOLUTION Writing each term in index form:

$$y = x^{\frac{1}{2}} + x^{-1}$$

Using $f^{(1)}(x) = nx^{n-1}$:

$$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} + (-1)x^{-1-1}$$
$$= \frac{1}{2}x^{-\frac{1}{2}} - x^{-2}$$
$$= \frac{1}{2\sqrt{x}} - \frac{1}{x^2}$$

EXAMPLE 16 Differentiate the following with respect to x:

(a) $y = 6x^5 + 4x^3 + \frac{3}{x^2}$

(b) $y = x^2 + \frac{1}{x^4} - 2\sqrt{x}$

SOLUTION

(a) Writing each term in index form:

$$y = 6x^5 + 4x^3 + 3x^{-2}$$

Using $f^{(1)}(x) = nx^{n-1}$ where $f(x) = x^n$:

$$\frac{dy}{dx} = 6(5x^{5-1}) + 4(3x^{3-1}) + 3(-2x^{-2-1})$$
$$= 30x^4 + 12x^2 - 6x^{-3}$$

(b) $y = x^2 + \frac{1}{x^4} - 2\sqrt{x}$

Writing each term in index form gives:

$$y = x^2 + x^{-4} - 2x^{\frac{1}{2}}$$

Using $f'(x) = nx^{n-1}$ where $f(x) = x^n$:

$$\frac{dy}{dx} = 2x^{2-1} + (-4)x^{-4-1} - 2\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$$
$$= 2x - 4x^{-5} - x^{-\frac{1}{2}}$$
$$= 2x - \frac{4}{x^5} - \frac{1}{\sqrt{x}}$$

EXAMPLE 17 Find the gradient function of these.

(a) $y = \frac{4x^2 + 3x - 2}{x^3}$

(b) $y = (4x + 2)(6x - 1)$

SOLUTION

(a) First we write each term in index form, using the rules of indices.

$$\begin{aligned} y &= (4x^2 + 3x - 2) \times x^{-3} \\ &= 4x^{2-3} + 3x^{1-3} - 2x^{-3} \\ &= 4x^{-1} + 3x^{-2} - 2x^{-3} \end{aligned}$$

Now we can differentiate each term, using $f'(x) = nx^{n-1}$ where $f(x) = x^n$:

$$\begin{aligned} \text{Therefore, } \frac{dy}{dx} &= 4(-1)(x^{-2}) + 3(-2)x^{-3} - 2(-3)x^{-4} \\ &= -4x^{-2} - 6x^{-3} + 6x^{-4} \\ &= -\frac{4}{x^2} - \frac{6}{x^3} + \frac{6}{x^4} \end{aligned}$$

(b) $y = (4x + 2)(6x - 1)$

We needed to write each term in the form x^n before differentiating. We expand the brackets:

$$\begin{aligned} y &= 24x^2 - 4x + 12x - 2 \\ &= 24x^2 + 8x - 2 \\ \frac{dy}{dx} &= 24(2x) + 8(1) \\ &= 48x + 8 \end{aligned}$$

Try these 13.5 Find the derivatives of these.

(a) $y = \dfrac{x^3 - x^2}{x^5}$

(b) $y = \dfrac{4x + 1}{\sqrt{x}}$

(c) $y = (x - 2)(3x + 4)$

(d) $y = (x + 5)(5x - 1)$

First principle and sums and differences of functions of x

EXAMPLE 18 Given that $f(x) = 4x - 6$, show using first principles that $f'(x) = 4$.

SOLUTION

By definition, $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$

Since $f(x) = 4x - 6$,

$$\begin{aligned} f(x + h) &= 4(x + h) - 6 \\ &= 4x + 4h - 6 \end{aligned}$$

Substituting into $f'(x)$, we have:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{(4x + 4h - 6) - (4x - 6)}{h} \\ &= \lim_{h \to 0} \frac{4x + 4h - 6 - 4x + 6}{h} \\ &= \lim_{h \to 0} \frac{4h}{h} \\ &= \lim_{h \to 0} 4 \\ &= 4 \end{aligned}$$

EXAMPLE 19 Given that $f(x) = x^2 - 2x + 3$, show, using first principles, that $f'(x) = 2x - 2$.

SOLUTION By definition, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Since $f(x) = x^2 - 2x + 3$

$$f(x+h) = (x+h)^2 - 2(x+h) + 3$$
$$= x^2 + 2hx + h^2 - 2x - 2h + 3$$

Substituting into $f'(x)$, we have:

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2hx + h^2 - 2x - 2h + 3) - (x^2 - 2x + 3)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - 2x - 2h + 3 - x^2 + 2x - 3}{h}$$
$$= \lim_{h \to 0} \frac{2hx + h^2 - 2h}{h}$$
$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 2)}{\cancel{h}}$$
$$= \lim_{h \to 0} (2x + h - 2)$$
$$= 2x + (0) - 2$$
$$= 2x - 2$$

Therefore, $f'(x) = 2x - 2$.

EXAMPLE 20 Given that $f(x) = \frac{1}{\sqrt{2x+1}}$, show that $f^{(1)}(x) = -\frac{1}{(2x+1)^{\frac{3}{2}}}$, using the definition of $f^{(1)}(x)$.

SOLUTION By definition, $f^{(1)}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Since $f(x) = \frac{1}{\sqrt{2x+1}}$

$$f(x+h) = \frac{1}{\sqrt{2(x+h)+1}}$$
$$= \frac{1}{\sqrt{2x+2h+1}}$$

Substituting for $f^{(1)}(x)$, we have:

$$f^{(1)}(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{2x+2h+1}} - \frac{1}{\sqrt{2x+1}}}{h} \qquad [1]$$

Let us work with the numerator of the fractional expression:

$$= \frac{1}{\sqrt{2x+2h+1}} - \frac{1}{\sqrt{2x+1}}$$
$$= \frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{(\sqrt{2x+2h+1})(\sqrt{2x+1})} \qquad \text{(LCM is } (\sqrt{2x+2h+1})(\sqrt{2x+1})\text{)}$$
$$= \frac{\sqrt{2x+1} - \sqrt{2x+2h+1}}{(\sqrt{2x+2h+1})(\sqrt{2x+1})} \times \frac{\sqrt{2x+1} + \sqrt{2x+2h+1}}{\sqrt{2x+1} + \sqrt{2x+2h+1}} \qquad \text{(Recall that } (a-b)(a+b) = a^2 - b^2\text{)}$$

$$= \frac{(2x + 1) - (2x + 2h + 1)}{(\sqrt{2x + 2h + 1})(\sqrt{2x + 1})(\sqrt{2x + 1} + \sqrt{2x + 2h + 1})}$$

$$= \frac{-2h}{(2x + 1)(\sqrt{2x + 2h + 1}) + (2x + 2h + 1)(\sqrt{2x + 1})}$$

Substituting into [1]:

$$f'(x) = \lim_{h \to 0} \frac{\dfrac{-2h}{(2x + 1)(\sqrt{2x + 2h + 1}) + (2x + 2h + 1)(\sqrt{2x + 1})}}{h}$$

$$= \lim_{h \to 0} \frac{-2}{(2x + 1)(\sqrt{2x + 2h + 1}) + (2x + 2h + 1)(\sqrt{2x + 1})}$$

$$= \frac{-2}{(2x + 1)\,(\sqrt{2x + 2(0) + 1}) + (2x + 2(0) + 1)\,(\sqrt{2x + 1})}$$ (Replacing h by 0)

$$= \frac{-2}{(2x + 1)(\sqrt{2x + 1}) + (2x + 1)(\sqrt{2x + 1})}$$

$$= \frac{-2}{(2x + 1)(2x + 1)^{\frac{1}{2}} + (2x + 1)(2x + 1)^{\frac{1}{2}}}$$

$$= \frac{-2}{2(2x + 1)(2x + 1)^{\frac{1}{2}}}$$

$$= \frac{-2}{2(2x + 1)^{\frac{3}{2}}}$$

$$= -\frac{1}{(2x + 1)^{\frac{3}{2}}}$$

Try these 13.6

Differentiate the following by using first principles.

(a) $f(x) = 2x - 4$

(b) $f(x) = 6x^2 + 2x + 1$

(c) $f(x) = \dfrac{1}{2x + 1}$

Rate of change

A function is said to have a rate of change at a point a, if the derivative of the function exists at $x = a$.

EXAMPLE 21 Find the rate of change of y with respect to x, given that $x = 1$ and $y = 4x^3 - x^2 + x + 2$.

SOLUTION

$$y = 4x^3 - x^2 + x - 2$$

$$\frac{dy}{dx} = 12x^2 - 2x + 1$$

Use $x = 1$:

$$\frac{dy}{dx} = 12(1)^2 - 2(1) + 1$$
$$= 12 - 2 + 1$$
$$= 11$$

The rate of change, given that $x = 1$, is 11.

EXERCISE 13A

1 Differentiate the following from first principles.

(a) $f(x) = 4x - 7$

(b) $f(x) = 3x + 9$

(c) $f(x) = x^2 + 2x + 5$

(d) $f(x) = 3x^2 - 4x + 1$

(e) $f(x) = 5x^2 + 2$

(f) $f(x) = x^3 + 3x + 1$

(g) $f(x) = \dfrac{1}{(x + 2)^2}$

2 Differentiate the following with respect to x.

(a) $4x^3 + 5x - 6$

(b) $x^5 - 3x^2 + 2$

(c) $x^5 + 7x^3 + 2x + 4$

(d) $4x + \dfrac{3}{x}$

(e) $6x^2 - \dfrac{7}{x^3}$

(f) $\dfrac{4}{x^3} + \dfrac{3}{x^2} - \dfrac{1}{x}$

(g) $4x + 3\sqrt{x} - 2$

(h) $6x + \dfrac{1}{\sqrt{x}} - 4$

(i) $2x^{\frac{5}{2}} - 3x^{\frac{3}{2}} + x^{\frac{1}{2}}$

(j) $6x\sqrt{x} + 5\sqrt{x}$

(k) $6x^2\sqrt{x} - \dfrac{3}{\sqrt{x}}$

(l) $4 + \dfrac{3}{x}$

3 Differentiate the following with respect to x.

(a) $\dfrac{x^2 + 3x}{x}$

(b) $\dfrac{4x^3 - 3x^2 + 2}{x^4}$

(c) $\dfrac{x^3 + 6x}{x^2}$

(d) $\dfrac{6x + 1}{\sqrt{x}}$

(e) $\dfrac{3x\sqrt{x} + 2}{\sqrt{x}}$

(f) $\dfrac{7x^2 - 4x + 5}{2x}$

4 For each of the following functions find $\dfrac{dy}{dx}$.

(a) $y = (x + 2)(7x - 1)$

(b) $y = (3x + 4)(2x + 7)$

(c) $y = (\sqrt{x} - 1)(x + 2)$

(d) $y = (6x + 2)(\sqrt{x} + 3)$

(e) $y = \dfrac{(4x - 1)(x + 3)}{x}$

(f) $y = \dfrac{(4x - 2)\sqrt{x}}{x}$

(g) $y = \left(3\sqrt{x} + 4x^{\frac{3}{2}}\right)x$

(h) $y = \dfrac{3x - 5}{\sqrt{x}}$

5 Find the gradients of the tangents to the curves at the given points.

(a) $y = 3x^3 - 4x^2 + 2$ at $(1, 1)$

(b) $y = 4x + 3x^2$ at $(1, 1)$

(c) $y = (2x + 1)(x - 2)$ at $(2, 0)$

(d) $y = \sqrt{x}\,(x + 4)$ at $x = 16$

6 Find the coordinates of the points on the curve $y = 4x^2 + x - 2$ at which the gradient is 17.

7 Find the coordinates of the points on the curve $y = 4x^3 - 3x^2 + 2x + 1$ at which the gradient is 8.

8 The gradient of the tangent to the curve $y = ax^3 + bx$ at $(1, 2)$ is 4. Find the values of a and b.

9 Given that the gradient of the curve $y = \dfrac{a}{x^2} + bx$ at the point $(2, -13)$ is -8. Find the value of a and the value of b.

10 The tangent to the curve $y = \frac{p}{x} + qx$ at $(3, 13)$ is parallel to the line $y = \frac{11}{3}x + 4$. Calculate the value of p and of q.

11 The equation of a curve is $y = 4x + \frac{1}{x}$.

(a) Find the gradient of the curve when $x = 1$.

(b) Find the coordinates of the points where the gradient is 0.

Chain rule

The chain rule is used to differentiate composite functions. This is the rule:

If $y = gf(x)$ and $u = f(x)$, then:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

When using the chain rule, we are differentiating a composite function. With this composite function we have one function inside of the other. The rule above is read as 'the differential of the function outside multiplied by the differential of the function inside'. That is:

If $y = g(f(x))$, then $\frac{dy}{dx} = g'(f(x)) \times f'(x)$. (Recall that $f(x)$ is inside)

EXAMPLE 22 Given that $y = (4x + 3)^8$, find $\frac{dy}{dx}$.

SOLUTION

Let $u = 4x + 3$, $y = u^8$.

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, we have:

$\frac{dy}{du} = 8u^7$ and $\frac{du}{dx} = 4$

Therefore, $\frac{dy}{dx} = 8u^7 \times 4$

$= 32u^7$

Replacing $u = 4x + 3$ gives:

$\frac{dy}{dx} = 32(4x + 3)^7$

outside

Short cut:

$y = (\underbrace{4x + 3}_{\text{inside}})^8$

$\frac{dy}{dx} = 8(4x + 3)^7 \times 4 = 32(4x + 3)^7$

Bring down the power; reduce the power by 1; and multiply by the differential of the function inside.

EXAMPLE 23 Find the rate of change of y with respect to x, given that $y = \frac{1}{\sqrt{7x - 3}}$.

SOLUTION

$y = \frac{1}{\sqrt{7x - 3}}$

$\Rightarrow y = (7x - 3)^{-\frac{1}{2}}$

Let $u = 7x - 3$

$\Rightarrow y = u^{-\frac{1}{2}}$

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$:

$\frac{dy}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$ and $\frac{du}{dx} = 7$

Therefore, $\frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \times 7$

$= -\frac{7}{2}u^{-\frac{3}{2}}$

Substituting $u = 7x - 3$ gives:

$\frac{dy}{dx} = -\frac{7}{2}(7x - 3)^{-\frac{3}{2}}$

Short cut:

$y = (7x - 3)^{-\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{1}{2}(7x - 3)^{-\frac{3}{2}} \times 7$ (Bring down the power, reduce the power by 1 and multiply by the differential of the inside of the brackets.)

$= -\frac{7}{2}(7x - 3)^{-\frac{3}{2}}$

EXAMPLE 24 Find the gradient function of $y = \frac{4}{(1 - 5x)^6}$.

SOLUTION The gradient of the function is $\frac{dy}{dx}$. Writing in index form:

$y = 4(1 - 5x)^{-6}$

Let $u = 1 - 5x$.

$\Rightarrow y = 4u^{-6}$

$\frac{dy}{du} = -24u^{-7}$ and $\frac{du}{dx} = -5$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= -24u^{-7} \times -5$

$= 120u^{-7}$

Replacing $u = 1 - 5x$ gives:

$\frac{dy}{du} = 120(1 - 5x)^{-7}$

$= \frac{120}{(1 - 5x)^7}$

Short cut:

$y = 4(1 - 5x)^{-6}$

$\frac{dy}{dx} = 4 \times (-6) \times (1 - 5x)^{-7} \times (-5)$

$= \frac{120}{(1 - 5x)^7}$

EXAMPLE 25 Find the gradient of the curve $y = (4x + 1)^{\frac{2}{3}}$, when $x = 2$.

SOLUTION We need to find $\frac{dy}{dx}$ when $x = 2$.

Let $u = 4x + 1$.

$\Rightarrow y = u^{\frac{2}{3}}$

$\frac{du}{dx} = 4$ and $\frac{dy}{du} = \frac{2}{3}u^{-\frac{1}{3}}$

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{2}{3}u^{-\frac{1}{3}} \times 4$$
$$= \frac{8}{3}u^{-\frac{1}{3}}$$

Substituting $u = 4x + 1$ gives:

$$\frac{dy}{dx} = \frac{8}{3}(4x + 1)^{-\frac{1}{3}}$$

Substituting $x = 2$, we have $\frac{dy}{dx} = \frac{8}{3}(4(2) + 1)^{\frac{-1}{3}} = 1.282$

You can try the shortcut on this example.

EXAMPLE 26 Find the value of $\frac{dy}{dx}$ at $x = 1$, when $y = (4 - 5x)^{10}$.

SOLUTION Let $u = 4 - 5x$.

$\Rightarrow y = u^{10}$

$\frac{dy}{du} = 10u^9$ and $\frac{du}{dx} = -5$

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 10u^9 \times (-5)$$
$$= -50u^9$$

Replacing $u = 4 - 5x$ gives:

$$\frac{dy}{dx} = -50(4 - 5x)^9$$

When $x = 1$, $\frac{dy}{dx} = -50(4 - 5)^9$

$$= -50(-1)^9$$
$$= 50$$

EXAMPLE 27 Given that $y = (3x^2 + 2x + 1)^{\frac{1}{2}}$, find $\frac{dy}{dx}$.

SOLUTION Let $u = 3x^2 + 2x + 1$

$\Rightarrow y = u^{\frac{1}{2}}$

$\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$ and $\frac{du}{dx} = 6x + 2$

Since $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{1}{2}u^{-\frac{1}{2}} \times (6x + 2)$$

Substituting $u = 3x^2 + 2x + 1$ gives:

$$\frac{dy}{dx} = \frac{1}{2}(6x + 2)(3x^2 + 2x + 1)^{-\frac{1}{2}}$$
$$= (3x + 1)(3x^2 + 2x + 1)^{-\frac{1}{2}}$$

Try these 13.7

(a) Find the derivative of these.

(i) $y = (6x + 2)^7$

(ii) $y = (3 - 2x)^{\frac{3}{4}}$

(b) Find the gradient of the curve $y = \frac{4}{\sqrt{3x - 2}}$ at $x = 6$.

Product rule

If $y = uv$, where u and v are two functions of x, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

This is known as the product rule for differentiation. We use this rule to differentiate functions that are multiplied by each other.

EXAMPLE 28 Given that $y = (x + 2)^2x^3$, find $\dfrac{dy}{dx}$.

SOLUTION Since y consists of two functions of x multiplied by each other, we use the product rule with:

$u = (x + 2)^2$, $v = x^3$

To differentiate $(x + 2)^2$, we use the chain rule:

$$\begin{aligned}\frac{d}{dx}[(x + 2)^2] &= 2(x + 2) \times (1)\\ &= 2(x + 2)\end{aligned}$$

We need $\dfrac{du}{dx} = 2(x + 2)$ and $\dfrac{dv}{dx} = 3x^2$

Substituting into $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ gives:

$$\begin{aligned}\frac{dy}{dx} &= (x + 2)^2(3x^2) + x^3(2(x + 2))\\ &= x^2(x + 2)(3(x + 2) + 2x)\\ &= x^2(x + 2)(5x + 6)\end{aligned}$$

EXAMPLE 29 Find the differential of $y = 4x(x^2 + 2)^5$.

SOLUTION Using the product rule with:

$u = 4x$, $v = (x^2 + 2)^5$

$\dfrac{du}{dx} = 4$

To differentiate $v = (x^2 + 2)^5$, use the chain rule.

Let $w = x^2 + 2$.

$\Rightarrow v = w^5$

$\dfrac{dv}{dw} = 5w^4$ and $\dfrac{dw}{dx} = 2x$

Therefore, $\dfrac{dv}{dx} = \dfrac{dv}{dw} \times \dfrac{dw}{dx} = 5w^4 \times 2x = 10xw^4$

Substituting $w = x^2 + 2$ gives:

$\dfrac{dv}{dx} = 10x(x^2 + 2)^4$

The short cut for the chain rule comes in very useful in this example:

$$\begin{aligned}\frac{d}{dx}[(x^2 + 2)^5] &= 5(x^2 + 2)^4(2x)\\ &= 10x(x^2 + 2)^4\end{aligned}$$

Bring down the power. Reduce the power by 1. Multiply by the differential of the function inside.

Substituting into $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ gives:

$$\frac{dy}{dx} = 4x(10x)(x^2 + 2)^4 + (x^2 + 2)^5(4)$$
$$= 4(x^2 + 2)^4(10x^2 + x^2 + 2) \qquad \text{(Factorising)}$$
$$= 4(x^2 + 2)^4(11x^2 + 2)$$

EXAMPLE 30 Find the gradient of curve $y = (x + 1)\sqrt{2x + 5}$ at the point $x = 2$.

SOLUTION

> **Short cut**
>
> $v = (2x + 5)^{\frac{1}{2}}$
>
> $\frac{dv}{dx} = \frac{1}{2}(2x + 5)^{-\frac{1}{2}} \times 2$
>
> $= (2x + 5)^{\frac{1}{2}}$
>
> $= \frac{1}{\sqrt{2x + 5}}$

Let $u = x + 1$, $v = \sqrt{2x + 5}$

$$\frac{du}{dx} = 1$$

$v = (2x + 5)^{\frac{1}{2}}$ and let $w = 2x + 5$.

$$\Rightarrow v = w^{\frac{1}{2}}$$

$$\frac{dv}{dw} = \frac{1}{2}w^{-\frac{1}{2}}, \frac{dw}{dx} = 2$$

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} = \frac{1}{2}w^{-\frac{1}{2}} \times 2 = w^{-\frac{1}{2}}$$

Substituting $w = 2x + 5$ gives:

$$\frac{dv}{dx} = (2x + 5)^{-\frac{1}{2}}$$

Substituting into $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ gives:

$$\frac{dy}{dx} = (x + 1)(2x + 5)^{-\frac{1}{2}} + (2x + 5)^{\frac{1}{2}}(1)$$

Substituting $x = 2$ gives:

$$\frac{dy}{dx} = (2 + 1)(4 + 5)^{-\frac{1}{2}} + (4 + 5)^{\frac{1}{2}}$$
$$= \frac{3}{\sqrt{9}} + \sqrt{9}$$
$$= \frac{3}{3} + 3$$
$$= 1 + 3$$
$$= 4$$

$\therefore$ when $x = 2$, $\frac{dy}{dx} = 4$.

Try these 13.8 Differentiate the following functions with respect to x.

(a) $y = x^2(3x + 2)^5$

(b) $y = (4x + 1)(\sqrt{6x - 2})$

(c) $y = (4x^2 + 6x + 1)^5(6x + 2)$

Quotient rule

If $y = \frac{u}{v}$, where u and v are two functions of x, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

This is the quotient rule for differentiation. We use this rule to differentiate functions of x that are divided by each other.

EXAMPLE 31 Given that $f(x) = \frac{x + 1}{2x + 3}$, find $f^{(1)}(x)$.

SOLUTION Using the quotient rule with $u = x + 1$, $v = 2x + 3$ gives:

$\frac{du}{dx} = 1, \frac{dv}{dx} = 2$

$$f^{(1)}(x) = \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2x + 3)(1) - (x + 1)(2)}{(2x + 3)^2}$$

$$= \frac{2x + 3 - 2x - 2}{(2x + 3)^2}$$

$$= \frac{1}{(2x + 3)^2}$$

EXAMPLE 32 Find the gradient of the curve $y = \frac{x}{x^2 + 1}$ at the point $x = 1$.

SOLUTION $y = \frac{x}{x^2 + 1}$

Using the quotient rule with $u = x$ and $v = x^2 + 1$ gives:

$\frac{du}{dx} = 1, \frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 + 1)(1) - (x)(2x)}{(x^2 + 1)^2}$$

$$= \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}$$

$$= \frac{1 - x^2}{(x^2 + 1)^2}$$

When $x = 1$, $\frac{dy}{dx} = \frac{1 - 1^2}{(1^2 + 1)^2}$

$$= \frac{0}{2^2}$$

$$= 0$$

$\therefore$ when $x = 1$, $\frac{dy}{dx} = 0$.

EXAMPLE 33 Find the gradient function of $y = \frac{4x + 5}{3x^2 + 2}$.

SOLUTION Let $u = 4x + 5$, $v = 3x^2 + 2$.

$\frac{du}{dx} = 4, \frac{dv}{dx} = 6x$

Quotient rule gives $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{(3x^2 + 2)4 - (4x + 5)(6x)}{(3x^2 + 2)^2}$$

$$= \frac{12x^2 + 8 - 24x^2 - 30x}{(3x^2 + 2)^2}$$

$$= \frac{-12x^2 - 30x + 8}{(3x^2 + 2)^2}$$

EXAMPLE 34 Find the value of a for which $\frac{dy}{dx} = \frac{a}{(3x + 2)^2}$ where $y = \frac{2x + 1}{3x + 2}$.

SOLUTION Using the quotient rule with $u = 2x + 1$ and $v = 3x + 2$ gives:

$\frac{du}{dx} = 2, \frac{dv}{dx} = 3$

Substituting into $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ gives:

$$\frac{dy}{dx} = \frac{(3x + 2)(2) - (2x + 1)(3)}{(3x + 2)^2}$$

$$= \frac{6x + 4 - 6x - 3}{(3x + 2)^2}$$

$$= \frac{1}{(3x + 2)^2}$$

Comparing with $\frac{dy}{dx} = \frac{a}{(3x + 2)^2}$ gives:

$a = 1$

Try these 13.9

(a) Find the derivatives of these.

(i) $y = \frac{6x + 2}{x^2 + 1}$

(ii) $y = \frac{3x^2 - 2}{x^3 + 4}$

(b) Find the gradient of the curve $y = \frac{x + 1}{\sqrt{x - 2}}$ at $x = 11$.

Differentiation of trigonometric functions

EXAMPLE 35 Given that $f(x) = \sin x$, show from first principles that $f^{(1)}(x) = \cos x$.

SOLUTION By definition $f^{(1)}(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$

Since $f(x) = \sin x$

$f(x + h) = \sin(x + h)$

Therefore $f^{(1)}(x) = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h}$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$ (Using $\sin(x + h) = \sin x \cos h + \cos x \sin h$)

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sin x \cos h - \sin x}{h} + \frac{\cos x \sin h}{h}\right)$$

$$= \lim_{h \to 0} \frac{\sin x \cos h - \sin x}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$
$$= \lim_{h \to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x(\sin h)}{h}$$
$$= \sin x \lim_{h \to 0} \frac{\cos(h - 1)}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

Recall that $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

Therefore, $f^{(1)}(x) = (\sin x)(0) + (\cos x)(1)$
$$= \cos x$$

Therefore, when $f(x) = \sin x$, $f^{(1)}(x) = \cos x$.

Try this 13.10 Prove that when $y = \cos x$, $\frac{dy}{dx} = -\sin x$.

EXAMPLE 36 Given that $y = \tan x$, show that $\frac{dy}{dx} = \sec^2 x$.

SOLUTION Since $\tan x = \frac{\sin x}{\cos x}$

$y = \frac{\sin x}{\cos x}$

Using the quotient rule with $u = \sin x$ and $v = \cos x$ gives:

$\frac{du}{dx} = \cos x$, $\frac{dv}{dx} = -\sin x$

Substituting into $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ gives:

$$\frac{dy}{dx} = \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} \qquad (\text{Since } \cos^2 x + \sin^2 x = 1)$$
$$= \sec^2 x$$

EXAMPLE 37 Given that $y = \sec x$, show that $\frac{dy}{dx} = \sec x \tan x$.

SOLUTION Since $y = \sec x$, $y = \frac{1}{\cos x}$.

$$y = (\cos x)^{-1}$$

Using the chain rule, $u = \cos x$ and $y = u^{-1}$.

$\frac{du}{dx} = -\sin x$, $\frac{dy}{du} = -u^{-2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -u^{-2} \times -\sin x$$
$$= \frac{\sin x}{u^2}$$

Substituting $u = \cos x$ gives:

$$\frac{dy}{dx} = \frac{\sin x}{\cos^2 x}$$
$$= \frac{\sin x}{\cos x} \times \frac{1}{\cos x}$$
$$= \tan x \sec x$$

Try these 13.11

(a) Given that $y = \cot x$, show that $\frac{dy}{dx} = -\text{cosec}^2 x$.

(b) Given that $y = \text{cosec}\, x$, show that $\frac{dy}{dx} = -\text{cosec}\, x \cot x$.

Standard derivatives of trigonometric functions

Function	Derivative
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\text{cosec}\, x$	$-\text{cosec}\, x \cot x$
$\cot x$	$-\text{cosec}^2 x$
$\sin(ax + b)$	$a\cos(ax + b)$, where a and b are constants
$\cos(ax + b)$	$-a\sin(ax + b)$, where a and b are constants
$\tan(ax + b)$	$a\sec^2(ax + b)$, where a and b are constants
$\sec(ax + b)$	$a\sec(ax + b)\tan(ax + b)$, where a and b are constants
$\text{cosec}(ax + b)$	$-a\,\text{cosec}(ax + b)\cot(ax + b)$, where a and b are constants
$\cot(ax + b)$	$-a\,\text{cosec}^2(ax + b)$, where a and b are constants

EXAMPLE 38 Find the derivative of $y = \sin\left(2x + \frac{\pi}{4}\right)$.

SOLUTION This function is of the form $\sin(ax + b)$ where $a = 2$ and $b = \frac{\pi}{4}$.

Therefore, $\frac{dy}{dx} = 2\cos\left(2x + \frac{\pi}{4}\right)$

We can also use the chain rule.

Let $u = 2x + \frac{\pi}{4}$.

$\Rightarrow y = \sin u$

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \cos u \times 2$$
$$= \cos\left(2x + \frac{\pi}{4}\right) \times 2 \qquad \left(\text{Replacing } u = 2x + \frac{\pi}{4}\right)$$
$$= 2\cos\left(2x + \frac{\pi}{4}\right)$$

EXAMPLE 39 Given that $y = \cos\left(4x - \frac{\pi}{4}\right)$, find $\frac{dy}{dx}$.

SOLUTION

Let $u = 4x - \frac{\pi}{4}$

$\Rightarrow y = \cos u$

Using the chain rule and $\frac{du}{dx} = 4$ and $\frac{dy}{du} = -\sin u$ gives:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\sin u \times 4$$
$$= -4\sin u$$
$$= -4\sin\left(4x - \frac{\pi}{4}\right) \quad \left(\text{Replacing } u = 4x - \frac{\pi}{4}\right)$$

EXAMPLE 40 Find the gradient function of $y = \sin(4x^2 + 2)$.

SOLUTION

This is a function of a function, so we use the chain rule.

Let $u = 4x^2 + 2$.

$\Rightarrow y = \sin u$

$\frac{du}{dx} = 8x$ and $\frac{dy}{du} = \cos u$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \cos u \times 8x$$
$$= 8x\cos u \quad \left(\text{Replacing } u = 4x^2 + 2\right)$$
$$= 8x\cos(4x^2 + 2)$$

Hence, the gradient function is $8x\cos(4x^2 + 2)$

EXAMPLE 41 Given that $f(x) = x^2\tan x$, find $f^{(1)}(x)$.

SOLUTION

> **Remember**
>
> $x^2 \tan x$ is the product of two functions of x.

Use the product rule with $u = x^2$ and $v = \tan x$.

$\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = \sec^2 x$

Substituting into $f^{(1)}(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ gives:

$$f^{(1)}(x) = x^2\sec^2 x + 2x\tan x$$

EXAMPLE 42 Find the gradient of curve $y = \frac{x+2}{\cos\left(x - \frac{\pi}{4}\right)}$ at $x = \frac{\pi}{4}$.

SOLUTION

Using the quotient rule with $u = x + 2$ and $v = \cos\left(x - \frac{\pi}{4}\right)$ gives:

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\sin\left(x - \frac{\pi}{4}\right)$

Substituting into $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ gives:

$$\frac{dy}{dx} = \frac{\cos\left(x - \frac{\pi}{4}\right) - (x+2)\left(-\sin\left(x - \frac{\pi}{4}\right)\right)}{\left(\cos\left(x - \frac{\pi}{4}\right)\right)^2}$$

$$= \frac{\cos\left(x - \frac{\pi}{4}\right) + (x + 2)\sin\left(x - \frac{\pi}{4}\right)}{\cos^2\left(x - \frac{\pi}{4}\right)}$$

Substituting $x = \frac{\pi}{4}$ gives:

$$\frac{dy}{dx} = \frac{\left(\cos\left(\frac{\pi}{4} - \frac{\pi}{4}\right)\right) + \left(\frac{\pi}{4} + 2\right)\sin\left(\frac{\pi}{4} - \frac{\pi}{4}\right)}{\cos^2\left(\frac{\pi}{4} - \frac{\pi}{4}\right)}$$

$$= \frac{\cos 0}{\cos^2 0}$$

$$= 1$$

EXAMPLE 43 The equation of a curve is given by $y = 4x \sin 2x$. Find the gradient of the curve at $x = a$.

SOLUTION Using the product rule with $u = 4x$ and $v = \sin 2x$ gives:

$\frac{du}{dx} = 4$ and $\frac{dv}{dx} = 2\cos 2x$

Substituting into

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 4x \times (2\cos 2x) + (\sin 2x) \times 4$$

$$= 8x\cos 2x + 4\sin 2x$$

When $x = a$, $\frac{dy}{dx} = 8a\cos 2a + 4\sin 2a$.

EXAMPLE 44 If $f(x) = \cos x \tan 2x$, find $f^{(1)}(x)$ and evaluate $f^{(1)}\left(\frac{\pi}{2}\right)$.

SOLUTION Using the product rule with $u = \cos x$ and $v = \tan 2x$ gives:

$\frac{du}{dx} = -\sin x$ and $\frac{dv}{dx} = 2\sec^2 2x$

Substituting into $f^{(1)}(x) = u\frac{dv}{dx} + v\frac{du}{dx}$ gives:

$$f^{(1)}(x) = \cos x \times 2\sec^2 2x + \tan 2x \times -\sin x$$

$$= 2\cos x \sec^2 2x - \sin x \tan 2x$$

When $x = \frac{\pi}{2}$,

$$f^{(1)}\left(\frac{\pi}{2}\right) = 2\cos\frac{\pi}{2} \times \frac{1}{\cos^2(2)\left(\frac{\pi}{2}\right)} - \sin\left(\frac{\pi}{2}\right)\tan 2\left(\frac{\pi}{2}\right)$$

$$= 0 - 0$$

$$= 0$$

EXAMPLE 45 Differentiate this with respect to x: $y = \sin^3 x$.

SOLUTION $y = (\sin x)^3$

Let $u = \sin x$ and $y = u^3$.

$\frac{du}{dx} = \cos x$, $\frac{dy}{dx} = 3u^2$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3u^2 \cos x$$

Substituting $u = \sin x$ gives:

$$\frac{dy}{dx} = 3\sin^2 x\cos x$$

Try these 13.12 Differentiate the following with respect to x.

(a) $y = \sin 4x$

(b) $y = \cos(4x + \pi)$

(c) $y = x^2\cos\left(3x - \frac{\pi}{2}\right)$

(d) $y = x\cot x$

(e) $y = x^2\sec x$

EXAMPLE 46 Given that $y = \tan^4 x$, find $\frac{dy}{dx}$.

SOLUTION Let $u = \tan x$ and $y = u^4$.

$$\frac{dy}{du} = 4u^3 \text{ and } \frac{du}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 4u^3 \times \sec^2 x$$

$$= 4(\tan x)^3\sec^2 x$$

$$= 4\tan^3 x\sec^2 x$$

Short cut

$y = \tan^4 x$

$\frac{dy}{dx} = 4(\tan^3 x) \times \sec^2 x$

$= 4\tan^3 x\sec^2 x$

EXAMPLE 47 Find the gradient of the tangent to the curve $y = x^3\cos^2 3x$.

SOLUTION Let $u = x^3$, $v = \cos^2 3x$ and $w = \cos 3x$.

$\Rightarrow v = w^2$

$$\frac{du}{dx} = 3x^2$$

$$\frac{dv}{dw} = 2w$$

$$\frac{dw}{dx} = -3\sin 3x$$

Short cut

$v = \cos^2 3x$

$\frac{dv}{dx} = 2(\cos 3x) \times (-3\sin 2x)$

$= -6\sin 3x\cos 3x$

If you use the shortcut when applying the chain rule, you will finish much faster.

First, we use the chain rule to find the differential of $\cos^2 3x$:

$$\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx}$$

$$\frac{dv}{dx} = 2w \times -3\sin 3x$$

$$= -6w\sin 3x$$

$$= -6\cos 3x\sin 3x \qquad (\text{Since } w = \cos 3x)$$

Using the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x^3 \times -6\cos 3x \sin 3x + \cos^2 3x \times 3x^2$$
$$= -6x^3 \cos 3x \sin 3x + 3x^2 \cos^2 3x$$
$$= 3x^2 \cos 3x(-2x \sin 3x + \cos 3x)$$

EXERCISE 13B

1 Differentiate the following functions with respect to x.

(a) $y = (6x + 1)^5$

(b) $y = (4 - 3x)^6$

(c) $y = \sqrt{2x + 9}$

(d) $y = \dfrac{4}{\sqrt{5x + 1}}$

(e) $y = (6x^2 + 3x + 1)^4$

(f) $y = (6x^3 + 5x)^{\frac{1}{4}}$

(g) $y = \dfrac{3}{\sqrt{7x^2 - 5}}$

(h) $y = (4\sqrt{x} + 5)^{10}$

(i) $y = (7x^{\frac{1}{2}} + 3x^5)^7$

(j) $y = (x^3 - 5x + 2)^{\frac{3}{4}}$

2 Differentiate the following functions with respect to t.

(a) $\theta = (3t + 1)(4t + 2)^8$

(b) $\theta = t^2(7t + 1)^3$

(c) $\theta = 4t\sqrt{2t - 1}$

(d) $\theta = 6t^2\sqrt{t^3 + 2t}$

(e) $\theta = (5t^2 + 2)(7 - 3t)^4$

(f) $\theta = t^3(4t^3 + 3t^2 + 1)^{\frac{1}{4}}$

(g) $\theta = (7t^{\frac{1}{2}} + 3t)\left(\dfrac{1}{\sqrt{3t^2 + 1}}\right)$

(h) $\theta = \sqrt{t}(t + 5)^{\frac{3}{4}}$

3 Find the rate of change of y with respect to x for each of the following.

(a) $y = \dfrac{4x + 2}{x - 1}$

(b) $y = \dfrac{3x - 5}{6x + 2}$

(c) $y = \dfrac{x^2 + 1}{2x + 5}$

(d) $y = \dfrac{3x + 2}{\sqrt{4x - 1}}$

(e) $y = \dfrac{x^3 - 2x + 1}{4x^2 + 5}$

(f) $y = \dfrac{x^4 - 2}{\sqrt{3x + 1}}$

(g) $y = \dfrac{x}{(x + 3)^3}$

(h) $y = \dfrac{x^4 + 2x}{x^2 - 7}$

(i) $y = \dfrac{(3x + 2)^2}{(2x - 1)^3}$

(j) $y = \dfrac{7x + 3}{5x - 1}$

4 Find the gradient of the tangents to the curves at the given points.

(a) $y = (2x - 1)^5$ at $(1, 1)$

(b) $y = x^2(x + 1)^3$ at $(0, 0)$

(c) $y = \dfrac{x^2 + 1}{x + 2}$ at $\left(2, \dfrac{5}{4}\right)$

(d) $y = (x^2 - 5x + 2)^4$ at $(0, 16)$

5 Differentiate the following with respect to x.

(a) $\sin 4x$

(b) $\sin 6x$

(c) $\cos 3x$

(d) $\cos 7x$

(e) $\cos 9x$

(f) $\sin\left(3x + \dfrac{\pi}{4}\right)$

(g) $\sin\left(\dfrac{\pi}{2} - 4x\right)$

(h) $\sin\left(\dfrac{3\pi}{2} + 4x\right)$

(i) $\sin(8x + 2)$

(j) $\cos(3x - \pi)$

(k) $\cos(5 - 7x)$

(l) $\cos(2\pi - 9x)$

6 Find the derivatives of the following functions.

(a) $\tan 2x$
(b) $\tan 5x$
(c) $\tan(2x + \pi)$
(d) $\tan\left(3x - \frac{\pi}{2}\right)$
(e) $\sec 4x$
(f) $\sec(4x + 3)$
(g) $\cot(6x - \pi)$
(h) $\cot\left(\frac{3}{4}x - \frac{\pi}{4}\right)$
(i) $\operatorname{cosec}\left(x - \frac{\pi}{4}\right)$
(j) $\operatorname{cosec}(7x - 4)$

7 Find the rate of change of each of the following.

(a) $y = \sin x^2$
(b) $\theta = \sin(t^2 + 3)$
(c) $\theta = \cos(4x^2 + \pi)$
(d) $y = \tan(7x^3 - 8)$
(e) $v = 8\tan(3x^3 - 4x + 5)$
(f) $y = (7x + 5)^{10}$
(g) $y = \dfrac{1}{(4x + 2)^3}$
(h) $y = \dfrac{8}{(7x^2 + 5x + 1)^6}$
(i) $x = \sqrt{t^3 - \dfrac{1}{4t^2}}$
(j) $y = \tan(5x + 1)^6$
(k) $y = 5\cos\left(6x^2 + \frac{1}{x}\right)$
(l) $y = \sec(x^3 + 5)$

8 Find the gradient function for each of the following.

(a) $y = x\sin x$
(b) $y = x\cos x$
(c) $y = x^2\tan x$
(d) $y = x^3\tan(3x + 2)$
(e) $y = (4x + 1)\sin 4x$
(f) $y = x\tan x^2$
(g) $y = \sin 2x\tan 2x$
(h) $y = (3x^2 + 1)\cos x$
(i) $y = \left(\frac{1}{3}x^3 - 2x\right)\sec x$
(j) $y = \left(\frac{4}{x} + 2x\right)\cot 4x$
(k) $y = \left(\dfrac{6}{x^2} - 3x + 2\right)\operatorname{cosec} x$

9 Find $\dfrac{dy}{dx}$ for the following functions of x.

(a) $y = \dfrac{x^2}{x + 2}$
(b) $y = \dfrac{x^3 + 4}{x^2 - 6x + 1}$
(c) $y = \dfrac{\sin x}{\cos x + 2}$
(d) $y = \dfrac{\tan x + 1}{\sec x + 3}$
(e) $y = \dfrac{x\cos x}{x^2 + 5}$
(f) $y = \dfrac{7x}{\sqrt{x + 1}}$
(g) $y = \dfrac{x + \sin x}{x + \cos x}$
(h) $y = \dfrac{\cos(3x + 2)}{4x^3 + 2}$
(i) $y = \dfrac{7x^2 + 2}{x - 4}$
(j) $y = \dfrac{x^4}{x^2 - 1}$

10 Find the first derivative of the following functions.

(a) $y = \cos^3 x$
(b) $y = \sin^3 x$
(c) $y = \tan^4 x$
(d) $y = x^2\cos^4 x$
(e) $y = \sec^3(x + 2)$
(f) $y = \tan^2(3x + 2)$

(g) $y = (2x + 1)\cot^2 x$

(h) $y = (4x^2 - x)\sin^2(x + 2)$

(i) $y = \dfrac{x}{\operatorname{cosec}^2 x + 1}$

(j) $y = \tan^4(3x)$

Higher derivatives

The second derivative is the derivative of the first derivative. The notation used for the second derivative is $f''(x)$, $\dfrac{d^2y}{dx^2}$ or $f^{(2)}(x)$.

Let $f(x) = 6x^2 + 4x - 2$. Then the first derivative is $f'(x) = 12x + 4$.

Differentiating again we get $f''(x) = 12$, which is the second derivative of $f(x)$ with respect to x. We can also find the third derivative in the same way and the notation used is $f'''(x)$, $\dfrac{d^3y}{dx^3}$, y''' or $f^3(x)$.

EXAMPLE 48 Find the second derivative of the function $f(x) = 4x^3 - 3x^2 + 2x + 5$.

SOLUTION

$y = 4x^3 - 3x^2 + 2x + 5$

Differentiating y with respect to x gives:

$\dfrac{dy}{dx} = 12x^2 - 6x + 2$

Differentiating $\dfrac{dy}{dx}$ with respect to x gives:

$\dfrac{d^2y}{dx^2} = 24x - 6$

EXAMPLE 49 Given that $y = x^2\cos x$, find $\dfrac{d^2y}{dx^2}$.

SOLUTION

$y = x^2\cos x$

Using the product rule and $u = x^2$ and $v = \cos x$ gives:

$\dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = -\sin x$

Substituting into $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$, we get:

$$\begin{aligned}\frac{dy}{dx} &= x^2(-\sin x) + (\cos x) \times (2x)\\ &= -x^2\sin x + 2x\cos x\end{aligned}$$

Differentiating again with respect to x, using the product rule on each term:

For $-x^2\sin x$, let $u = -x^2$ and $v = \sin x$.

$\dfrac{du}{dx} = -2x$ and $\dfrac{dv}{dx} = \cos x$

$$\begin{aligned}\frac{d(-x^2\sin x)}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx}\\ &= -x^2\cos x + (\sin x) \times (-2x)\\ &= -x^2\cos x - 2x\sin x\end{aligned}$$

For $2x\cos x$, let $u = 2x$ and $v = \cos x$.

$\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = -\sin x$

$$\frac{d(x\cos x)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 2x \times (-\sin x) + (\cos x) \times (2)$$
$$= -2x\sin x + 2\cos x$$

Therefore, $\dfrac{d^2y}{dx^2} = (-x^2\cos x - 2x\sin x) + (-2x\sin x + 2\cos x)$

$$= -x^2\cos x - 4x\sin x + 2\cos x$$

EXAMPLE 50 Given that $y = \dfrac{x}{x+2}$, show that $x^2\dfrac{d^2y}{dx^2} + (2x - 4y)\dfrac{dy}{dx} = 0$.

SOLUTION Using the quotient rule to find $\dfrac{dy}{dx}$, let $u = x$ and $v = x + 2$:

$$\frac{dv}{dx} = 1 \text{ and } \frac{du}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(x+2) \times 1 - x \times 1}{(x+2)^2}$$
$$= \frac{x + 2 - x}{(x+2)^2}$$
$$= \frac{2}{(x+2)^2}$$

Using the chain rule, let $w = x + 2$.

$$\Rightarrow \frac{dy}{dw} = 2w^{-2}$$
$$\frac{d^2y}{dw^2} = -4w^{-3}$$
$$\Rightarrow \frac{d^2y}{dx^2} = -4(x+2)^{-3}$$
$$= \frac{-4}{(x+2)^3}$$

Hence, $x^2\dfrac{d^2y}{dx^2} = \dfrac{-4x^2}{(x+2)^3}$. (Multiplying both sides by x^2)

Now, $(2x - 4y)\dfrac{dy}{dx} = \left(2x - \dfrac{4x}{x+2}\right)\left(\dfrac{2}{(x+2)^2}\right)$

$$= \left(\frac{2x(x+2) - 4x}{(x+2)}\right)\left(\frac{2}{(x+2)^2}\right)$$
$$= \frac{4x(x+2) - 8x}{(x+2)^3}$$
$$= \frac{4x^2}{(x+2)^3}$$

Therefore, $x^2\dfrac{d^2y}{dx^2} + (2x - 4y)\dfrac{dy}{dx} = \dfrac{-4x^2}{(x+2)^3} + \dfrac{4x^2}{(x+2)^3} = 0$.

EXAMPLE 51 Find the value of $\dfrac{d^2y}{dx^2}$ at the point $x = \dfrac{\pi}{2}$, where $y = x\sin x$.

SOLUTION Using the product rule, $u = x$ and $v = \sin x$, gives:

$$\frac{du}{dx} = 1 \text{ and } \frac{dv}{dx} = \cos x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x\cos x + \sin x$$

Using the product rule to differentiate $x\cos x$, and let $u = x$ and $v = \cos x$, gives:

$\frac{du}{dx} = 1$ and $\frac{dv}{dx} = -\sin x$

$$\frac{d^2y}{dx^2} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= x \times -\sin x + \cos x \times 1 + \cos x$$
$$= -x\sin x + 2\cos x$$

When, $x = \frac{\pi}{2}$,

$$\frac{d^2y}{dx^2} = -\frac{\pi}{2} \times \sin\frac{\pi}{2} + 2 \times \cos\frac{\pi}{2}$$
$$= -\frac{\pi}{2} \times 1 + 2 \times 0 = -\frac{\pi}{2}$$

Try these 13.13

(a) Find $\frac{d^2y}{dx^2}$ where $y = (2x + 1)\cos x^2$.

(b) Given that $y = \frac{x + 2}{2x - 1}$, find $\frac{d^2y}{dx^2}$ when $x = 0$.

EXERCISE 13C

In questions **1** to **10**, find $\frac{d^2y}{dx^2}$ for each of these functions.

1 $y = \sqrt{4x + 7}$

2 $y = (2x + 3)^{10}$

3 $y = \frac{1}{(5x - 3)^3}$

4 $y = (x + 2)\sin x$

5 $y = \cos x^2$

6 $y = \frac{x^2 + 2}{x + 1}$

7 $y = \sin(2x^2 + 5x + 1)$

8 $y = x^3\cos^2 x$

9 $y = \cos 3x + \sin 4x$

10 $y = \frac{3x - 2}{x - 4}$

11 If $y = \frac{3\cos 2x}{x}$, prove that $x\frac{d^2y}{dx^2} + \frac{2dy}{dx} + 4xy = 0$.

12 Find, in terms of x, both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $y = \sqrt{x^2 + 1}$.

13 If $y = 2x^2 + \frac{4}{x}$, prove that $\frac{x^2d^2y}{dx^2} = 2y$.

14 If $y = \frac{x + 2}{x - 1}$, prove that $\frac{3d^2y}{dx^2} - 2(x - 1)\left(\frac{dy}{dx}\right)^2 = 0$.

15 Given that $y = \cos x + \sin x$, show that $\frac{d^2y}{dx^2} + y = 0$.

16 If $y = x^2\cos^2 x$, find $\frac{d^2y}{dx^2}$ when $x = \frac{\pi}{2}$.

17 Given that $y = \frac{x^2 + 1}{x^2 - 1}$, find $\frac{d^2y}{dx^2}$ when $x = 0$.

18 Prove that $(x + 4)\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$, if $y = \frac{2x}{x + 4}$.

19 If $y = \frac{1}{x(x - 1)}$, show that $\frac{d^2y}{dx^2}$ is $\frac{7}{4}$ when $x = 2$.

20 If $y = \frac{1}{\sqrt{1 + x^2}}$, prove that $(1 + x^2)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0$.

SUMMARY

Differentiation

Notation for differentiation

$f'(x)$, $\frac{dy}{dx}$, $f'(x), y'$, $\frac{dy}{dx}(f(x))$

Definition:

$f'(x) = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$\frac{d}{dx}(x^n) = nx^{n-1}$

$\frac{d}{dx}(ax^n) = anx^{n-1}$, where a is a constant

$\frac{d}{dx}((ax+b)^n) = an(ax+b)^{n-1}$, where a is a constant

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$

$\frac{d}{dx}(\mathrm{cosec}\, x) = -\mathrm{cosec}\, x \cot x$

$\frac{d}{dx}(\cot x) = -\mathrm{cosec}^2 x$

Standard results

Chain rule:

$\frac{d}{dx}(fg(x)) = f(g'(x)) \times g'(x)$

Product rule:

$y = uv$, where u and v are functions of x

$\Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient rule:

$y = \frac{u}{v}$, where u and v are functions of x

$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Checklist

Can you do these?

- Demonstrate the concept of the derivative at $x = c$ as the gradient of the tangent at $x = c$.
- Define the derivative at a point as a limit.
- Use the different notations for derivative i.e. $f'(x)$, $\frac{dy}{dx}$, $f^{(1)}(x)$.
- Differentiate from first principles: $f(x) = x^n$, $f(x) = k$, $f(x) = \sin x$.

- Use the derivative of x^n.
- Calculate derivatives of polynomials.
- Calculate derivatives of trigonometric functions.
- Use the product rule and quotient rule for differentiation.
- Differentiate composite functions using the chain rule.

Review Exercise 13

In questions **1** to **6**, differentiate each function from first principles.

1 $f(x) = x + \frac{1}{x}$

2 $f(x) = 2x^2 - 5x + 2$

3 $f(x) = x^3 - 2x + 1$

4 $f(x) = \frac{1}{2x + 3}$

5 $f(x) = \sin 2x$

6 $f(x) = \cos 2x$

7 If $y = \frac{5x - 2}{3x + 7}$, find $\frac{d^2y}{dx^2}$ in its simplest form.

8 Given that $y = A\cos 5x + B\sin 5x$, show that $\frac{d^2y}{dx^2} + 25y = 0$.

9 Differentiate with respect to x.

(a) $x^2 \sin 5x$

(b) $\cos^4 4x$

(c) $\left(\frac{1 - 2x}{1 + 2x}\right)^2$

10 Find the first derivative of the following.

(a) $y = (2x + 1)\tan(4x + 5)$

(b) $y = \sin(2x + 3) + x\tan 4x$

(c) $y = \sin^3(x^2 + 4x + 2)$

(d) $y = \frac{1 + \cos 2\theta}{\sin 2\theta}$

(e) $t = \tan^2\theta \sin^2\theta$

(f) $r = \sin^2 x + 2\cos^3 x$

11 Find the gradient of the curve $y = x^3\cos(2x^2 + \pi)$ at $x = 0$.

12 Given that $y = \sqrt{4x - 3}$, show that $\frac{d^2y}{dx^2} = -\frac{4}{y^3}$.

13 Given that $y = \frac{x}{\sqrt{x^2 + 32}}$, show that $\frac{dy}{dx} = \frac{32}{(x^2 + 32)^{\frac{3}{2}}}$.

14 If $y = \frac{\cos x}{1 - \sin x}$, show that $\frac{dy}{dx} = 2 + \sqrt{2}$ when $x = \frac{\pi}{4}$.

15 If $y = \frac{1 + \cos x}{1 - \sin x}$, show that $\frac{dy}{dx} = 2(9 + 5\sqrt{3})$ when $x = \frac{\pi}{3}$.

16 $y = \frac{x}{1 - 5x}$

(a) Find $\frac{dy}{dx}$.

(b) Show that $x^2\frac{dy}{dx} = y^2$.

(c) Hence, show that $x^2\frac{d^2y}{dx^2} = \frac{10y^3}{x}$.

17 $y = \dfrac{x+1}{x-1}$

(a) Find $\dfrac{dy}{dx}$.

(b) Show that $(x+1)^2\dfrac{dy}{dx} = -2y^2$.

(c) Hence, show that $(x+1)^2\dfrac{d^2y}{dx^2} + 2(x+2y+1)\dfrac{dy}{dx} = 0$.

18 If $y = \dfrac{x^2}{2-3x^2}$ show that $x^3\dfrac{dy}{dx} = 4y^2$.

19 $y = \cos^3 x \sin x$

(a) Find $\dfrac{dy}{dx}$.

(b) Show that $\dfrac{dy}{dx} = -\dfrac{1}{2}$, when $x = \dfrac{\pi}{4}$.

20 Differentiate the following with respect to t.

(a) $\theta = \sin\sqrt{2t-\pi}$

(b) $\theta = t\sqrt{4t^2 - 3t + 2}$

(c) $\theta = \dfrac{1+\cos 2t}{1-\sin 2t}$

21 If $y = \sqrt{1+\cos x}$, show that $2y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + y^2 = 1$.

CHAPTER 14

Applications of Differentiation

At the end of this chapter you should be able to:

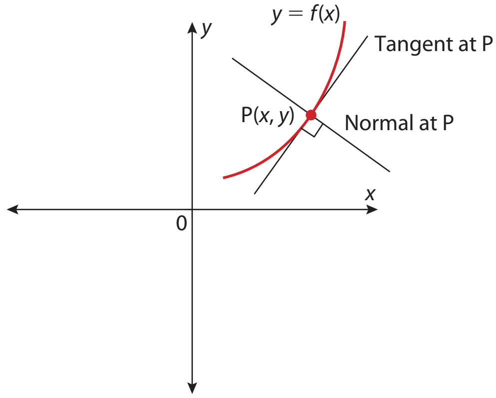
- Find the gradient of the tangent to a curve
- Find the gradient of the normal to a curve
- Find the equation of a tangent
- Find the equation of a normal
- Find the region for which a function is increasing or decreasing
- Identify stationary points
- Identify maximum points, minimum points, points of inflexion
- Use the first derivative to classify maximum points, minimum points and points of inflexion
- Use the second derivative test to identify maximum, minimum points
- Solve practical problems involving maximum and minimum
- Sketch the graphs of polynominals
- Solve graphically $f(x) = g(x)$, $f(x) \leqslant g(x)$, $f(x) \geqslant g(x)$
- Sketch the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$
- Sketch the graphs of $y = \sin kx$, $y = \cos kx$ and $y = \tan kx$, $k \in \mathbb{Q}$
- Identify the periodicity, symmetry and amplitude of $y = \sin kx$, $y = \cos kx$ and $y = \tan kx$
- Solve rate of change problems.
- Identify the properties of a curve and sketch the curve

KEYWORDS/TERMS

differentiation • gradient • tangent • normal •
increasing • decreasing • stationary points •
maximum • minimum • points of inflexion •
rate of change

Tangents and normals

Consider the function $y = f(x)$ with a point (x, y) lying on the graph of the function. The tangent line to the function at (x, y) is a line that touches the curve at one point. Both the graph of $y = f(x)$ and the tangent line pass through this point. The gradient of the tangent line and the gradient of the function have the same value at this point. The gradient of the tangent at $x = a$ is the value of $\frac{dy}{dx}$ when $x = a$. The normal is perpendicular to the tangent. If the gradient of the tangent is $\frac{dy}{dx}$, then the gradient of the normal is $-\frac{1}{\frac{dy}{dx}}$ since the product of the gradients of the perpendicular lines is -1.



EXAMPLE 1 Find the gradient of the tangent to the curve $y = x^2 - 3x + 2$ at the point $x = 0$.

SOLUTION The gradient of the tangent is $\frac{dy}{dx}$ at this point.

$y = x^2 - 3x + 2$

Differentiating with respect to x gives:

$$\frac{dy}{dx} = 2x - 3$$

When $x = 0$, $\frac{dy}{dx} = -3$.

Therefore, the gradient of the tangent at $x = 0$ is -3.

EXAMPLE 2 Find the gradient of the tangent to the curve $y = \sqrt{4x + 1}$ at the point $x = 2$.

SOLUTION Since $y = \sqrt{4x + 1} = (4x + 1)^{\frac{1}{2}}$, we use the chain rule.

Let $u = 4x + 1$.

$$\Rightarrow y = u^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{2}u^{-\frac{1}{2}} \times 4$$

$$= 2(4x + 1)^{-\frac{1}{2}}$$

Substituting $x = 2$ gives:

$$\frac{dy}{dx} = 2(4(2) + 1)^{-\frac{1}{2}} = 2(9)^{-\frac{1}{2}} = \frac{2}{3}$$

EXAMPLE 3 Find the gradient of the normal to the curve $y = 4x^3 - 3x^2 + x + 1$ at the point $x = 1$.

SOLUTION We find the gradient of the tangent first, that is $\frac{dy}{dx}$, when $x = 1$.

$y = 4x^3 - 3x^2 + x + 1$

$$\frac{dy}{dx} = 12x^2 - 6x + 1$$

When $x = 1$,

$$\frac{dy}{dx} = 12(1)^2 - 6(1) + 1$$
$$= 7$$

Since the gradient of the tangent = 7,

$$\text{gradient of the normal} = -\frac{1}{\text{gradient of tangent}}$$
$$= -\frac{1}{7}$$

EXAMPLE 4 Given that $y = 4x^3 - \frac{6}{x^2}$, find the gradient of the tangent to the curve at $x = 2$.

SOLUTION We need to find $\frac{dy}{dx}$ when $x = 2$.

Since $y = 4x^3 - \frac{6}{x^2}$

$$y = 4x^3 - 6x^{-2}$$

Now, $\frac{dy}{dx} = 12x^2 - 6(-2)x^{-3}$

$$= 12x^2 + \frac{12}{x^3}$$

When $x = 2$, $\frac{dy}{dx} = 12(2)^2 + \frac{12}{2^3}$

$$= 48 + \frac{12}{8}$$
$$= 48 + \frac{3}{2}$$
$$= 49\frac{1}{2}$$

Equations of tangents and normals

Let m be the gradient of the tangent at the point (x_1, y_1). Using the equation of a straight line, we have the equation of the tangent is $(y - y_1) = m(x - x_1)$.

Since the tangent and normal are perpendicular to each other, the gradient of the normal is $-\frac{1}{m}$ and the equation of the normal is $y - y_1 = -\frac{1}{m}(x - x_1)$

EXAMPLE 5 For the curve $y = 2x^2 - 3x + 1$, find the equation of the tangent to the curve and the equation of the normal to the curve at the point $x = 2$.

SOLUTION Find the gradient of the tangent:

$$y = 2x^2 - 3x + 1$$
$$\frac{dy}{dx} = 4x - 3$$

When $x = 2$,

$$\frac{dy}{dx} = 4(2) - 3$$
$$= 8 - 3$$
$$= 5$$

To find the equation of the tangent, we need the value of y when $x = 2$.

$x = 2$

$$\Rightarrow y = 2(2)^2 - 3(2) + 1$$
$$= 8 - 6 + 1$$
$$= 3$$

We have $m = 2$ and the point (2, 3).

Hence, the equation of the tangent is:

$y - 3 = 5(x - 2)$

$y - 3 = 5x - 10$

$y = 5x - 7$

Now we find the equation of the normal.

The gradient of the tangent at $x = 2$ is 5.

Therefore, the gradient of the normal $= -\frac{1}{5}$, since $5 \times -\frac{1}{5} = -1$.

Hence, the equation of the normal at (2, 3) is:

$$y - 3 = -\frac{1}{5}(x - 2)$$
$$y = -\frac{1}{5}x + \frac{2}{5} + 3$$
$$y = -\frac{1}{5}x + \frac{17}{5}$$

EXAMPLE 6 For the curve $y = x^2 \cos x$, find the equation of the tangent and the equation of the normal to the curve when $x = \frac{\pi}{2}$.

SOLUTION We find the gradient of the curve at $x = \frac{\pi}{2}$ by differentiating $y = x^2 \cos x$, using product rule:

Let $u = x^2$ and $v = \cos x$.

$$\frac{du}{dx} = 2x, \qquad \frac{dv}{dx} = -\sin x$$
$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{dy}{dx} = -x^2 \sin x + 2x \cos x$$

When $x = \frac{\pi}{2}$,

$$\frac{dy}{dx} = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right)$$
$$= -\left(\frac{\pi}{2}\right)^2 \times 1 + 2\left(\frac{\pi}{2}\right) \times 0$$
$$= -\frac{\pi^2}{4}$$

When $x = \frac{\pi}{2}$, $\qquad y = \left(\frac{\pi}{2}\right)^2 \cos\frac{\pi}{2} = 0$

We have $m = -\frac{\pi^2}{4}$ and the point (2, 0).

Hence, the equation of the tangent is:

$$y - 0 = -\frac{\pi^2}{4}\left(x - \frac{\pi}{2}\right)$$

Therefore, $y = -\frac{\pi^2}{4}x + \frac{\pi^3}{8}$.

Since the gradient of the tangent is $-\frac{\pi^2}{4}$, the gradient of the normal is $\frac{4}{\pi^2}$.

Hence, the equation of the normal is:

$$y - 0 = \frac{4}{\pi^2}\left(x - \frac{\pi}{2}\right)$$

$$y = \frac{4}{\pi^2}x - \frac{2}{\pi}$$

EXAMPLE 7 Find the gradient of the tangent to the curve $y = \frac{2x+1}{x-2}$ at the point where $y = 3$. Hence, find the equation of the normal to the curve at this point.

SOLUTION Find $\frac{dy}{dx}$, using the quotient rule.

Let $u = 2x + 1$ and $v = x - 2$.

$$\frac{du}{dx} = 2, \frac{dv}{dx} = 1$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(x-2)(2) - (2x+1)}{(x-2)^2} = \frac{2x - 4 - 2x - 1}{(x-2)^2} = \frac{-5}{(x-2)^2}$$

When $y = 3$,

$$\frac{2x+1}{x-2} = 3 \Rightarrow 3x - 6 = 2x + 1$$

$$x = 7$$

Therefore, $\frac{dy}{dx} = \frac{-5}{(7-2)^2} = \frac{-5}{25} = \frac{-1}{5}$.

The gradient of the tangent $= \frac{-1}{5}$.

Hence, the gradient of the normal $= 5$.

When $x = 7$, $y = 3$

Hence, the equation of the normal at $(7, 3)$ is:

$$y - 3 = 5(x - 7)$$

$$y - 3 = 5x - 35$$

$$y = 5x - 32$$

Try these 14.1

(a) Find the equation of the tangent to the curve $y = \frac{x-1}{x+1}$ at the point $x = 2$.

(b) Find the equation of the normal to the curve $y = x^2 \sin x$ at the point $x = 0$.

Increasing and decreasing functions

DEFINITION

A function $f(x)$ is increasing over an interval, where $f'(x) > 0$.
A function $f(x)$ is decreasing over an interval, where $f'(x) < 0$.
When $f'(x) > 0$ the graph rises as you move from left to right and the graph falls when $f'(x) < 0$.

Let $f(x)$ be a differentiable function over the interval (a, b).

If $f'(x) > 0$ over the interval $a < x < b$, then $f(x)$ is increasing over the interval $a \leq x \leq b$.

If $f'(x) < 0$ over the interval $a < x < b$, then $f(x)$ is decreasing over the interval $a \leq x \leq b$.

If $f'(x) = 0$ for every x over some interval, then $f(x)$ is constant over the interval.

EXAMPLE 8 Find the range of values of x for which the function $f(x) = x^2 + 2x - 1$ is increasing.

SOLUTION For an increasing function, $f'(x) > 0$.

$f(x) = x^2 + 2x - 1$

$f'(x) = 2x + 2$

$f'(x) > 0$

$\Rightarrow 2x + 2 > 0$

$x > -1$

$\therefore f(x)$ is increasing for $x \geq -1$.

EXAMPLE 9 Given that $f(x) = x^5 - 5x$, find the following.

(a) $f'(x)$

(b) The range of values of x for which $f(x)$ is decreasing

SOLUTION

(a) $f(x) = x^5 - 5x$

$f'(x) = 5x^4 - 5$

(b) For a decreasing function, $f'(x) < 0$

$\Rightarrow 5x^4 - 5 < 0$

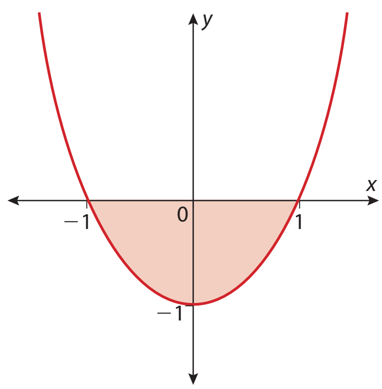
$\Rightarrow x^4 - 1 < 0$

$\Rightarrow (x^2 + 1)(x - 1)(x + 1) < 0$

Since $x^2 + 1$ is always positive, we identify where $(x - 1)(x + 1)$ is negative:

$\Rightarrow -1 \leq x \leq 1$

We can use a graph.
For the sign of $(x - 1)(x + 1)$:



$(x - 1)(x + 1)$ is negative for $-1 \leq x \leq 1$.

Hence, $f(x)$ is decreasing in the region $-1 \leq x \leq 1$.

EXAMPLE 10 Find the range of values of x for which $f(x) = 6x - 2x^3$ is an increasing function.

SOLUTION

$f(x) = 6x - 2x^3$

$f'(x) = 6 - 6x^2$

For an increasing function, $f'(x) > 0$.

$\Rightarrow 6 - 6x^2 > 0$

$\Rightarrow (1 - x^2) > 0$

$\Rightarrow x^2 - 1 < 0$

$(x - 1)(x + 1) < 0$

From the example above the solution set is $-1 \leq x \leq 1$.

Therefore, $f'(x)$ is increasing for $-1 \leq x \leq 1$.

EXAMPLE 11

(a) Solve the inequality $x^3 - x^2 - 2x > 0$.

(b) Hence, find:

(i) the interval for which $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing

(ii) the region for which $f(x)$ is decreasing.

SOLUTION

(a) $x^3 - x^2 - 2x > 0$

$\Rightarrow x(x^2 - x - 2) > 0$

$= x(x - 2)(x + 1) > 0$

The critical values of $x^3 - x^2 - 2x$ are $-1, 0, 2$.

Sign table:

	x	$x - 2$	$x + 1$	$x(x - 2)(x + 1)$
$x < -1$	$-$ve	$-$ve	$-$ve	$-$ve
$-1 < x < 0$	$-$ve	$-$ve	$+$ve	$+$ve
$0 < x < 2$	$+$ve	$-$ve	$+$ve	$-$ve
$x > 2$	$+$ve	$+$ve	$+$ve	$+$ve

Therefore, the solution set is:

$\{x: -1 \leq x \leq 0\} \cup \{x: x \geq 2\}$

(b) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x$

(i) For an increasing function, $f'(x) > 0$.

$\Rightarrow 12x^3 - 12x^2 - 24x > 0$

$\Rightarrow x^3 - x^2 - 2x > 0$

$\Rightarrow x(x^2 - x - 2) > 0$

$\Rightarrow x(x - 2)(x + 1) > 0$

From (a) $\{x: -1 \leq x \leq 0\} \cup \{x: x \geq 2\}$

(ii) For a decreasing function, $f'(x) < 0$.

$\Rightarrow x(x - 2)(x + 1) < 0$

$\Rightarrow \{x: x \leq -1\} \cup \{x: 0 \leq x \leq 2\}$

Try these 14.2

(a) Find the interval for which the following functions are increasing.

(i) $f(x) = x^2 + 2x + 3$

(ii) $f(x) = x^3 - 2x^2 + 5$

(iii) $f(x) = x^4 - x$

(b) Find the interval for which each of the following functions is decreasing.

(i) $f(x) = 4x^2 + 6x + 2$

(ii) $f(x) = \dfrac{x+1}{x-2}$

EXERCISE 14A

1 Find the range of values of x for which the function $y = 6x^2 - 2x$ is increasing.

2 In what interval must t lie if the function $x = t^4 - t^2$ is decreasing?

3 Find the range of values of x for which the function $y = \frac{1}{3}x + \frac{2}{x}$ is increasing.

4 For what values of t is the function $s = 2 - 3t + t^2$ decreasing?

5 Find the interval in which x lies if the function $y = 2x^3 + 3x^2 - 12x + 4$ is increasing.

In questions **6** to **10**, find the equation of the tangent to the curve at the given point.

6 $y = 4x^2 + 3x + 1$ at the point where $x = 1$

7 $y = \dfrac{4}{2x+3}$ at the point where $x = 2$

8 $y = \dfrac{4}{1-2x}$ at the point where $x = 1$

9 $y = \dfrac{4}{x^2}$ at the point where $y = 1$

10 $y = x\cos x$ at the point where $x = \dfrac{\pi}{2}$

In questions **11** to **15**, find the equation of the normal to the curve at the given point.

11 $y = \sin(2x - \pi)$ at the point where $x = \pi$

12 $y = x\tan x$ at the point where $x = \dfrac{\pi}{4}$

13 $y = \dfrac{2x+1}{\sin^2 x}$ at the point where $x = \dfrac{\pi}{2}$

14 $y = \dfrac{x-2}{2x+1}$ at the point where $y = 1$

15 $y = \dfrac{1}{\sqrt{3x-2}}$ at the point where $x = 1$

16 The tangent to the curve $y = x^2 - 4x + 5$ at a certain point is parallel to the line with equation $y + 3x = 4$. Find the equation of the tangent.

17 The normal to the curve $y = \dfrac{4}{(2x-1)^2}$ at the point $(1, 4)$ meets the x-axis at A and the y-axis at B. Find the length of AB.

18 Show that the normal to the curve $y = 2x - \dfrac{3}{1-x}$, where $x = 2$, meets the curve again at $(4, 9)$.

19 The normal to the curve $y = 3 + 4x - \frac{1}{2}x^2$ at $(2, 9)$ meets the curve again at the point A.

(a) Find the equation of the normal.

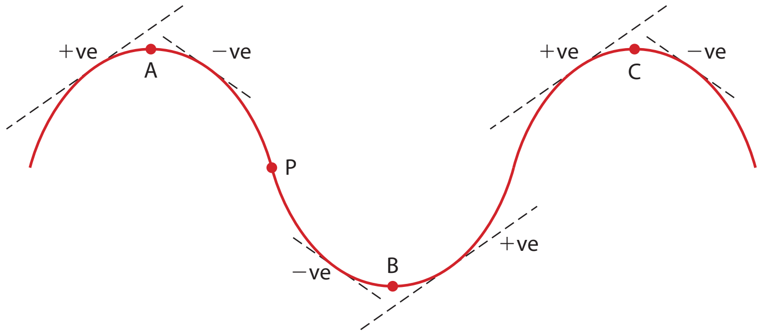
(b) Find the coordinates of A.

(c) Find the equation of the tangent at A.

20 The equation of a curve is $y = 4x^2\cos(2x)$. Find the equation of the tangent to the curve at the point $x = \pi$.

Stationary points/second derivatives

Maximum and minimum values



The slope of the curve at point A is 0. Just before A, the slope of the curve is positive. Just after A, the slope of the curve is negative. Point A is a stationary point and is called a **maximum point** and the curve has a **maximum value** at A.

Just before P, $f(x)$ is curved downwards and $f(x)$ is said to be concave down in that region. Just after P, $f(x)$ is curved upwards and $f(x)$ is said to be concave upwards in that region. P is called a **point of inflexion**.

Point B is another stationary point since the slope of the curve is 0. Just before B, the gradient of the curve is negative. Just after B, the gradient of the curve is positive. Point B is called a **minimum point** and the curve has a **minimum value** at B.

Point C is a maximum point, since the gradient just before C is positive and the gradient of the curve just after C is negative while the gradient at C is zero.

Points A, B and C are called **turning points** of the curve. A and C are maximum points and B is a minimum value. Point P is a point of inflexion.

Stationary points

Let $y = f(x)$. Stationary points of y occur at $\frac{dy}{dx} = 0$.

EXAMPLE 12 Find the stationary point of $y = x^2 + 2x - 1$.

SOLUTION

$y = x^2 + 2x - 1$

Differentiating with respect to x:

$\frac{dy}{dx} = 2x + 2$

Since stationary points exist when $\frac{dy}{dx} = 0$:

$\Rightarrow 2x + 2 = 0$

$x = -1$

When $x = -1$, $y = (-1)^2 + 2(-1) - 1$

$= 1 - 2 - 1 = -2$

Therefore, the coordinates of the stationary point are $(-1, -2)$.

EXAMPLE 13 Find the coordinates of the stationary points of $y = x^3 - 3x + 1$.

SOLUTION

$y = x^3 - 3x + 1$

Differentiating with respect to x:

$\frac{dy}{dx} = 3x^2 - 3$

Since stationary points exist when $\frac{dy}{dx} = 0$:

$\Rightarrow 3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1$, $y = (1)^3 - 3(1) + 1$

$= 1 - 3 + 1$

$= -1$

When $x = -1$, $y = (-1)^3 - 3(-1) + 1$

$= -1 + 3 + 1$

$= 3$

Therefore, the coordinates of the stationary points are $(1, -1)$ and $(-1, 3)$.

EXAMPLE 14 Find the coordinates of the stationary point of $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$.

SOLUTION

$y = \cos x$

$\frac{dy}{dx} = -\sin x$

At stationary points $\frac{dy}{dx} = 0$.

$\Rightarrow -\sin x = 0$

$\Rightarrow x = \sin^{-1} 0$

$= 0$

When $x = 0$, $y = \cos 0 = 1$.

Therefore, the coordinates of the stationary point are $(0, 1)$.

EXAMPLE 15 Find the coordinates of the stationary points of the curve $y = 2\cos 2\theta + \sin 2\theta$, $0 \leq \theta \leq \frac{\pi}{2}$.

SOLUTION

$y = 2\cos 2\theta + \sin 2\theta$

$\frac{dy}{dx} = -4\sin 2\theta + 2\cos 2\theta$

At stationary points $\frac{dy}{dx} = 0$.

$\Rightarrow -4\cos 2\theta + 2\sin 2\theta = 0$

$2\sin 2\theta = 4\cos 2\theta$

$\frac{\sin 2\theta}{\cos 2\theta} = \frac{4}{2}$

$\tan 2\theta = 2$

$2\theta = \tan^{-1}2$

$\theta = \frac{1}{2}\tan^{-1}2$

$\quad = 0.554$ radians

When $\theta = 0.554^c$, $y = 1.79$

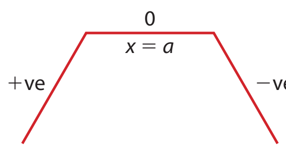
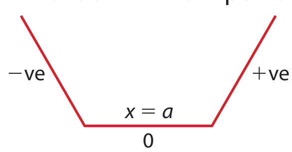
Therefore, the coordinates of the stationary point are $(0.554^c, 1.79)$.

Classification of turning points

First derivative test

Let $x = a$ be a turning point of $y = f(x)$.

We can classify a turning point as a maximum or a minimum point by using the sign of $\frac{dy}{dx}$ before and after the turning point.

Maximum point	**Minimum point**	**Point of inflexion**
For $x < a$, if $\frac{dy}{dx} > 0$	For $x < a$, if $\frac{dy}{dx} < 0$	For $x < a$, if $\frac{dy}{dx} < 0$ For $x > a$, if $\frac{dy}{dx} < 0$
For $x > a$, if $\frac{dy}{dx} < 0$	For $x > a$, if $\frac{dy}{dx} > 0$	For $x < a$, if $\frac{dy}{dx} > 0$ For $x > a$, if $\frac{dy}{dx} > 0$
$\Rightarrow x = a$ is a maximum point 	$\Rightarrow x = a$ is a minimum point 	Since $\frac{dy}{dx}$ does not change sign as it passes through the turning point, then the point is a point of inflexion.

EXAMPLE 16 Find and classify the stationary point of $y = 3x^2 + 2x - 1$.

SOLUTION

$y = 3x^2 + 2x - 1$

$\frac{dy}{dx} = 6x + 2$

At stationary points $\frac{dy}{dx} = 0$.

$\Rightarrow 6x + 2 = 0$

$x = -\frac{2}{6}$

$\quad = -\frac{1}{3}$

When $x = -\frac{1}{3}$, $y = 3\left(-\frac{1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) - 1$

$= \frac{1}{3} - \frac{2}{3} - 1$

$= -\frac{4}{3}$

There is a stationary point at $\left(-\frac{1}{3}, -\frac{4}{3}\right)$.

Let us classify this point using the first derivative test:

Since the point is at $x = -\frac{1}{3}$:

For $x < -\frac{1}{3}$, using $x = -\frac{1}{2}$:

$\frac{dy}{dx} = 6\left(-\frac{1}{2}\right) + 2$

$= -3 + 2$

$= -1$

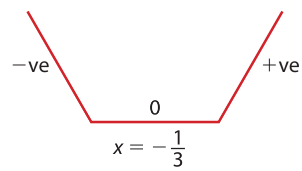
$-1 < 0$

For $x > -\frac{1}{3}$, using $x = 0$:

$\frac{dy}{dx} = 6(0) + 2$

$= 2$

$2 > 0$



Since there is a negative gradient followed by a positive gradient, the stationary point is a minimum point.

Hence, $\left(-\frac{1}{3}, \frac{4}{3}\right)$ is a minimum point.

EXAMPLE 17 Find and classify the turning points of $y = 2x^3 + 7x^2 + 4x - 3$.

SOLUTION

$y = 2x^3 + 7x^2 + 4x - 3$

$\frac{dy}{dx} = 6x^2 + 14x + 4$

At stationary points $\frac{dy}{dx} = 0$.

$6x^2 + 14x + 4 = 0$

$3x^2 + 7x + 2 = 0$

$(3x + 1)(x + 2) = 0$

$\Rightarrow x = -\frac{1}{3}, x = -2$

When $x = -\frac{1}{3}, y = 2\left(-\frac{1}{3}\right)^3 + 7\left(-\frac{1}{3}\right)^2 + 4\left(-\frac{1}{3}\right) - 3$

$= -\frac{2}{27} + \frac{7}{9} - \frac{4}{3} - 3$

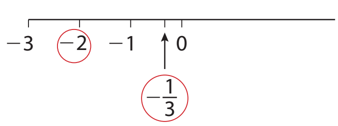
$= -\frac{98}{27}$

When $x = -2, y = 2(-2)^3 + 7(-2)^2 + 4(-2) - 3$

$= -16 + 28 - 8 - 3$

$= 1$

The turning points are at $\left(-\frac{1}{3}, -\frac{98}{27}\right)$ and $(-2, 1)$.



To classify the turning points:

We can check the sign of $\frac{dy}{dx}$ within the region as follows:

Start with $x = -2$.

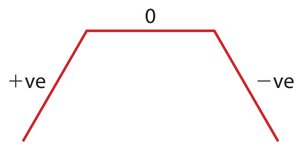
When checking the sign of $\frac{dy}{dx}$ for $x > -2$, we use a value between -2 and $-\frac{1}{3}$, since our next turning point occurs at $-\frac{1}{3}$.

For $x < -2$, using $x = -3$:

$$\begin{aligned}\frac{dy}{dx} &= 6(-3)^2 + 14(-3) + 4\\ &= 54 - 42 + 4\\ &= 16\\ 16 &> 0\end{aligned}$$

For $x > -2$, using $x = -1$:

$$\begin{aligned}\frac{dy}{dx} &= 6(-1)^2 + 14(-1) + 4\\ &= 6 - 14 + 4\\ &= -4\\ -4 &< 0\end{aligned}$$



Since $\frac{dy}{dx}$ changes sign from positive to negative, the point at $x = -2$ is a maximum point.

Now, we look at $x = -\frac{1}{3}$.

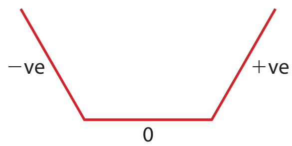
(For $x < -\frac{1}{3}$, we could use the same sign that was used for $x > -2$, i.e. $\frac{dy}{dx} < 0$.)

For $x < -\frac{1}{3}$, using $-\frac{1}{2}$:

$$\begin{aligned}\frac{dy}{dx} &= 6\left(-\frac{1}{2}\right)^2 + 14\left(-\frac{1}{2}\right) + 4\\ &= \frac{3}{2} - 7 + 4\\ &= -\frac{3}{2}\\ -\frac{3}{2} &< 0\end{aligned}$$

For $x > -\frac{1}{3}$, using $x = 0$:

$$\begin{aligned}\frac{dy}{dx} &= 6(0)^2 + 14(0) + 4\\ &= 4\\ 4 &> 0\end{aligned}$$



Since $\frac{dy}{dx}$ changes sign from negative to positive, the point at $x = -\frac{1}{3}$ is a minimum point.

EXAMPLE 18 Find and classify the stationary point(s) of $y = 8x + \frac{1}{2x^2}$.

SOLUTION

$$y = 8x + \frac{1}{2}x^{-2}$$

$$\frac{dy}{dx} = 8 - x^{-3}$$

At the stationary points, $\frac{dy}{dx} = 0$.

$\Rightarrow 8 - \frac{1}{x^3} = 0$

$\Rightarrow 8 = \frac{1}{x^3}$

$\Rightarrow x^3 = \frac{1}{8}$

$\Rightarrow x = \sqrt[3]{\frac{1}{8}}$

$= \frac{1}{2}$

When $x = \frac{1}{2}$, $y = 8\left(\frac{1}{2}\right) + \frac{1}{2\left(\frac{1}{2}\right)^2}$

$= 4 + 2$

$= 6$

There is a stationary point at $\left(\frac{1}{2}, 6\right)$.

For $x < \frac{1}{2}$, using $x = \frac{1}{4}$, $\frac{dy}{dx} = 8 - \frac{1}{\left(\frac{1}{4}\right)^3}$

$= 8 - 64$

$= -56$

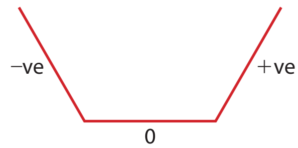
$-56 < 0$

For $x > \frac{1}{2}$, using $x = 1$, $\frac{dy}{dx} = 8 - \frac{1}{1^3}$

$= 8 - 1$

$= 7$

$7 > 0$



Therefore, $\left(\frac{1}{2}, 6\right)$ is a minimum point.

EXAMPLE 19 Find the coordinates of the stationary points of $y = x + \sin x$ for $0 \leq x \leq \pi$.

SOLUTION

$y = x + \sin x$

$\frac{dy}{dx} = 1 + \cos x$

When $\frac{dy}{dx} = 0$,

$1 + \cos x = 0$

$\Rightarrow \cos x = -1$

$x = \pi$

When $x = \pi$, $\quad y = \pi + \sin \pi$

$= \pi + 0$

$= \pi$

The stationary point is at (π, π).

Second derivative test

We can classify turning points by using the second derivative test. Let the point at $x = a$ be a stationary point of $y = f(x)$.

(i) If $\frac{d^2y}{dx^2} > 0$ when $x = a$, then there is a minimum point at $x = a$.

(ii) If $\frac{d^2y}{dx^2} < 0$ when $x = a$, then there is a maximum point at $x = a$.

(iii) If $\frac{d^2y}{dx^2} = 0$, you must test further (in this case you can go back to the first derivative test).

EXAMPLE 20 Find and classify the stationary points of $y = 4x^2 - 3x + 2$.

SOLUTION

$y = 4x^2 - 3x + 2$

$\frac{dy}{dx} = 8x - 3$

At stationary points $\frac{dy}{dx} = 0$.

$\Rightarrow 8x - 3 = 0$

$x = \frac{3}{8}$

To classify:

$\frac{d^2y}{dx^2} = 8$

$8 > 0$

Therefore, there is a minimum point when $x = \frac{3}{8}$.

When $x = \frac{3}{8}$, $y = 4\left(\frac{3}{8}\right)^2 - 3\left(\frac{3}{8}\right) + 2$

$= \frac{23}{16}$

There is a minimum point at $\left(\frac{3}{8}, \frac{23}{16}\right)$.

EXAMPLE 21 Find and classify the stationary points of $y = -3x^2 + 2x + 5$.

SOLUTION

$y = -3x^2 + 2x + 5$

$\frac{dy}{dx} = -6x + 2$

When $\frac{dy}{dx} = 0$, $-6x + 2 = 0$

$\Rightarrow x = \frac{2}{6} = \frac{1}{3}$

$\frac{d^2y}{dx^2} = -6$

$-6 < 0$

Therefore, when $x = \frac{1}{3}$ there is a maximum point.

When $x = \frac{1}{3}$, $y = -3\left(\frac{1}{3}\right)^2 + 2\left(\frac{1}{3}\right) + 5$

$$= -\frac{1}{3} + \frac{2}{3} + 5$$

$$= \frac{16}{3}$$

There is a maximum point at $\left(\frac{1}{2}, \frac{16}{3}\right)$.

EXAMPLE 22 Determine the nature of the turning points of the curve $y = 7x^3 - 4x^2 - 5x + 6$.

SOLUTION

Turning points occur when $\frac{dy}{dx} = 0$.

$\frac{dy}{dx} = 21x^2 - 8x - 5$

$\Rightarrow 21x^2 - 8x - 5 = 0$

$\Rightarrow (3x + 1)(7x - 5) = 0$

$\Rightarrow x = -\frac{1}{3}, x = \frac{5}{7}$

Using the second derivative test:

$\frac{d^2y}{dx^2} = 42x - 8$

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 42\left(-\frac{1}{3}\right) - 8$

$$= -14 - 8$$

$$= -22$$

$-22 < 0$

Since $\frac{d^2y}{dx^2} < 0$ at $x = -\frac{1}{3}$, there is a maximum point at $x = -\frac{1}{3}$.

When $x = \frac{5}{7}$, $\frac{d^2y}{dx^2} = 42\left(\frac{5}{7}\right) - 8$

$$= 30 - 8$$

$$= 22$$

$22 > 0$

Since $\frac{d^2y}{dx^2} > 0$ at $x = \frac{5}{7}$, there is a minimum point at $x = \frac{5}{7}$.

EXAMPLE 23 Find and classify the turning points of $y = 4\cos 2\theta + 3\sin\theta$, where $0 \leq \theta \leq \frac{\pi}{2}$.

SOLUTION

$y = 4\cos 2\theta + 3\sin\theta$

Differentiate with respect to θ:

$\frac{dy}{d\theta} = -8\sin 2\theta + 3\cos\theta$

At stationary points $\frac{dy}{d\theta} = 0$.

$\Rightarrow -8\sin 2\theta + 3\cos\theta = 0$

Since $\sin 2\theta = 2\sin\theta\cos\theta$, we have:

$-16\sin\theta\cos\theta + 3\cos\theta = 0$

$\cos\theta(-16\sin\theta + 3) = 0$

$\Rightarrow \cos\theta = 0, \sin\theta = \frac{3}{16}$

$\Rightarrow \theta = \frac{\pi}{2}, \theta = 0.19$ radians

Using the second derivative test:

$$\frac{dy}{d\theta} = -8\sin 2\theta + 3\cos\theta$$

$$\frac{d^2y}{d\theta^2} = -16\cos 2\theta - 3\sin\theta$$

When $\theta = \frac{\pi}{2}$, $\frac{d^2y}{d\theta^2} = -16\cos\pi - 3\sin\frac{\pi}{2}$

$$= -16(-1) - 3(1)$$

$$= 13$$

$$13 > 0$$

Therefore, there is a minimum point when $\theta = \frac{\pi}{2}$.

When $\theta = 0.19^c$, $\frac{d^2y}{d\theta^2} = -16\cos(2)(0.19^c) - 3\sin 0.19^c$

$$= -15.4$$

$$-15.4 < 0$$

Therefore, there is a maximum point when $\theta = 0.19^c$.

EXAMPLE 24 Classify the stationary points of $y = \frac{3x}{(x-1)(x-4)}$.

SOLUTION

$$y = \frac{3x}{(x-1)(x-4)}$$

$$= \frac{3x}{x^2 - 5x + 4}$$

Using the quotient rule with $u = 3x$ and $v = x^2 - 5x + 4$:

$$\frac{du}{dx} = 3, \frac{dv}{dx} = 2x - 5$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2 - 5x + 4)(3) - (3x)(2x - 5)}{(x^2 - 5x + 4)^2}$$

$$= \frac{3x^2 - 15x + 12 - 6x^2 + 15x}{(x^2 - 5x + 4)^2}$$

$$= \frac{-3x^2 + 12}{(x^2 - 5x + 4)^2}$$

At stationary points $\frac{dy}{dx} = 0$.

$$\Rightarrow \frac{-3x^2 + 12}{(x^2 - 5x + 4)^2} = 0$$

$$\Rightarrow -3x^2 + 12 = 0$$

$$\Rightarrow 3x^2 = 12$$

$$x^2 = 4$$

$$\Rightarrow x = \pm 2$$

Use the second derivative test to classify the turning points.

$$\frac{dy}{dx} = \frac{-3x^2 + 12}{(x^2 - 5x + 4)^2}$$

Let $u = -3x^2 + 12$ and $v = (x^2 - 5x + 4)^2$.

$$\frac{du}{dx} = -6x, \frac{dv}{dx} = 2(2x - 5)(x^2 - 5x + 4)$$

$$\frac{d^2y}{dx^2} = \frac{-6x(x^2 - 5x + 4)^2 - (-3x^2 + 12)(2)(2x - 5)(x^2 - 5x + 4)}{[(x^2 - 5x + 4)^2]^2}$$

$$= \frac{(x^2 - 5x + 4)[-6x(x^2 - 5x + 4) - (-3x^2 + 12)(4x - 10)]}{[(x^2 - 5x + 4)^2]^2}$$

$$= \frac{(x^2 - 5x + 4)(-6x^3 + 30x^2 - 24x + 12x^3 - 48x - 30x^2 + 120)}{(x^2 - 5x + 4)^4}$$

$$= \frac{(x^2 - 5x + 4)(6x^3 - 72x + 120)}{(x^2 - 5x + 4)^4}$$

$$= \frac{6x^3 - 72x + 120}{(x^2 - 5x + 4)^3}$$

When $x = 2$, $\frac{d^2y}{dx^2} = \frac{6(2)^3 - 72(2) + 120}{(2^2 - 5(2) + 4)^3}$

$$= \frac{48 - 144 + 120}{-8}$$

$$= \frac{24}{-8}$$

$$= -3$$

$$-3 < 0$$

Therefore, there is a maximum point at $x = 2$.

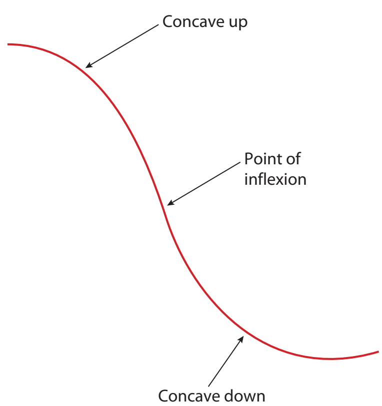
When $x = -2$, $\frac{d^2y}{dx^2} = \frac{1}{27}$

$$\frac{1}{27} > 0$$

Therefore, there is a minimum point at $x = -2$.

Inflexion points

Inflexion points occur where the curve changes concavity. A positive second derivative corresponds to concave up and a negative second derivative corresponds to concave down. When the function changes from concave up to concave down or vice versa, $f''(x) = 0$ at this point. For a point to be an inflexion point, the second derivative must be 0 at this point. This is not a necessary condition for a point of inflexion and therefore we must check the concavity of the function to verify that there is an inflexion point.



EXAMPLE 25 Find the point of inflexion on the curve $f(x) = x^3 + 1$.

SOLUTION We first find the values of x where $f''(x) = 0$.

$f(x) = x^3 + 1$
$f'(x) = 3x^2$
$f''(x) = 6x$

$f''(x) = 0$
$\Rightarrow 6x = 0$
$\Rightarrow x = 0$

We now verify that $f''(x)$ changes sign as it passes through $x = 0$.

Let us choose a value of x just before 0, say $x = -1$.

When $x = -1, f''(-1) = -6$ and the function is concave down at $x = -1$.

Next we choose a value of x just after 0, say $x = 1$.

When $x = 1, f''(1) = 6$ and the function is concave up at $x = 1$.

Since the function has different concavities on either side of $x = 0$, there is a point of inflexion at $x = 0$.

EXAMPLE 26 Find and classify the stationary points and points of inflexion of $y = 3x^5 - 20x^3$.

SOLUTION $y = 3x^5 - 20x^3$

Differentiating with respect to x:

$$\frac{dy}{dx} = 15x^4 - 60x^2$$

Differentiating again with respect to x:

$$\frac{d^2y}{dx^2} = 60x^3 - 120x$$

When $\frac{dy}{dx} = 0$, we get:

$15x^4 - 60x^2 = 0$
$\therefore\ 15x^2(x^2 - 4) = 0$
$\Rightarrow x^2 = 0, x^2 - 4 = 0$
$\Rightarrow x = 0, 2, -2$

When $\frac{d^2y}{dx^2} = 0$, we get:

$60x^3 - 120x = 0$
$60x(x^2 - 2) = 0$
$\Rightarrow x = 0, x^2 - 2 = 0$
$\Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$

We can classify the turning points by using the second derivative test.

When $x = 2$:
$$\frac{d^2y}{dx^2} = 60(2)^3 - 120(2) = 240 > 0$$
$\Rightarrow$ there is a minimum point at $x = 2$.

When $x = -2$:
$$\frac{d^2y}{dx^2} = 60(-2)^3 - 120(-2) = -240 < 0$$
$\Rightarrow$ there is a maximum point at $x = -2$.

When $x = 0$:

For $x < 0$, $\frac{d^2y}{dx^2}$ is negative.

For $x > 0$, $\frac{d^2y}{dx^2}$ is positive.

Since there is a sign change of $\frac{d^2y}{dx^2}$ as x passes through 0, there is a point of inflexion at $x = 0$.

When $x = \sqrt{2}$:

For $x < \sqrt{2}$, $\frac{d^2y}{dx^2}$ is negative.

$x > \sqrt{2}$, $\frac{d^2y}{dx^2}$ is positive.

Since there is a sign change of $\frac{d^2y}{dx^2}$ as x passes through $\sqrt{2}$, there is a point of inflexion at $x = \sqrt{2}$.

When $x = -\sqrt{2}$:

For $x < -\sqrt{2}$, $\frac{d^2y}{dx^2}$ is negative.

$x > -\sqrt{2}$, $\frac{d^2y}{dx^2}$ is positive.

Since there is a sign change of $\frac{d^2y}{dx^2}$ as x passes through $-\sqrt{2}$, there is a point of inflexion at $x = -\sqrt{2}$.

Hence, there is a maximum point when $x = -2$, a minimum point when $x = 2$, and points of inflexion when $x = 0, \sqrt{2}, -\sqrt{2}$.

EXAMPLE 27 Find and classify the nature of the stationary points and points of inflexion on the curve $y = \cos 2x - 2\sin x$, where $0° < x \le 90°$.

SOLUTION

$y = \cos 2x - 2\sin x$

Differentiating with respect to x gives:
$$\frac{dy}{dx} = -2\sin 2x - 2\cos x$$
Differentiating again gives:
$$\frac{d^2y}{dx^2} = -4\cos 2x + 2\sin x$$
At stationary points, $\frac{dy}{dx} = 0$.

$\Rightarrow -2\sin 2x - 2\cos x = 0$

Replacing $\sin 2x = 2\sin x\cos x$

$\Rightarrow -4\sin x\cos x - 2\cos x = 0$

$\Rightarrow -2\cos x\,(2\sin x + 1) = 0$

$\therefore\ \cos x = 0,\ 2\sin x + 1 = 0$

$\Rightarrow \cos x = 0,\ \sin x = -\frac{1}{2}$

$\Rightarrow x = 90°$ within the interval.

$\frac{d^2y}{dx^2} = 0, \qquad -4\cos 2x + 2\sin x = 0$

Replacing $\cos 2x = 1 - 2\sin^2 x$ gives:
$-4(1 - 2\sin^2 x) + 2\sin x = 0$
$\Rightarrow 8\sin^2 x + 2\sin x - 4 = 0$
$\Rightarrow 4\sin^2 x + \sin x - 2 = 0$

Using the quadratic formula:
$\sin x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
$\sin x = \frac{-1 \pm \sqrt{1 + 32}}{8} = \frac{-1 \pm \sqrt{33}}{8} = 0.593, -0.843$
Hence, $x = 36.4°$ in the given range.

Now we classify the turning points.

When $x = 90°$:
$\frac{d^2y}{dx^2} = -4\cos 180° + 2\sin 90° = 6 > 0$
Hence, there is a minimum point at $= 90°$.

When $x = 36.4°$:

For $x < 36.4$, $\frac{d^2y}{dx^2}$ is negative.

For $x > 36.4°$, $\frac{d^2y}{dx^2}$ is positive.

Since there is a sign change of $\frac{d^2y}{dx^2}$ as x passes through 36.4°, there is a point of inflexion at $x = 36.4°$.

Practical maximum and minimum problems

EXAMPLE 28 A rectangular area is to be fenced using 36 metres of wire. Find the length and breadth of the rectangle if it is to enclose the maximum area.

SOLUTION Let the sides of the rectangle be l and w.

Perimeter of the rectangle $= 2l + 2w$
$\therefore\ 2l + 2w = 36$
$\Rightarrow l + w = 18 \qquad [1]$

Area of the rectangle:

$A = lw$ [2]
$l = 18 - w$ Rearranging [1]
$A = (18 - w)w$ Substituting [1] into [2]
$\Rightarrow A = 18w - w^2$

Now that we have the area in terms of one variable, we can differentiate:

$$\frac{dA}{dw} = 18 - 2w$$

When $\frac{dA}{dw} = 0$:

$\Rightarrow 18 - 2w = 0$

$\Rightarrow w = \frac{18}{2} = 9\,\text{m}$

Classify the turning point:

$$\frac{d^2A}{dw^2} = -2$$

$-2 < 0$

Therefore, the area is at its maximum.

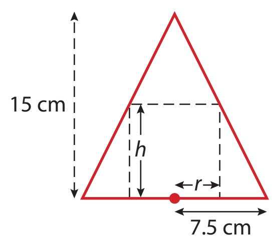
Since $l = 18 - w$:

when $w = 9$, $l = 18 - 9 = 9$

Therefore, the length and width giving a maximum area are $l = 9\,\text{m}$ and $w = 9\,\text{m}$.

EXAMPLE 29 Rajeev has a cone of height 15 cm and base radius 7.5 cm. Rajeev wishes to cut a cylinder of radius r cm and height h cm from this cone. What is the height of the cylinder of maximum volume which Rajeev can cut from this cone?

SOLUTION We use the fact that the ratio of the corresponding sides in similar triangles are equal.



$$\frac{r}{7.5} = \frac{15 - h}{15}$$

$$r = \frac{15 - h}{15} \times 7.5$$

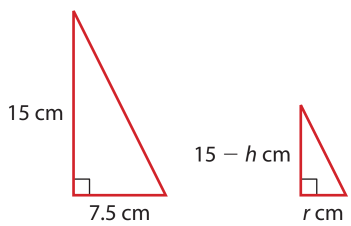
$$= \frac{15}{2} - \frac{h}{2}$$

$$V = \pi r^2 h$$

$$V = \pi\left(\frac{15}{2} - \frac{h}{2}\right)^2 h$$

$$= \frac{\pi}{4}(225 - 30h + h^2)h$$

$$= \frac{\pi}{4}(225h - 30h^2 + h^3)$$



$$\frac{dV}{dh} = \frac{\pi}{4}(225 - 60h + 3h^2)$$

At stationary points, $\frac{dV}{dh} = 0$.

$\Rightarrow 225 - 60h + 3h^2 = 0$
$h^2 - 20h + 75 = 0$
$(h - 15)(h - 5) = 0$
$h = 5, 15$

$$\frac{d^2V}{dh^2} = \frac{\pi}{4}(-60 + 6h)$$

When $h = 5, \frac{d^2V}{dh^2} = \frac{\pi}{4}(-60 + 30) < 0 \Rightarrow$ maximum point

When $h = 15, \frac{d^2V}{dh^2} = \frac{\pi}{4}(-60 + 90) > 0 \Rightarrow$ minimum point

Therefore, the height of the cylinder with the maximum volume is 5 cm.

EXAMPLE 30 A circular cylinder, open at one end, is constructed of a thin sheet of metal, whose surface area is $600\pi\,\text{cm}^2$. The height of the cylinder is h cm and the radius of the cylinder is r cm.

(a) Show that the volume, $V\,\text{cm}^3$, contained by the cylinder is $V = \frac{\pi r}{2}(600 - r^2)$

(b) Evaluate the value of V, and determine whether this value is a maximum or minimum.

SOLUTION

(a) Surface area of the cylinder $= \pi r^2 + 2\pi rh$

$$600\pi = \pi r^2 + 2\pi rh$$
$$600 - r^2 = 2rh$$
$$h = \frac{600 - r^2}{2r}$$
$$V = \pi r^2 h$$
$$= \pi r^2\left(\frac{600 - r^2}{2r}\right)$$
$$= \frac{\pi r}{2}(600 - r^2)$$

(b) $V = 300\pi r - \frac{\pi}{2}r^3$

$$\frac{dV}{dr} = 300\pi - \frac{3}{2}\pi r^2$$

At stationary points, $\frac{dV}{dr} = 0$.

$$\Rightarrow 300\pi - \frac{3}{2}\pi r^2 = 0$$
$$\Rightarrow 300\cancel{\pi} = \frac{3}{2}\cancel{\pi}r^2$$

Therefore, $r^2 = 200$

$r = \sqrt{200}$ cm

Since $V = \frac{\pi r}{2}(600 - r^2)$:

when $r = \sqrt{200}$, $V = \frac{\pi \times \sqrt{200}}{2}(600 - 200)$

$$= 200\pi\sqrt{200}$$
$$= 2000\sqrt{2}\pi\,\text{cm}^3 \quad (\text{Since } \sqrt{200} = \sqrt{2 \times 100} = 10\sqrt{2})$$

$$\frac{d^2V}{dr^2} = \frac{\pi}{2} \times -6r$$

When $r = \sqrt{200}, \frac{d^2V}{dr^2} = \frac{\pi}{2}(-6\sqrt{200}) < 0$

Hence, the stationary value of V is a maximum.

EXAMPLE 31 Tristan has 150 cm^2 of paper to make an open rectangular box of length $2x$ cm, width x cm and height h cm. Express h in terms of x. Find the value of x which will make the volume of the box a maximum. Hence, find the volume.

SOLUTION

Surface area of the box $= (2x \times x) + (2xh)(2) + 2(hx)$

$= 6xh + 2x^2$

$6xh + 2x^2 = 150$

$3xh + x^2 = 75$

$h = \dfrac{75 - x^2}{3x}$

$V = 2x \times x \times h$

$= 2x^2\left(\dfrac{75 - x^2}{3x}\right)$

$= 50x - \dfrac{2}{3}x^3$

$\dfrac{dV}{dx} = 50 - 2x^2$

$\dfrac{d^2V}{dx^2} = -4x$

At turning points, $\dfrac{dV}{dx} = 0$.

$\Rightarrow 50 - 2x^2 = 0$

$\Rightarrow x^2 = \dfrac{50}{2}$

$\Rightarrow x^2 = 25$

$\Rightarrow x = \sqrt{25}$

$= 5$

When $x = 5$, $\dfrac{d^2V}{dx^2} = -4(5) = -20 < 0 \Rightarrow$ maximum point

Therefore, the volume is maximum.

When $x = 5$, $V = 50(5) - \dfrac{2}{3}(5)^3$

$= 166\dfrac{2}{3}\ \text{cm}^3$

EXAMPLE 32 The volume of a right-solid cylinder of radius r cm is $500\pi\ \text{cm}^3$, find the value of r for which the total surface area of the solid is a minimum.

SOLUTION

Volume of cylinder $= \pi r^2 h$

$\Rightarrow \pi r^2 h = 500\pi$

$h = \dfrac{500\pi}{\pi r^2}$

$= \dfrac{500}{r^2}$

Total surface area $A = 2\pi rh + 2\pi r^2$

$A = 2\pi r\left(\dfrac{500}{r^2}\right) + 2\pi r^2 \quad \left(\text{Substituting } h = \dfrac{500}{r^2}\right)$

$= \dfrac{1000\pi}{r} + 2\pi r^2$

$\dfrac{dA}{dr} = -\dfrac{1000\pi}{r^2} + 4\pi r$

At minimum points, $\frac{dA}{dr} = 0$.

$\Rightarrow -\frac{1000\pi}{r^2} + 4\pi r = 0$

$\Rightarrow 4\pi r = \frac{1000\pi}{r^2}$

$4\pi r^3 = 1000\pi$

$r^3 = \frac{1000\pi}{4\pi}$

$r^3 = 250$

$r = \sqrt[3]{250}$

$r = 6.3\,\text{cm}$

$\frac{d^2A}{dr^2} = \frac{2000\pi}{r^3} + 4\pi$

When $r = 6.3$, $\frac{d^2A}{dr^2} = \frac{2000\pi}{(6.3)^3} + 4\pi = 12\pi > 0$

Therefore, the surface area is a minimum when $r = 6.3\,\text{cm}$.

EXERCISE 14B

In questions 1 to 7, find and classify the points of inflexion of each function.

1 $y = x^2 - 2x + 1$

2 $y = x^3 - 3x + 1$

3 $y = 2x^3 - 3x^2 - 12x + 1$

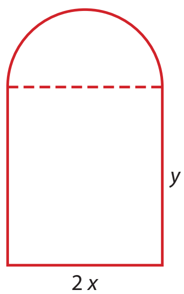
4 $y = x^3 - 6x^2 + 9x - 2$

5 $y = x^4 - 2x^2 + 3$

6 $y = \frac{x}{x^2 + 1}$

7 $y = \frac{x^2 - 4}{x^2 + 4}$

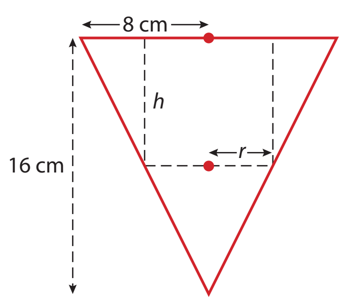
8 A factory wishes to make a large number of cylindrical containers using a thin metal, each to hold $20\,\text{cm}^3$. The surface area of each container is $A\,\text{cm}^2$. Find A as a function of r, the radius of the cylinder. Hence, find the radius and height of the cylinder so that the total area of the metal used is a minimum.



9 A window is in the shape of a rectangle surmounted by a semicircle whose diameter is the width of the window (see diagram). The perimeter is 15 m. Find the width of the window when the area is a maximum.

10 The dimensions of a rectangular sheet of cardboard are 16 m by 10 m. Equal squares of length x m are cut away from the four corners of the cardboard; the remaining edges are folded to form a rectangular open box of volume $V\,\text{cm}^3$. Find the volume V in terms of x and hence, find the maximum volume of the box.

11 The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 8 cm and height 16 cm. The cylinder just fits inside the cone as shown.



(a) Express h in terms of r and hence, show that the volume, $V\,\text{cm}^3$, of the cylinder is given by $V = \frac{2}{3}\pi r^2(8 - r)$.

(b) Given that r varies, find the value of V.

12 A wire 120 cm long, is cut into two pieces. one piece is bent to form a square of side y cm and the other is bent to form a circle of radius x cm. The total area of the square and the circle is A cm^2.

(a) Show that $A = \dfrac{(4 + \pi)y^2 + 3600 - 240y}{\pi}$.

(b) Given that x and y can vary, show that A has a maximum value when $y = \dfrac{120}{4 + \pi}$.

13 A solid rectangular block has a base which measures $3x$ cm by $2x$ cm. The height of the block is h cm and the volume of the block is 144 cm^3.

(a) Express h in terms of x and show that the total surface area A cm^2, of the block is given by $A = 12x^2 + \dfrac{240}{x}$.

(b) Given that x can vary, find the following.

(i) Find the value of x for which A has a value.

(ii) Find the value and determine whether it is a maximum or a minimum.

14 A box with a square base and open top must have a volume of 64 000 cm^3. Find the dimensions of the box that minimise the amount of material used.

Parametric differentiation

Let $x = f(t)$, $y = g(t)$. We can find $\dfrac{dy}{dx}$ using the chain rule as follows.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \text{ and by extension } \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

EXAMPLE 33 The equation of a curve is defined by $x = 6t^2 + 5$, $y = t^3 + 2t + 1$. Find the gradient of the tangent to the curve at $t = 1$ and the value of $\dfrac{d^2y}{dx^2}$ at $t = 1$.

SOLUTION We need to find $\dfrac{dy}{dx}$ when $t = 1$.

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

$\dfrac{dx}{dt} = 12t$, $\dfrac{dy}{dt} = 3t^2 + 2$

$\therefore \dfrac{dy}{dx} = \dfrac{3t^2 + 2}{12t}$

When $t = 1$, $\dfrac{dy}{dx} = \dfrac{3(1)^2 + 2}{12(1)} = \dfrac{5}{12}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}\left[\frac{3t^2 + 2}{12t}\right] = \frac{d}{dt}\left[\frac{1}{4}t + \frac{1}{6t}\right] = \frac{1}{4} - \frac{1}{6t^2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{1}{4} - \frac{1}{6t^2}}{12t}$$

When $t = 1$, $\dfrac{d^2y}{dx^2} = \dfrac{\frac{1}{4} - \frac{1}{6}}{12} = \dfrac{1}{144}$.

EXAMPLE 34 The equation of a curve is defined by $x = \sin 4t$, $y = 2\cos 4t - 1$. Show that $\dfrac{dy}{dx} = -2\tan 4t$ and find $t = 1$, $\dfrac{d^2y}{dx^2}$ in terms of t.

SOLUTION

$$\frac{dx}{dt} = 4\cos 4t, \frac{dy}{dt} = -8\sin 4t$$

Since $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we have:

$$\frac{dy}{dx} = \frac{-8\sin 4t}{4\cos 4t} = -2\tan 4t$$

Now $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}$

$$\frac{d}{dt}\left[\frac{dy}{dx}\right] = \frac{d}{dt}[-2\tan 4t] = -8\sec^2 4t$$

$$\therefore \frac{d^2y}{dx^2} = \frac{-8\sec^2 4t}{4\cos 4t} = -\frac{2}{\cos^3 4t} = -2\sec^3 4t$$

Try these 14.3

(a) Given the parametric equations of a curve as $x = t^3 + 4t - 1$, $y = t^2 + 7t + 9$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of t.

(b) Find the value of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{4}$ for the curve with parametric equations $x = \tan t$, $y = 2\sin t + 1$.

Rate of change

Given three variables x, y and t: $\dfrac{dy}{dx}$ represents the rate of change of y with respect to x, $\dfrac{dx}{dt}$ represents the rate of change of x with respect to t and $\dfrac{dy}{dt}$ represents the rate of change of y with respect to t. The connected rate of change for the three variables are:

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

When solving problems involving rate of change, follow these steps.

(i) Identify all variables clearly.

(ii) State any rates that are given.

(iii) State clearly what needs to be found.

(iv) Identify the connection among the three variables.

EXAMPLE 35 A cube is expanding in such a way that its sides are changing at a rate of $4\,\text{cm s}^{-1}$. Find the rate of change of the total surface area of the cube when its volume is $216\,\text{cm}^3$.

SOLUTION Let us identify our variables:

Let x be the length of one side of the cube

A be the total surface area

V be the volume of the cube

t time

What do we need to find?

$\frac{dA}{dt}$ = rate of change of the surface area.

We need to find $\frac{dA}{dt}$ when $V = 216\,\text{cm}^3$.

What is given?

We have $\frac{dx}{dt} = 4\,\text{cm s}^{-1}$

Our connection:

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$

To find $\frac{dA}{dt}$ we need a connection between A and x.

Since A is the surface area of the cube:

$A = 6 \times x \times x = 6x^2$

$\frac{dA}{dx} = 12x$

We have not used the volume of the cube.

Since $V = x \times x \times x = x^3$

When $V = 216\,\text{cm}^3$

$x^3 = 216$

$x = \sqrt[3]{216}$

$= 6\,\text{cm}$

Therefore, when $x = 6\,\text{cm}$, $\frac{dA}{dx} = 12(6) = 72$.

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$$
$$= 72 \times 4$$
$$= 288\,\text{cm}^2\,\text{s}^{-1}$$

Note the connections:
$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt}$
A is connected to x by the surface area of the cube.

EXAMPLE 36 Two variables x and y are connected by the equation $y = 4x(x + 2)$. If x is increasing at a rate of 2 units per second, find the rate of change of y at the instant when $x = 6$ units.

SOLUTION We need to find $\frac{dy}{dt}$ when $x = 6$.

We are given $\frac{dx}{dt} = 2$.

Rate of change connection:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Since $y = 4x(x + 2)$

$y = 4x^2 + 8x$

$\frac{dy}{dx} = 8x + 8$

When $x = 6$, $\frac{dy}{dx} = 8(6) + 8 = 56$.

Hence, $\frac{dy}{dt} = 56 \times 2$

$= 112$ units per second.

EXAMPLE 37 A spherical ball is being inflated at a rate of $40\,\text{cm}^3\,\text{s}^{-1}$. Find the rate of increase of its radius given that the surface area is $200\pi\,\text{cm}^2$.

SOLUTION Let r be the radius at time t

A be the surface area at time t

V be the volume of the sphere

We need to find $\frac{dr}{dt}$ when $A = 200\pi$.

We are given $\frac{dV}{dt} = 40\,\text{cm}^3\,\text{s}^{-1}$

Connected rate of change:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

Since we are dealing with a sphere:

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 3 \times \frac{4}{3}\pi r^2$

$\frac{dV}{dr} = 4\pi r^2$

When $A = 200\pi$, $\frac{dV}{dr} = 200\pi$ (Since surface area of a sphere $= 4\pi r^2$)

Now $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\Rightarrow 40 = 200\pi \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{40}{200\pi}$

$= \frac{1}{5\pi}\,\text{cm}\,\text{s}^{-1}$

EXAMPLE 38 A gas in a container changes its volume according to the law $PV = c$, where P is the number of units of pressure, V is the number of units of volume and c is a constant. Given that P is increasing at a rate of 40 units per second at the instant when $P = 80$, calculate in terms of c the rate of change of the volume at this instant.

SOLUTION We need to find $\frac{dV}{dt}$.

We are given $\frac{dP}{dt} = 40$ when $P = 80$.

Connected rate of change: $\frac{dV}{dt} = \frac{dV}{dP} \times \frac{dP}{dt}$.

$PV = c$

$V = \frac{c}{P}$

$\frac{dV}{dP} = -\frac{c}{p^2}$

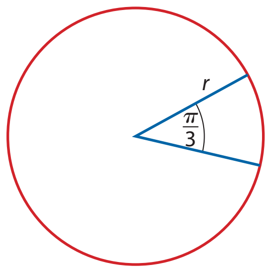
Therefore, $\frac{dV}{dP} = -\frac{c}{(80)^2}$

$= -\frac{c}{6\,400}$

$\frac{dV}{dt} = -\frac{c}{6\,400} \times 40$

$= -\frac{c}{160}$

EXAMPLE 39 A sector of a circle of radius r and centre O has an angle of $\frac{\pi}{3}$ radians. Given that r increases at a constant rate of 8 cm s^{-1}, calculate, the rate of increase of the area of the sector when $r = 4\text{ cm}$.



SOLUTION Let A be the area of sector.

We need to find $\frac{dA}{dt}$ when $r = 4\text{ cm}$.

We are given $\frac{dr}{dt} = 8\text{ cm s}^{-1}$.

Connected rate of change:

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

Since we are dealing with the area of a sector:

$A = \frac{1}{2}r^2\theta$

When $\theta = \frac{\pi}{3}$, $A = \frac{\pi}{6}r^2$.

$\Rightarrow \frac{dA}{dr} = 2 \times \frac{\pi}{6}r = \frac{\pi}{3}r$

When $r = 4\text{ cm}$, $\frac{dA}{dr} = \frac{4\pi}{3}$.

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$= \frac{4\pi}{3} \times 8$

$= \frac{32}{3}\pi\text{ cm}^2\text{ s}^{-1}$

EXERCISE 14C

1 The variables x and y are related by the equation $x^3y = 10$. If x increases at a rate of 0.5 units per second, find the rate of change of y when:

(a) $x = 5$

(b) $y = 2$.

2 The variables x and y are related by $\frac{1}{y^2} = \frac{1}{50} - \frac{1}{x^2}$ if x increases at a rate of 5 cm s^{-1}, calculate the rate of increase change of y when $x = 10\text{ cm}$.

3 Air is being pumped into a spherical balloon at a rate of $0.04\,\text{cm}^3/\text{s}$. Find the rate of increase of the surface area of the balloon when the volume is $150\pi\,\text{cm}^3$.

4 A tank, initially empty, is being filled with water. The depth of the tank is x cm and its volume $V\,\text{cm}^3$ is given by $V = x^6(x^2 + 4)$. Given that the depth of the water increases at a rate of $4\,\text{cm}\,\text{s}^{-1}$, find the rate of increase of the volume when $x = 1$ cm.

5 The radius of a circle is increasing at a rate of $2\,\text{cm}\,\text{s}^{-1}$. Find the rate of increase of the circumference of the circle when the area of the circle is $4\pi\,\text{cm}^2$.

6 A cube is expanding in such a way that its sides are changing at a rate of $0.05\,\text{cm}\,\text{s}^{-1}$. Find the rate of change of the total surface area of the cube when its volume is $64\,\text{cm}^3$.

7 The circumference of a circle is increasing at a rate of $4\,\text{cm}\,\text{s}^{-1}$.

(a) Find the rate of increase of the radius.

(b) Find the rate of increase of the area, at the instant when the radius is 64 cm.

8 The surface area of a sphere is increasing at a constant rate of $10\,\text{cm}^2\,\text{s}^{-1}$. Find the rate of increase of the following.

(a) The radius

(b) The volume, at the instant when the radius is 4 cm

9 Variables x and y are related by $y = \dfrac{4x + 2}{x + 1}$. Given that x and y are functions of t and that $\dfrac{dy}{dt} = 0.3$. Find the corresponding rate of change of x when $y = 5$.

10 A metal cube is being expanded by heat. At the instant when the length of an edge is 4 cm, the volume of the cube is increasing at the rate of $0.024\,\text{cm}^3\,\text{s}^{-1}$. At what rate is one length of the edge increasing at this instant?

11 A certain gas, under varying pressure, changes its volume according to the law $PV = 600$, where P is the number of units of pressure and V is the number of units of volume.

(a) Find the rate at which P increases with V.

(b) What is $\dfrac{dp}{dv}$ when $V = 20\,\text{cm}^3$?

12 Q is a fixed point on the circumference of a circle, centre O, radius 6 cm. A variable point P moves round the circumference such that θ increases at a constant rate of $\dfrac{\pi}{3}$ radians per second.

(a) Find the rate of change of the arc length from P to Q.

(b) Find the rate of change of the area of the sector POQ.

13 Liquid is poured into a container at a rate of $16\,\text{cm}^3\,\text{s}^{-1}$. The volume of liquid in the container is $V\,\text{cm}^3$, where $V = 2\left(x^2 - \frac{7}{2}x\right)$ and x cm is the height of liquid in the container. Find the following, when $V = 4$.

(a) The value of x

(b) The rate at which x is increasing

Curve sketching

Polynomials, rational functions, trigonometric functions

Polynomials

When sketching a curve, we gather as much information as possible about the function and then sketch the graph.

For a polynomial we can find

(i) The stationary points

(ii) The points where the curve cuts the axes

(iii) The intervals on which the function is increasing or decreasing

EXAMPLE 40 Sketch the graph of $y = x^3 - x$.

SOLUTION In order to sketch the graph, we need to do the following.

- Identify the domain of the function.
- Find the x and y intercepts where possible.
- Identify maximum and minimum points.
- Identify concavity and points of inflexion.

Remember

- $f''(x)$ can be positive and/or negative.
- Inflexion points occur when the curve has a concavity.
- If $\frac{d^2y}{dx^2}$ is positive, concave upwards.
- If $\frac{d^2y}{dx^2}$ is negative, concave downwards.

First we find the intercepts.

When $x = 0$, $y = 0$.

Therefore, $(0, 0)$ lies on the graph.

When $y = 0$, $x^3 - x = 0$.

$\Rightarrow x(x^2 - 1) = 0$

$\Rightarrow x = 0, x = 1, -1$

Therefore, $(1, 0)$ and $(-1, 0)$ also lie on the graph.

Now we find the turning points of the curve.

$y = x^3 - x$

$\frac{dy}{dx} = 3x^2 - 1$

At the turning points, $\frac{dy}{dx} = 0$.

$\Rightarrow 3x^2 - 1 = 0$

$x^2 = \frac{1}{3}$

$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

When $x = \frac{\sqrt{3}}{3}$, $y = \left(\frac{\sqrt{3}}{3}\right)^3 - \frac{\sqrt{3}}{3}$

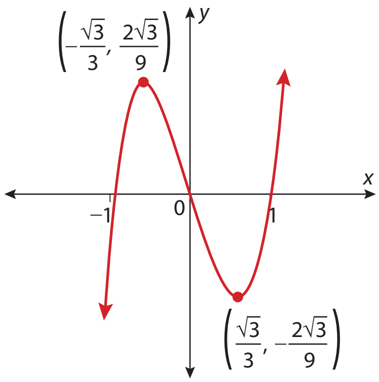
$$= \frac{3\sqrt{3}}{27} - \frac{\sqrt{3}}{3}$$

$$= \frac{\sqrt{3}}{9} - \frac{\sqrt{3}}{3}$$

$$= -2\frac{\sqrt{3}}{9}$$

When $x = -\frac{\sqrt{3}}{3}$, $y = \left(-\frac{\sqrt{3}}{3}\right)^3 + \frac{\sqrt{3}}{3}$

$$= +2\frac{\sqrt{3}}{9}$$



Classify the turning points:

$\frac{d^2y}{dx^2} = 6x$

When $x = \frac{\sqrt{3}}{3}$, $\frac{d^2y}{dx^2} = 6\frac{\sqrt{3}}{3} > 0 \Rightarrow$
The turning point is a minimum.

When $x = -\frac{\sqrt{3}}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{\sqrt{3}}{3}\right) < 0 \Rightarrow$ The turning point is a maximum.

EXAMPLE 41 Sketch the graph of $y = 4x^3 - 15x^2 + 12x - 8$.

SOLUTION When $x = 0$, $y = -8$, the graph passes through $(0, -8)$.

Stationary points:

$\frac{dy}{dx} = 12x^2 - 30x + 12$

At stationary points $\frac{dy}{dx} = 0$.

$\Rightarrow 12x^2 - 30x + 12 = 0$

$2x^2 - 5x + 2 = 0$

$(2x - 1)(x - 2) = 0$

$\Rightarrow x = \frac{1}{2}, x = 2$

When $x = \frac{1}{2}$, $y = 4\left(\frac{1}{2}\right)^3 - 15\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) - 8 = -\frac{21}{4}$

When $x = 2$, $y = 4(2)^3 - 15(2)^2 + 12(2) - 8 = -12$

Classify the stationary points:

$\frac{d^2y}{dx^2} = 24x - 30$

When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 24\left(\frac{1}{2}\right) - 30 = 12 - 30 = -18 < 0 \Rightarrow$ There is maximum point when $x = \frac{1}{2}$.

When $x = 2$, $\frac{d^2y}{dx^2} = 24(2) - 30 = 48 - 30 = 18 > 0 \Rightarrow$ There is a minimum point when $x = 2$.

$$\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow 24x - 30 = 0$$

$$\Rightarrow x = \frac{30}{24} = \frac{5}{4}$$

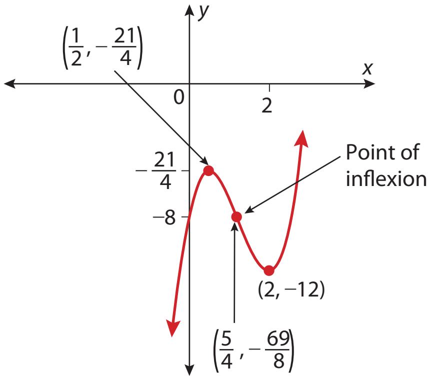
When $x = \frac{5}{4}$, $y = 4\left(\frac{5}{4}\right)^3 - 15\left(\frac{5}{4}\right)^2 + 12\left(\frac{5}{4}\right) - 8 = -\frac{69}{8}$

Sign change of $\frac{d^2y}{dx^2}$:

For $x < \frac{5}{4}$, $\frac{d^2y}{dx^2}$ is negative.

For $x > \frac{5}{4}$, $\frac{d^2y}{dx^2}$ is positive.

Since the concavity of $\frac{d^2y}{dx^2}$ changes as x passes through $\frac{5}{4}$, when $x = \frac{5}{4}$ there is a point of inflexion.



EXAMPLE 42 Let $y = 3x^4 - 4x^3 - 12x^2 + 8$.

(a) Find $\frac{dy}{dx}$.

(b) Find the coordinate of the stationary points of the curve.

(c) Classify the stationary points.

(d) Find the points for which $\frac{d^2y}{dx^2} = 0$ and classify these points.

(e) Sketch the graph of y.

SOLUTION

(a) $y = 3x^4 - 4x^3 - 12x^2 + 8$

$$\frac{dy}{dx} = 12x^3 - 12x^2 - 24x$$

(b) At stationary points $\frac{dy}{dx} = 0$.

$$\Rightarrow 12x^3 - 12x^2 - 24x = 0$$
$$12x(x^2 - x - 2) = 0$$
$$12x(x - 2)(x + 1) = 0$$
$$\Rightarrow x = 0, x = 2, x = -1$$

When $x = 0$, $y = 8$ (0, 8)

When $x = -1$, $y = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 8$
$= 3 + 4 - 12 + 8$
$= 3$
$(-1, 3)$

$x = 2$, $y = 3(2)^4 - 4(2)^3 - 12(2)^2 + 8$
$= 48 - 32 - 48 + 8$
$= -24$
$(2, -24)$

The stationary points are at $(-1, 3)$, $(0, 8)$ and $(2, -24)$.

(c) Using the second derivative test gives:

$$\frac{d^2y}{dx^2} = 36x^2 - 24x - 24$$

When $x = 0, \frac{d^2y}{dx^2} = -24 < 0 \Rightarrow$ There is a maximum point at $(0, 8)$.

When $x = -1, \frac{d^2y}{dx^2} = 36(-1)^2 - 24(-1) - 24 = 36 > 0 \Rightarrow$ There is a minimum point at $(-1, 3)$.

When $x = 2, \frac{d^2y}{dx^2} = 36(2)^2 - 24(2) - 24 = 72 > 0 \Rightarrow$ There is minimum point at $(2, -24)$.

(d) $\frac{d^2y}{dx^2} = 0$

$\Rightarrow 36x^2 - 24x - 24 = 0$

$3x^2 - 2x - 2 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{2^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{28}}{6}$$

$$= \frac{2 \pm 2\sqrt{7}}{6}$$

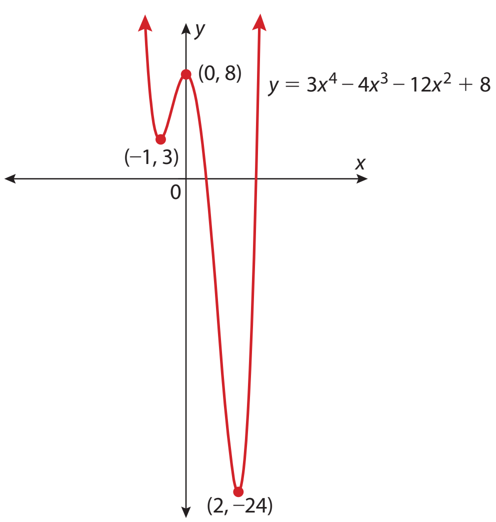
$$= \frac{1 \pm \sqrt{7}}{3}$$

When $x = \frac{1 + \sqrt{7}}{3}, y = -10.4$

When $x = \frac{1 - \sqrt{7}}{3}, y = 5.3$

These are the points of inflexion.

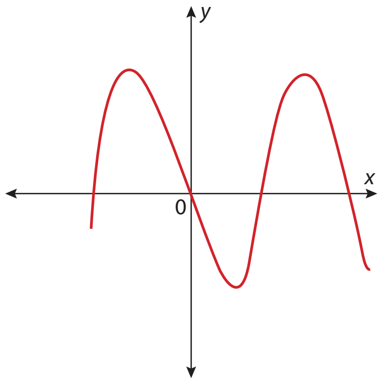
(e)



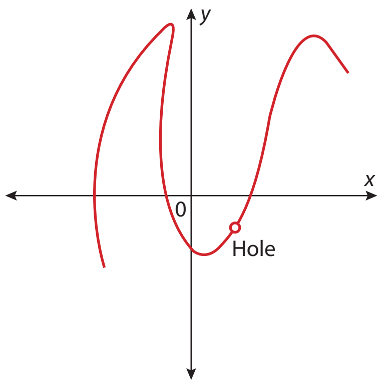
Graph of a polynomial

Both linear functions and quadratic functions are polynomial. The graph of a linear function $y = mx + c$ is a straight line with gradient m and y-intercept c. The graph of a quadratic $y = ax^2 + bx + c$, where $a \neq 0$, is a parabola with a minimum turning point if $a > 0$ and a maximum turning point if $a < 0$. These two graphs are smooth and continuous. The graph of every polynomial is both smooth and continuous. There are no sharp corners or holes or gaps on a polynomial graph. We can draw one continuous curve without lifting the pencil when drawing the graph of a polynomial.

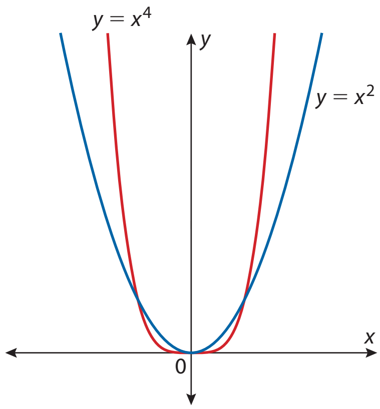
This is a graph of a polynomial since the curve is smooth and continuous.



This graph cannot be the graph of a polynomial, since there is a hole.



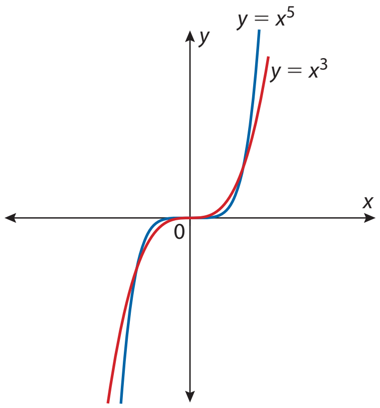
Graphs of functions of the form $f(x) = x^n$ where n is an even integer



When n is even, the graphs of $y = x^2, y = x^4, y = x^6, y = x^8\dots$ each have a minimum turning point at the origin but increase rapidly as the power of x increases. All the curves will touch the x-axis at (0, 0) and pass through the points $(-1, 1)$ and (1, 1). The graphs are all symmetric about the y-axis.

Graphs of functions of the form $f(x) = x^n$ where n is an odd integer greater than 1

All the curves will pass through the points (0, 0) and pass through the points $(-1, -1)$ and (1, 1). The graphs are symmetric with respect to the origin. The graph increases or decreases rapidly as n increases.



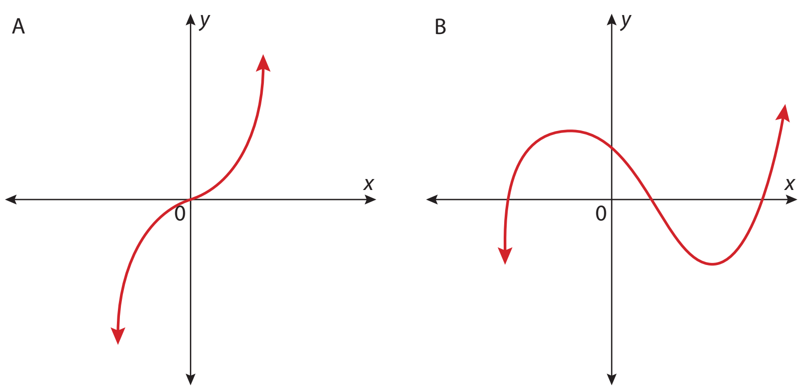
Graphs of polynomials

The graph of a polynomial will depend on the sign of the leading term in the polynomial and the degree of the polynomial. Recall that the degree of a polynomial is the highest power of x in the polynomial. The behaviour of the graph depends on whether the polynomial is of odd degree or even degree.

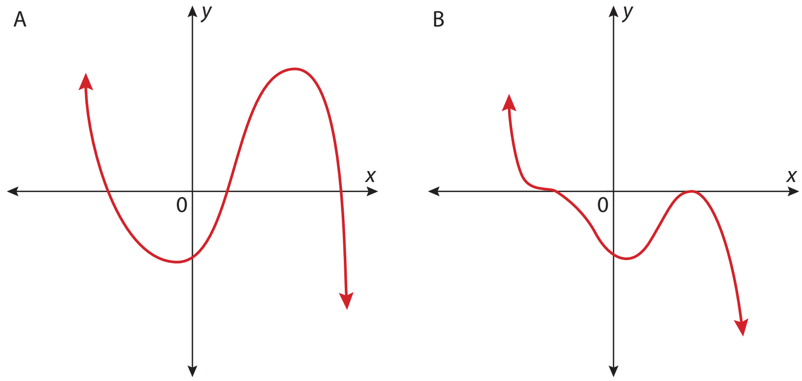
Polynomials of odd degree

For a polynomial with a degree that is an odd number, the behaviour of the end points of the polynomial can be deduced as follows.

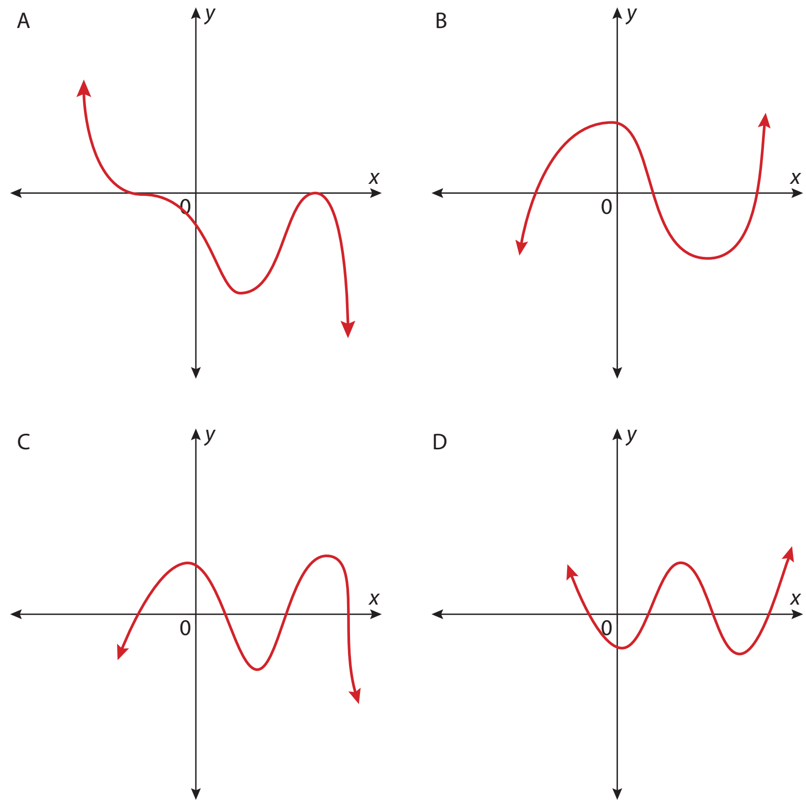
With a positive leading coefficient, the graph falls to the left and rises to the right as seen below.



With a negative leading coefficient, the graph rises to the left and falls to the right as shown below.



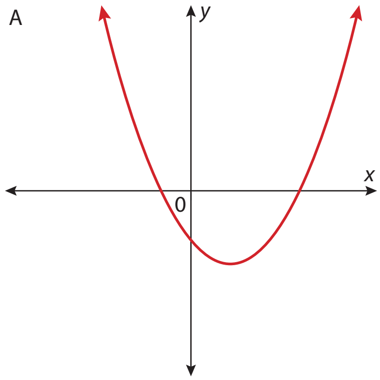
EXAMPLE 43 Which of the following could be the graph of a polynomial whose leading term is $-4x^5$?

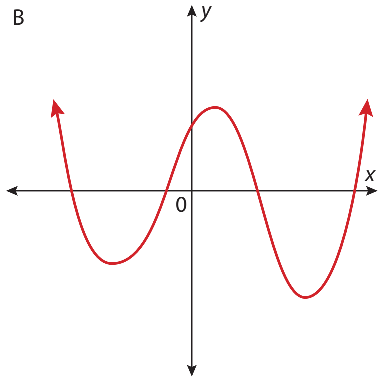


SOLUTION Since the leading polynomial is of odd degree and the coefficient of the leading polynomial is negative, the graph rises to the left and falls to the right. The answer is A.

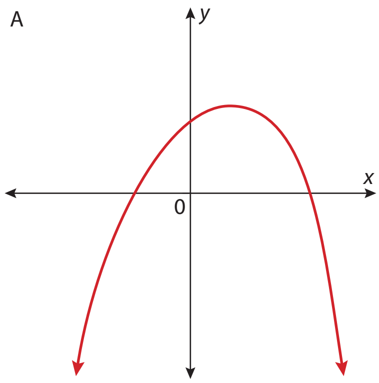
Polynomials of even degree

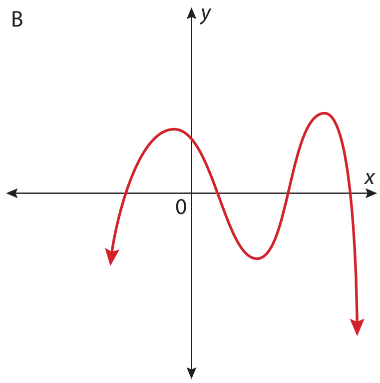
With a positive leading coefficient and a polynomial of even degree, the graph will rise on both ends as shown below.



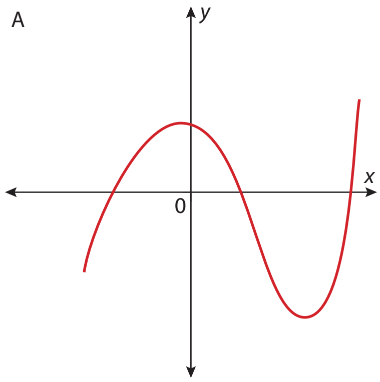


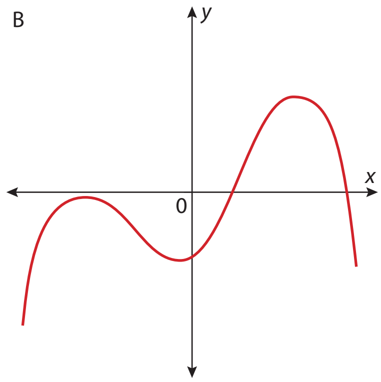
With a negative leading coefficient and a polynomial of even degree, the graph falls on both ends as shown later.

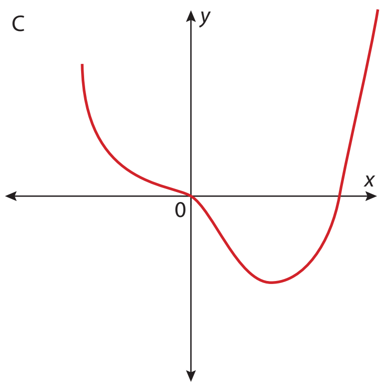


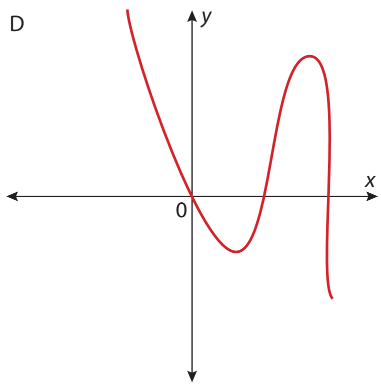


EXAMPLE 44 Which of the following functions could be the graph of $y = 2x^4 - 4x$?









SOLUTION The degree of the polynomial is even and the coefficient of the leading term is positive. With a positive coefficient and even degree, the graph rises on both ends. The answer is C.

Zeros of a polynomial

DEFINITION

If $f(x)$ is a polynomial in x and $f(a) = 0$, then a is a zero of $f(x)$.

Note

(i) If a is a zero of $f(x) = 0$, then $x - a$ is a factor of $f(x)$.
(ii) All the zeros of a polynomial are the x-intercept of the graph of $f(x)$.
(iii) If the factor $(x - a)$ occurs more than once, a is called a repeated zero or a repeated root of the polynomial equation.

If we can locate all the zeros of a polynomial, then the graph will cut the x-axis at these points. The curve will either be above the x-axis or below the x-axis between each zero.

EXAMPLE 45 Sketch the graph of $y = x(x - 1)(x - 2)$.

SOLUTION Let us find the zero of the function.

When $y = 0$, $x(x - 1)(x - 2) = 0$

$\Rightarrow x = 0, x - 1 = 0, x - 2 = 0$

$\Rightarrow x = 0, x = 1, x = 2$

The zero of the polynomial will split the x-axis into the following intervals:

$[-\infty, 0], [0, 1], [1, 2]$ and $[2, \infty]$

As the graph passes through the zeros, it will be either above the x-axis or below the x-axis. We can make a decision by looking at points within each interval.

When $x = -1$, $\quad y = (-1)(-1 - 1)(-1 - 2) = -6$, $\qquad (-1, -6)$

$x = \frac{1}{2}$, $\quad y = \left(\frac{1}{2}\right)\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{8}$ $\qquad \left(\frac{1}{2}, \frac{3}{8}\right)$

$x = \frac{3}{2}$, $\quad y = \left(\frac{3}{2}\right)\left(\frac{3}{2} - 1\right)\left(\frac{3}{2} - 2\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) = -\frac{3}{8}$ $\qquad \left(\frac{3}{2}, -\frac{3}{8}\right)$

$x = 3$, $\quad y = 3(3 - 1)(3 - 2) = 6$ $\qquad (3, 6)$

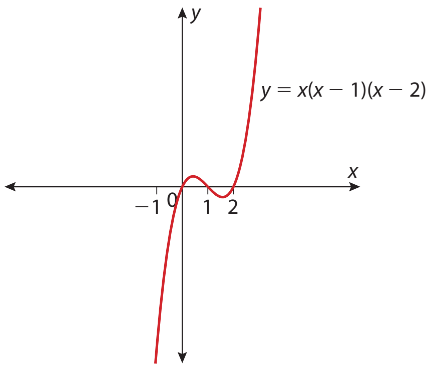
Points on the curve are $(-1, -6), \left(\frac{1}{2}, \frac{3}{8}\right), \left(\frac{3}{2}, -\frac{3}{8}\right), (3, 6)$.

For $x < 0$ the graph is below the x-axis.

For $0 < x < 1$ the curve is above the x-axis.

For $1 < x < 2$ the curve is below the x-axis and for $x > 2$, the curve is above the x-axis.

Draw a smooth curve passing through the points. The graph must turn and either go up or down to pass through the x-axis. We have two turning points, one maximum and one minimum.



EXAMPLE 46 **(a)** Find the values of x for which $2x^3 + x^2 - 8x = 4$.

(b) Sketch the graph of $f(x) = 2x^3 + x^2 - 8x - 4$.

SOLUTION **(a)** $f(x) = 2x^3 + x^2 - 8x - 4$

$$x = 2, \qquad f(2) = 2(2)^3 + (2)^2 - 8(2) - 4$$
$$= 16 + 4 - 16 - 4$$
$$= 0$$

Since $f(2) = 0$, $x - 2$ is a factor of $f(x)$.

$$\begin{array}{r}
2x^2 + \ \ 5x - 2 \\
x-2\overline{)\,2x^3 + \ \ x^2 - \ \ 8x - 4} \\
-\underline{(2x^3 - 4x^2)} \qquad\qquad \\
5x^2 - \ \ 8x \qquad \\
-\underline{(5x^2 - 10x)} \qquad \\
2x - 4 \\
-\underline{(2x + 4)} \\
0
\end{array}$$

$$\therefore f(x) = (x - 2)(2x^2 + 5x + 2)$$
$$= (x - 2)(x + 2)(2x + 1)$$

Now

$(x - 2)(x + 2)(2x + 1) = 0$

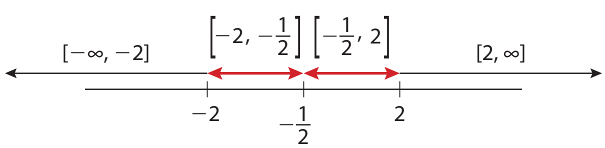
$x - 2 = 0, \qquad x + 2 = 0, \qquad 2x + 1 = 0$

$x = 2, \qquad x = -2, \qquad x = -\frac{1}{2}$

(b) Since $f(x) = 0$ when $x = -2$, $x = -\frac{1}{2}$ and $x = 2$, the graph of $f(x)$ cuts the x-axis at $(-2, 0)$, $\left(-\frac{1}{2}, 0\right)$ and $(2, 0)$.

We can divide the x-axis into these intervals.

$[-\infty, -2]$, $\left[-2, -\frac{1}{2}\right]$, $\left[-\frac{1}{2}, 2\right]$ and $[2, \infty]$



We next find the sign of $f(x)$ in each region.

When:

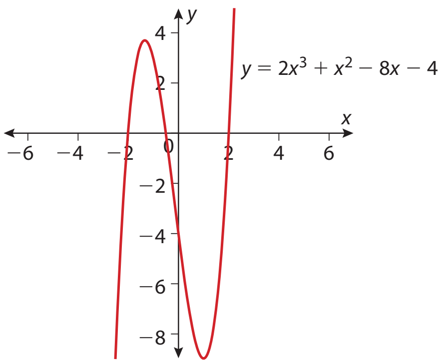
$x = -3, \quad f(x) = 2(-3)^3 + (-3)^2 - 8(-3) - 4 = -54 + 9 + 24 - 4 = -25$

$x = -1, \quad f(x) = 2(-1)^3 + (-1)^2 - 8(-1) - 4 = -2 + 1 + 8 - 4 = 3$

$x = 0, \quad f(x) = 2(0)^3 + (0)^2 - 8(0) - 4 = -4$

$x = 3, \quad f(3) = 2(3)^3 + (3)^2 - 8(3) - 4 = 35$

We can now draw the graph of $y = 2x^3 + x^2 - 8x - 4$.



Graphing functions

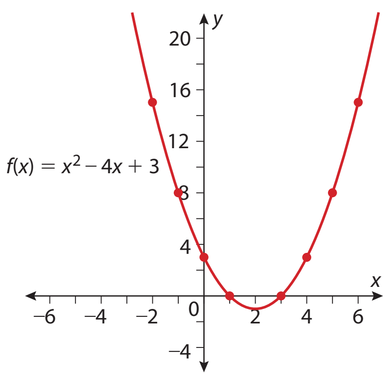
Graphing functions with a table of values

EXAMPLE 47 Let $f(x) = x^2 - 4x + 3$. Complete the table of values and draw the graph of $f(x)$ using the table of values.

x	-2	-1	0	1	2	3	4	5	6
$f(x)$									15

SOLUTION

x	$f(x) = x^2 - 4x + 3$
-2	$f(-2) = (-2)^2 - 4(-2) + 3 = 15$
-1	$f(-1) = (-1)^2 - 4(-1) + 3 = 8$
0	$f(0) = (0)^2 - 4(0) + 3 = 3$
1	$f(1) = (1)^2 - 4(1) + 3 = 0$
2	$f(2) = (2)^2 - 4(2) + 3 = -1$
3	$f(3) = (3)^2 - 4(3) + 3 = 0$
4	$f(4) = (4)^2 - 4(4) + 3 = 3$
5	$f(5) = (5)^2 - 4(5) + 3 = 8$



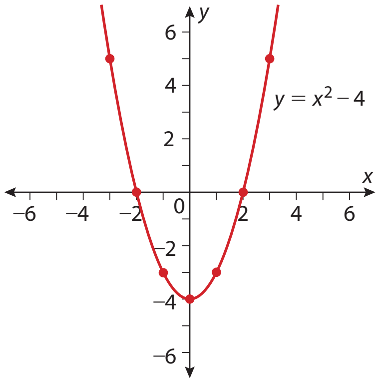
EXAMPLE 48 Given that $f(x) = x^2 - 4$, complete the table of values.

x	−3	−2	−1	0	1	2	3
$f(x)$							

Hence, draw the graph of $f(x)$ using the table of values.

SOLUTION

x	$f(x) = x^2 - 4$
−3	$f(-3) = (-3)^2 - 4 = 9 - 4 = 5$
−2	$f(-2) = (-2)^2 - 4 = 4 - 4 = 0$
−1	$f(-1) = (-1)^2 - 4 = 1 - 4 = -3$
0	$f(0) = 0^2 - 4 = -4$
1	$f(1) = 1^2 - 4 = -3$
2	$f(2) = (2)^2 - 4 = 0$
3	$f(3) = (3)^2 - 4 = 5$

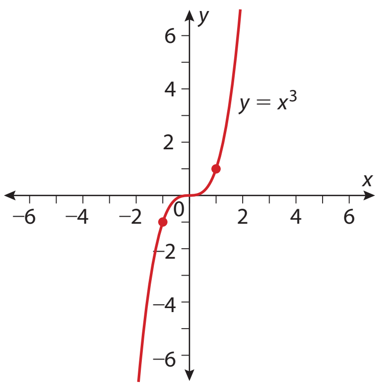


x	−3	−2	−1	0	1	2	3
$f(x)$	5	0	−3	−4	−3	0	5

EXAMPLE 49 Graph the function $y = x^3$ by using a table of values. Identify the domain and range of the function.

SOLUTION

x	$f(x) = x^3$
−3	$f(-3) = (-3)^3 = -27$
−2	$f(-2) = (-2)^3 = -8$
−1	$f(-1) = (-1)^3 = -1$
0	$f(0) = 0^3 = 0$
1	$f(1) = 1^3 = 1$
2	$f(2) = 2^3 = 8$
3	$f(3) = 3^3 = 27$

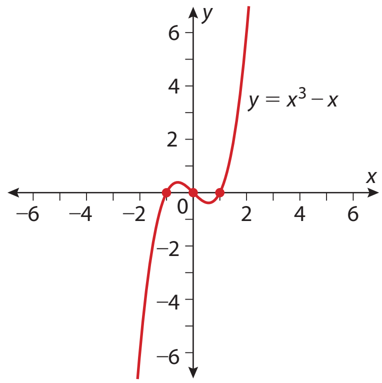


Domain and range are the set of real numbers $x \in \mathbb{R}$, $y \in \mathbb{R}$.

EXAMPLE 50 Graph the function $f(x) = x^3 - x$, using a table of values.

SOLUTION

x	$f(x) = x^3 - x$
-3	$f(-3) = (-3)^3 - (-3) = -24$
-2	$f(-2) = (-2)^3 - (-2) = -6$
-1	$f(-1) = (-1)^3 - (-1) = 0$
0	$f(0) = 0^3 - 0 = 0$
1	$f(1) = (1)^3 - 1 = 0$
2	$f(2) = (2)^3 - 2 = 6$
3	$f(3) = (3)^3 - 3 = 24$

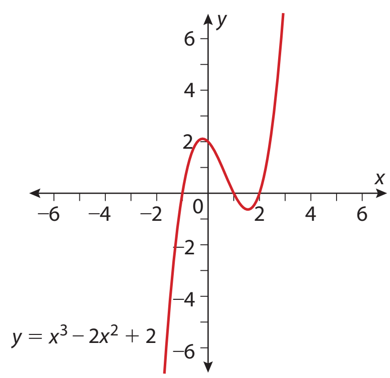


EXAMPLE 51 Fill out the following table for $f(x) = x^3 - 2x^2 - x + 2$ and graph the function $f(x)$.

x	-3	-2	-1	0	1	2	3
$f(x)$							

SOLUTION

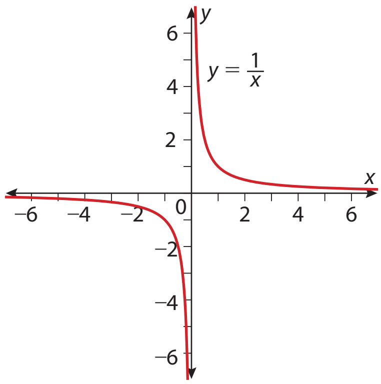
x	$f(x) = x^3 - 2x^2 - x + 2$
-3	$f(-3) = (-3)^3 - 2(-3)^2 - (-3) + 2$ $= -40$
-2	$f(-2) = (-2)^3 - 2(-2)^2 - (-2) + 2$ $= -12$
-1	$f(-1) = (-1)^3 - 2(-1)^2 - (-1) + 2$ $= 0$
0	$f(0) = (0)^3 - 2(0)^2 - (0) + 2 = 2$
1	$f(1) = (1)^3 - 2(1)^2 - (1) + 2 = 0$
2	$f(2) = (2)^3 - 2(2)^2 - (2) + 2 = 0$
3	$f(3) = (3)^3 - 2(3)^2 - (3) + 2 = 8$



EXAMPLE 52 Graph the function $g(x) = \frac{1}{x}$, using a table of values. Identify the domain and range of $g(x)$.

SOLUTION The function $g(x)$ is undefined at $x = 0$.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x) = \frac{1}{x}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	undefined	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$



The domain of $g(x)$, $x \in \mathbb{R}, x \neq 0$.

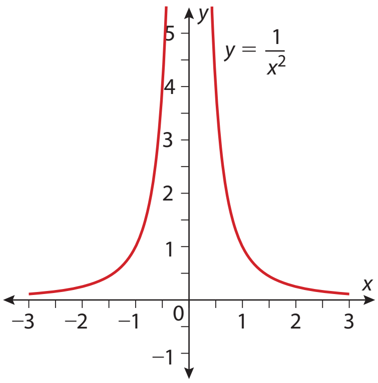
The range of $g(x)$, $y \in \mathbb{R}, y \neq 0$.

EXAMPLE 53 The function $g(x) = \frac{1}{x^2}$ identify the value of x for which $f(x)$ is undefined. Complete the table for $f(x)$. Using the table of $f(x)$, draw the graph of $f(x)$.

x	-3	-2	-1	0	1	2	3
$f(x)$							

SOLUTION The function is undefined at $x = 0$.

x	$f(x) = \frac{1}{x^2}$
-3	$\left(\frac{1}{-3}\right)^2 = \frac{1}{9}$
-2	$\left(\frac{1}{-2}\right)^2 = \frac{1}{4}$
-1	$\left(\frac{1}{-1}\right)^2 = 1$
0	undefined
1	$\left(\frac{1}{+1}\right)^2 = 1$
2	$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
3	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$



Solving simultaneous equations graphically

EXAMPLE 54 By sketching the graphs of $y = 2x + 1$ and $2y = 3x - 5$, find the solution of the simultaneous equations $y = 2x + 1$ and $2y = 3x - 5$.

SOLUTION To draw a straight-line graph we need two points on the line.

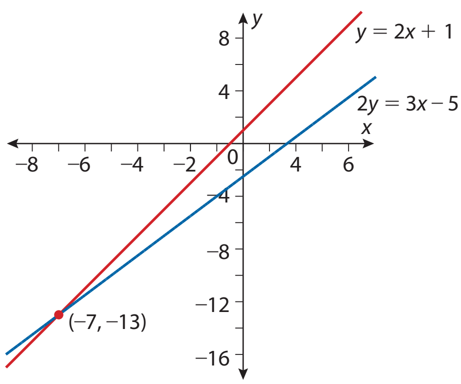
For $y = 2x + 1$:

x	0	−0.5
y	1	0

For $2y = 3x - 5$:

x	0	$\frac{5}{3}$
y	$-\frac{5}{2}$	0

Plot the points on the same graph and draw each line. Where the two lines intersect will be the solution to the equations.



The solution to the simultaneous equations is $x = -7$ and $y = -13$.

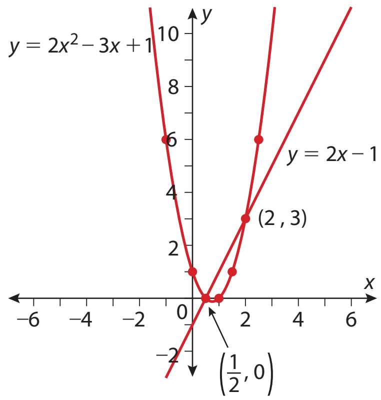
EXAMPLE 55 Draw the graphs of $y = 2x^2 - 3x + 1$ and $y = 2x - 1$. Hence, write down the solutions of the simultaneous equation.

SOLUTION For $y = 2x - 1$:

x	0	0.5
y	−1	0

For $y = 2x^2 - 3x + 1$:

x	-1	0	0.5	1	1.5	2	2.5	3
$2x^2$	2	0	0.5	2	4.5	8	12.5	18
$-3x$	3	0	-1.5	-3	-4.5	-6	-7.5	-9
1	1	1	1	1	1	1	1	1
y	6	1	0	0	1	3	6	10



The graphs intersect at $\left(\frac{1}{2}, 0\right)$ and (2, 3).

The solutions of the equations are $x = \frac{1}{2}$, $y = 0$, and $x = 2$, $y = 3$.

EXAMPLE 56 Solve graphically for x the simultaneous equations $y = x^2 - 2x + 4$ and $y = 3x - 2$.

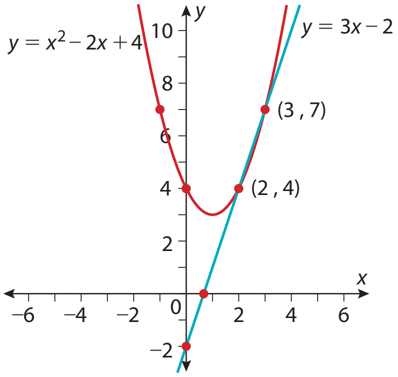
SOLUTION Sketch the two graphs on the same axes.

For $y = 3x - 2$:

x	0	$\frac{2}{3}$
y	-2	0

For $y = x^2 - 2x + 4$:

x	-1	0	0.5	1	1.5	2	2.5	3	3.5
x^2	1	0	0.25	1	2.25	4	6.25	9	12.25
$-2x$	2	0	-1	-2	-3	-4	-5	-6	-7
$+4$	4	4	4	4	4	4	4	4	4
y	7	4	3.25	3	3.25	4	5.25	7	9.25



EXAMPLE 57

(a) Fill in the following table.

x	−3	−2	−1	0	1	2	3
x^3							
$-x^2$							
$y = x^3 - x^2$							

(b) Draw the graph of $y = x^3 - x^2$, using the table of values.

(c) On the same axes draw the graph of $y = 8x - 12$.

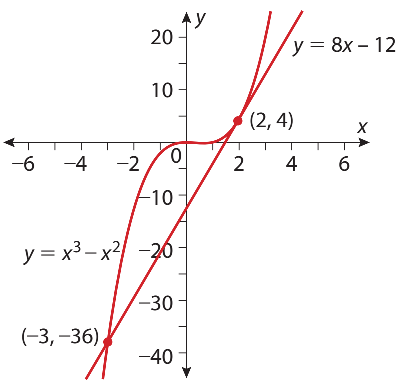
(d) Hence, solve the equation $x^3 - x^2 - 8x + 12 = 0$.

SOLUTION

(a)

x	−3	−2	−1	0	1	2	3
x^3	−27	−8	−1	0	1	8	27
$-x^2$	−9	−4	−1	0	−1	−4	−9
$y = x^3 - x^2$	−36	−12	−2	0	0	4	18

(b) and **(c)**



(d) $x^3 - x^2 - 8x + 12 = 0$

$\Rightarrow x^3 - x^2 = 8x - 12$

The solutions of the equation are the points of intersection of the two graphs $y = x^3 - x^2$ and $y = 8x - 12$

Read off the points of intersection: (2, 4) and (−3, −36)

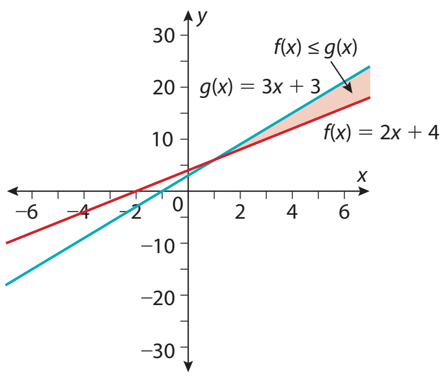
The solution of the equation is $x = 2, -3$.

Solving inequalities graphically

To solve the inequality $f(x) \le g(x)$, we first sketch the graphs of $y = f(x)$ and $y = g(x)$. We then shade the region for which $f(x) \le g(x)$. We can read off f the solution set from the graph.

EXAMPLE 58 Solve the inequality $f(x) \le g(x)$, where $f(x) = 2x + 4$ and $g(x) = 3x + 3$.

SOLUTION Sketch on the same axes the graphs of $f(x) = 2x + 4$ and $g(x) = 3x + 3$.



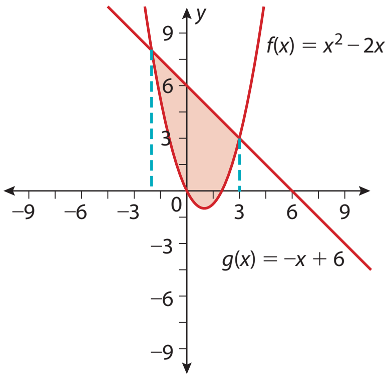
From the graph, the solution set is $\{x: x \ge 1\}$.

EXAMPLE 59 Solve the inequalities $f(x) \le g(x)$ where $f(x) = x^2 - 2x$, $g(x) = -x + 6$.

SOLUTION Draw the graph of $y = x^2 - 2x$ by using a table of values.

x	−2	−1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	−2	−4	−6	−8
y	8	3	0	−1	0	3	8

Draw the graph of $g(x) = -x + 6$.



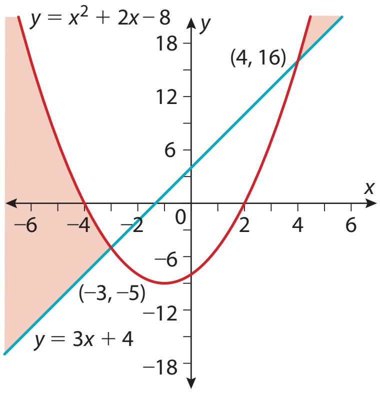
Note that the curve is below the line for $\{x: -2 \leq x \leq 3\}$ and hence, this becomes the solution set.

EXAMPLE 60 Find the solution set to $f(x) \geq g(x)$ when $f(x) = x^2 + 2x - 8$, $g(x) = 3x + 4$.

SOLUTION Draw the graph of $y = x^2 + 2x - 8$ and $y = 3x + 4$ on the same axes.

For $y = x^2 + 2x - 8$:

x	−3	−2	−1	0	1	2	3	4
x^2	9	4	1	0	1	4	9	16
$2x$	−6	−4	−2	0	2	4	6	8
−8	−8	−8	−8	−8	−8	−8	−8	−8
y	−5	−8	−9	−8	−5	0	7	16



The solution set is $\{x: x \leq -3\} \cup \{x: x \geq 4\}$.

Review of trigonometry

Trigonometry has a variety of applications in different aspects of our lives. Some of the fields that make use of trigonometry are architecture, astronomy, engineering, economics and computer graphics.

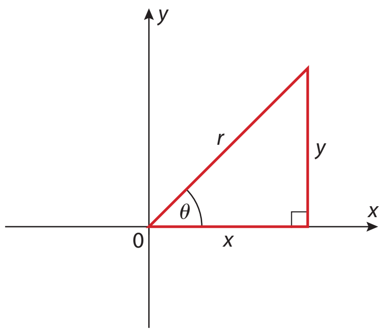
How do engineers make an exact structure or a blueprint? How do we measure the distance of the stars? How does the computer recognise music? How do pilots find their way in the sky?

The simplest sound – a pure tone – is represented by $f(t) = A\sin(2\pi wt)$. This equation represents that of a sinusoidal wave. Whenever you listen to music on an iPod, you are listening to sound waves. These waves take the shape of sine waves. As the period of the sine wave changes, the sound changes. When since functions for different notes are added together, you will hear more than one note at a time. Next time you listen to music, think of the sine curves at work.

Did you know that 'noise reducing' headphones also make use of since curves?

Let us review some of the work you have previously done on trigonometry.

$r = \sqrt{x^2 + y^2}$, Pythagoras' theorem



$\sin\theta = \dfrac{y}{r}$

$\cos\theta = \dfrac{x}{r}$

$\tan\theta = \dfrac{y}{x}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\cot\theta = \dfrac{1}{\tan\theta}$

Also, $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

Sine, cosine and tangent of 45°, 30° and 60°

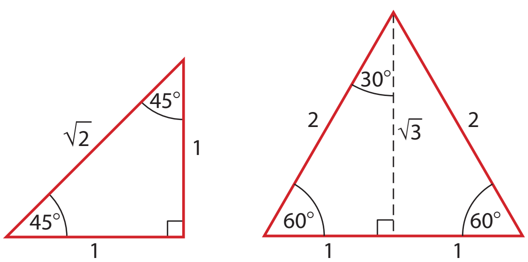
$\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$\tan 45° = 1$

$\sin 30° = \dfrac{1}{2}$, $\cos 30° = \dfrac{\sqrt{3}}{2}$, $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

$\sin 60° = \dfrac{\sqrt{3}}{2}$, $\cos 60° = \dfrac{1}{2}$, $\tan 60° = \sqrt{3}$



EXAMPLE 61 Express $\cot 30° + \cos 30°$ in terms of $\sqrt{3}$, without the use of a calculator.

SOLUTION

$$\cot 30° = \frac{1}{\tan 30°}$$

$$= \frac{1}{\frac{1}{\sqrt{3}}} \qquad \left(\text{Since } \tan 30° = \frac{1}{\sqrt{3}}\right)$$

$$= \sqrt{3}$$

$$\cos 30° = \frac{\sqrt{3}}{2}$$

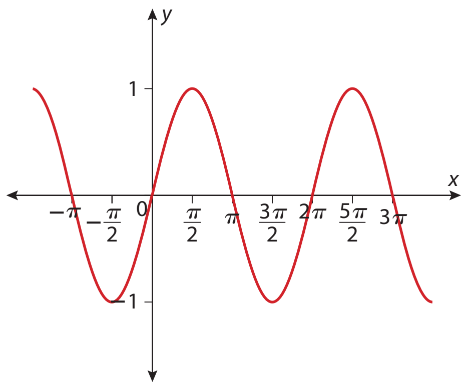
$$\text{Therefore, } \cot 30° + \cos 30° = \sqrt{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{2\sqrt{3} + \sqrt{3}}{2}$$

$$= \frac{3\sqrt{3}}{2}$$

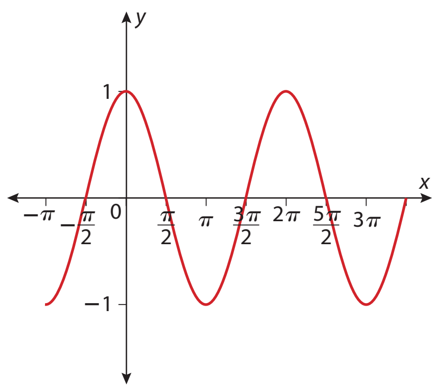
Graph of $y = \sin x$

$\sin x$ is periodic with period 2π radians. The maximum value of $\sin x$ is 1, and the minimum value is -1. The graph is symmetric with respect to the origin (0, 0).



Graph of $y = \cos x$

$\cos x$ is periodic with period 2π radians. The maximum value of $\cos x$ is 1, and the minimum value is -1. The graph is symmetric with respect to the y-axis.



Graph of $y = \tan x$

$\tan x$ is periodic with period π radians. The range of $\tan x$ is all real numbers. The graph is symmetric about the origin (0, 0). $\tan x$ has asymptotes at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}\ldots$

Graph of cosec x

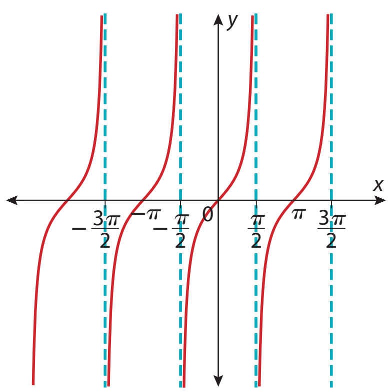
Since $\operatorname{cosec} x = \frac{1}{\sin x}$ we can look at the properties of $\operatorname{cosec} x$ using the graph of $\sin x$.

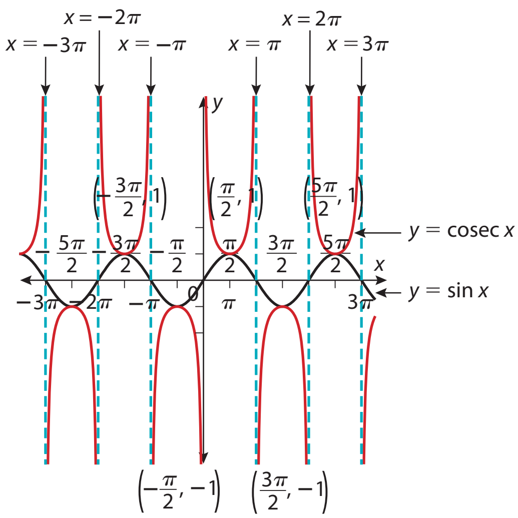
(i) When $\sin x$ is positive, $\operatorname{cosec} x$ is also positive and when $\sin x$ is negative, $\operatorname{cosec} x$ is also negative.

(ii) Since $\operatorname{cosec} x = \frac{1}{\sin x}$, when $\sin x$ is maximum, $\operatorname{cosec} x$ is minimum and when $\sin x$ is minimum $\operatorname{cosec} x$ is maximum.

(iii) As $x \to 0$, $\sin x \to 0$ and $\operatorname{cosec} x \to \infty$.

As $x \to \pi$, $\sin x \to 0$ and $\operatorname{cosec} x \to \infty$.





The graph of cosec x will repeat itself every 2π radians along the x-axis.

Graph of sec x

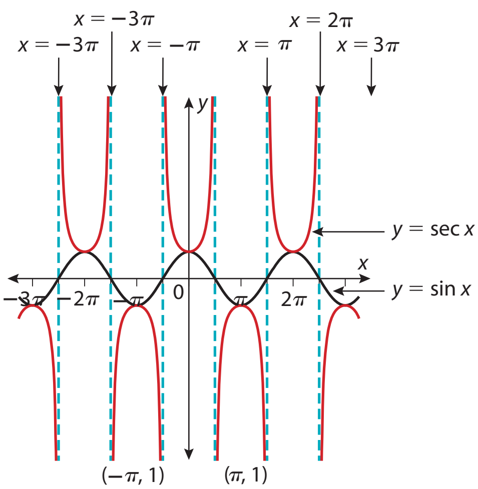
Using the graph of cos x we can derive the graph of sec x since $\sec x = \frac{1}{\cos x}$.

(i) Whenever cos x is positive, sec x is positive and whenever cos x is negative, cosec x is negative.

(ii) When cos x is maximum, sec x is minimum and when cos x is minimum, sec x is maximum.

(iii) As $x \to \frac{\pi}{2}$, $\cos x \to 0$ and $\text{cosec}\, x \to \infty$.

As $x \to \frac{3\pi}{2}$, $\cos x \to 0$ and $\text{cosec}\, x \to \infty$.



The graph repeats itself every 2π radians along the x-axis.

Graph of cot x

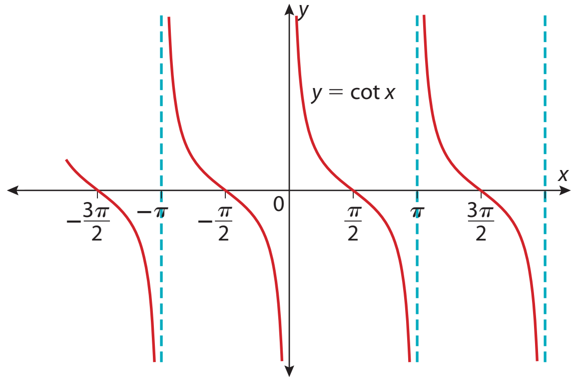
Since $\cot x = \frac{1}{\tan x}$ we can derive the graph of cot x using the properties of tan x.

(i) When tan x is positive, cot x is positive and when tan x is negative, cot x is negative.

(ii) As $x \to 0$, $\tan x \to 0$ and $\frac{1}{\tan x} \to \infty$.

As $x \to \pi$, $\tan x \to 0$ and $\frac{1}{\tan x} \to \infty$.

As $x \to \frac{\pi}{2}$, $\tan x \to \infty$ and $\frac{1}{\tan x} \to \infty$.



DEFINITION

A function $f(x)$ is said to be periodic with period k if and only if $f(x) = f(x + k)$. The period k is the x-distance between any point and the next point at which the same pattern of y-values repeats itself.

Properties and graphs of trigonometric functions

The graph of $y = \sin x$ is periodic and it is this periodicity that makes the trigonometric functions important in the study of electric currents, sound waves, fluid motion, vibration of a spring and so on. The graph of $\sin x$ reaches up to 1 and down to -1 and we say that the amplitude of $y = \sin x$ is 1. The amplitude of the curve represents the maximum y-value of the curve.

The graphs of functions of the form $y = a \sin(bx)$ or $y = a \cos(bx)$ are called sinusoidal graphs.

EXAMPLE 62 Show that $f(x) = \sin x$ is periodic with period 2π.

SOLUTION

Since $f(x) = \sin x$:

$f(x + 2\pi) = \sin(x + 2\pi)$

Recall that $\sin(x + 2\pi) = \sin x$.

Therefore, $f(x + 2\pi) = \sin(x + 2\pi) = \sin x$.

Hence, $f(x + 2\pi) = f(x)$.

By definition, $f(x)$ is periodic with period 2π.

Periodic functions are functions that repeat a particular pattern or cycle. The sine function and cosine function are two trigonometric functions which can be used to model the repetitive behaviour of tidal waves and blood pressure.

Graph of $y = a\sin x$ and $y = a\cos x$

The amplitude of $y = a\sin x$ or $y = a\cos x$ is $|a|$ and the period of these functions is 2π.

EXAMPLE 63 Plot the graph of $y = 2\sin x$.

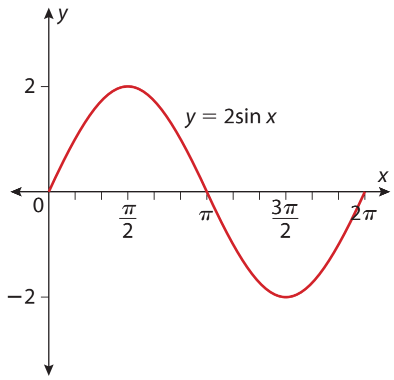
SOLUTION

Draw up a table of values.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	0	1	1.73	2	1.73	1	0	-1	-1.73	-2	-1.73	-1	0

Note

$y = 2 \sin x$ is also a stretch of $y = \sin x$ along the y-axis by scale factor 2.



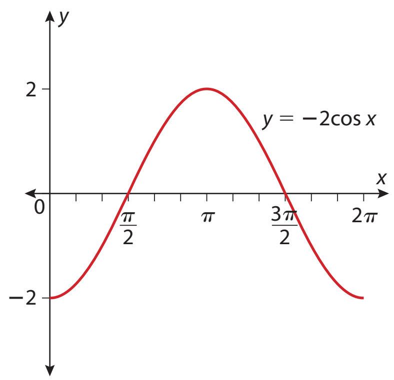
The amplitude of the function is 2 and the graph of $y = 2 \sin x$ has the same shape as $y = \sin x$ except the maximum and minimum of $y = 2 \sin x$ is twice that of $y = \sin x$. The period of the function is 2π.

EXAMPLE 64 Plot the graph of $y = -2 \cos x$. State the period and amplitude of the function.

SOLUTION Draw up a table of values.

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y	−2	−1.73	−1	0	1	1.73	2	1.73	1	0	−1	−1.73	−2

The period of the function is 2π and the amplitude is $|-2| = 2$.



Note

$y = -2 \cos x$ is also a stretch of $y = \cos x$ along the y-axis by scale factor 2, followed by a reflection in the x-axis.

Try these 14.4 Find the period and amplitude of the following functions and plot the graph of each for the interval $0 \le x \le 2\pi$.

(a) $y = 4 \sin x$

(b) $y = 3 \cos x$

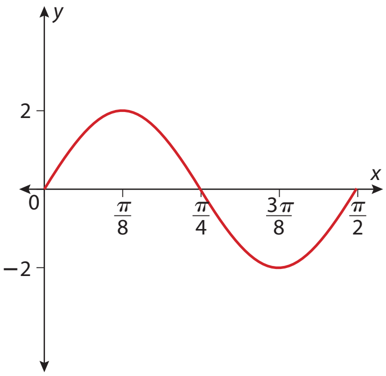
Graph of $y = a \sin bx$ and $y = a \cos bx$

The graphs of $y = a \sin bx$ and $y = a \cos bx$ both have amplitude $|a|$ and period $\frac{2\pi}{b}$. We can sketch graphs of this form using the amplitude and period of the functions.

EXAMPLE 65 Sketch $y = 2 \sin 4x$ for $0 \leq x \leq 2\pi$.

SOLUTION

The amplitude of the function is 2 and the period is $\frac{2\pi}{4} = \frac{\pi}{2}$. The graph repeats itself every $\frac{\pi}{2}$ units along the x-axis.



This graph will lie between 2 and -2 on the y-axis and one cycle will lie between 0 and $\frac{\pi}{2}$.

The interval $\left[0, \frac{\pi}{2}\right]$ can be divided into four subintervals each of length $\frac{\frac{\pi}{2}}{4} = \frac{\pi}{8}$ to obtain the following x- and y-values.

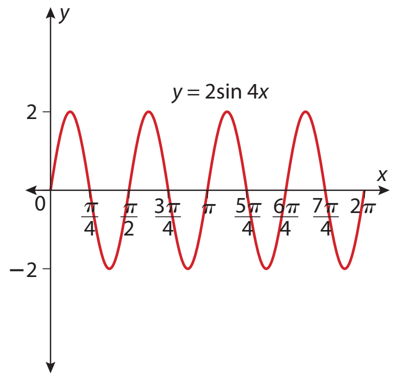
x:	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$
y:	0	2	0	-2	0

We can draw the graph for one cycle and then repeat this cycle.

The diagram shows one cycle of $y = 2 \sin 4x$.

Note

$y = 2 \sin 4x$ is a stretch of $y = \sin x$ along the x-axis by scale factor $\frac{1}{4}$ followed by a stretch along the y-axis by scale factor 2. We get four cycles of $\sin x$ within the interval $0 \leq x \leq 2\pi$.



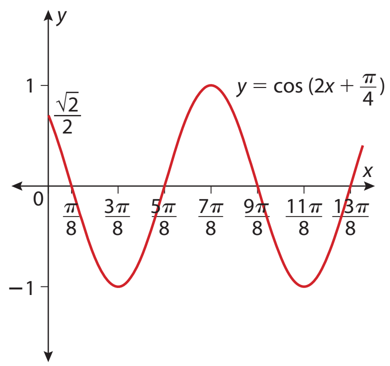
Graph of $y = a \sin(bx + c)$ and $y = a \cos(bx + c)$

In the function $y = a\sin(bx + c)$ and $y = a\cos(bx + c)$ the value of c is called the phase angle. The quantity $-\frac{c}{b}$ is called the displacement (or phase shift). Recall, the curve shifts to the left if $-\frac{c}{b} < 0$ and to the right if $-\frac{c}{b} > 0$.

EXAMPLE 66 Find the amplitude, period and displacement of $y = \cos\left(2x + \frac{\pi}{4}\right)$. Hence, sketch the graph of $y = \cos\left(2x + \frac{\pi}{4}\right)$ for $0 \le x \le 2\pi$.

SOLUTION Amplitude $= 1$

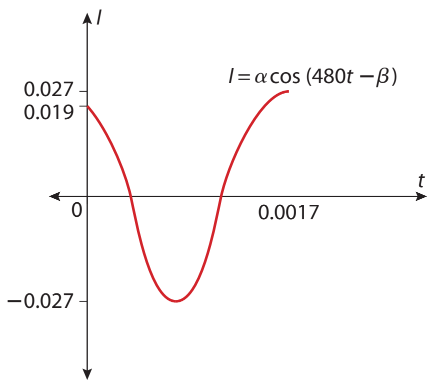
Period $= \frac{2\pi}{2} = \pi$



EXAMPLE 67 Sketch one cycle of the acoustical intensity I of the sound wave for which $I = \alpha \cos(480t - \beta)$ given that t is in seconds, $\alpha = 0.027\,\text{W cm}^{-2}$, $\beta = 0.80$.

SOLUTION Amplitude $= 0.027$

Period $= \frac{2\pi}{480} = \frac{\pi}{240}$



Try these 14.5 Determine the amplitude, period, and displacement for each of the following.

(a) $y = \cos\left(3x - \frac{\pi}{12}\right)$

(b) $y = 40\cos\left(2\pi x - \frac{\pi}{8}\right)$

(c) $y = 2\sin\left(x - \frac{\pi}{3}\right)$

Transformations of trigonometric functions

Like other functions, we can use transformations such as translation, stretch, etc. on the graphs of trigonometric functions. You should be able to identify the amplitude, period and symmetries related to these graphs, and so be able to sketch them.

EXAMPLE 68 Sketch the graph of $y = \sin 2x$ for $0 \le x \le 2\pi$.

SOLUTION Using $f(x) = \sin x$ gives:

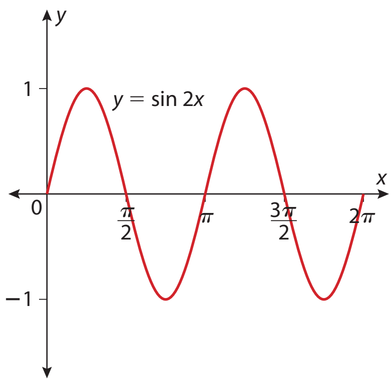
$f(2x) = \sin 2x$ is a stretch along the x-axis by factor $\frac{1}{2}$.

There will be two complete sine curves within the interval 0 to 2π.

Amplitude = 1

Period = π

x-values obtained by intervals of $\frac{\pi}{4}$ (period divided by 4). One cycle can be drawn by looking at the values of x and y:



x:	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y:	0	1	0	−1	0

Amplitude = 1

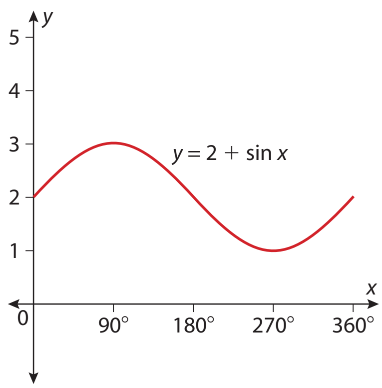
Period = π

EXAMPLE 69 Sketch the graph of $f(x) = 2 + \sin x$ for $0° \le x \le 360°$. Identify the periodicity and the amplitude of the function.

SOLUTION Using $f(x) = \sin x$, $f(x) = 2 + \sin x$ is a shift upwards of $f(x)$ by 2 units.

Amplitude = 1

Period = 360°



EXAMPLE 70 Sketch the graph of $f(x) = 3 + 2\sin x$ for $0° \leq x \leq 360°$.

SOLUTION The graph of $f(x)$ can be sketched using the graph of $y = \sin x$.

The graph of $\sin x$ is stretched along the y-axis by factor 2 and shifted upwards by 3 units.

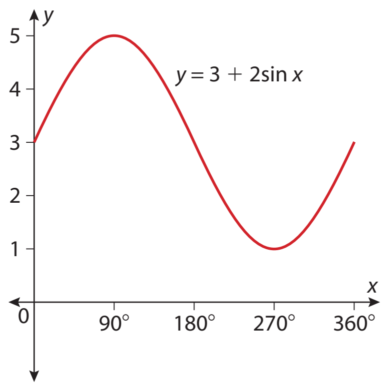
When $\sin x = 1$, $f(x) = 3 + 2 = 5$

When $\sin x = -1$, $f(x) = 3 - 2 = 1$

$\therefore$ maximum of $f(x)$ is 5, minimum is 1.

Amplitude $= 2$

Period $= 360°$

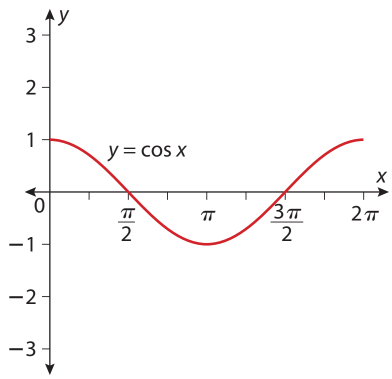


EXAMPLE 71 Sketch the graph of $f(x) = 2 - \cos 2x$ for $0 \leq x \leq 2\pi$.

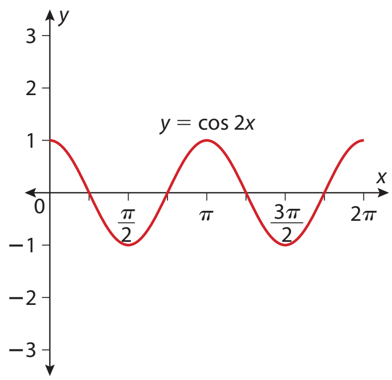
SOLUTION We can go through a sequence of transformations to sketch the graph, starting with $y = \cos x$ for $0 \leq x \leq 2\pi$.

Amplitude $= 1$

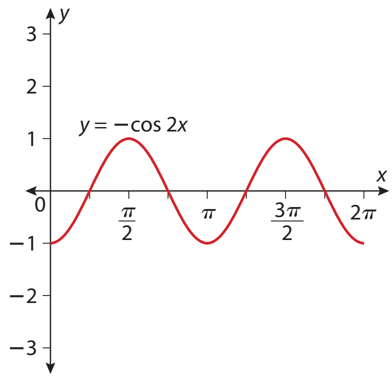
Period $= \frac{2\pi}{2} = \pi$



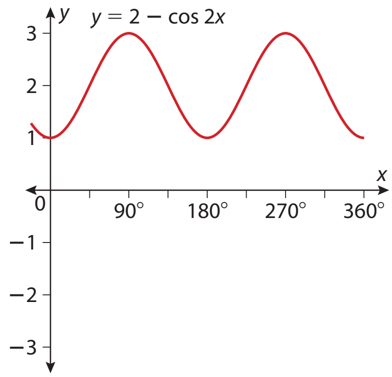
$y = \cos 2x$ is a stretch along the x-axis by factor $\frac{1}{2}$. We get two complete cycles within the same interval.



For $y = -\cos 2x$ we reflect $y = \cos 2x$ along the x-axis.



Shift $y = -\cos 2x$ up the y-axis by 2 units and we get $y = 2 - \cos 2x$.



EXAMPLE 72 Describe the sequence of transformations which maps the graph of $f(x) = \sin x$ onto the graph of $g(x) = 3 + 2\sin\frac{1}{2}x$.

SOLUTION Let $f(x) = \sin x$.

$\Rightarrow f\left(\frac{1}{2}x\right) = \sin\left(\frac{1}{2}x\right)$

$f\left(\frac{1}{2}x\right)$ is a stretch of $f(x)$ along the x-axis by scale factor 2.

$2f\left(\frac{1}{2}x\right) = 2\sin\left(\frac{1}{2}x\right)$

$2f\left(\frac{1}{2}x\right)$ is a stretch of $f\left(\frac{1}{2}x\right)$ along the y-axis by scale factor 2.

$3 + 2f\left(\frac{1}{2}x\right) = 3 + 2\sin\left(\frac{1}{2}x\right)$

$3 + 2f\left(\frac{1}{2}x\right)$ moves the graph of $2f\left(\frac{1}{2}x\right)$ up the y-axis by 3 units.

Try these 14.6

(a) Sketch one cycle of $y = \sin 3x$.

(b) Describe the sequence of transformations which maps the graph of $y = \sin x$ onto the graph of $y = 2 - \sin x$. Hence, sketch $y = 2 - \sin x$ for $0 \le x \le 2\pi$.

(c) Sketch the graph of $y = |\cos 2x|$ for $0 \le x \le 2\pi$.

$y = a\sin(bx) + c$ and $y = a\cos(bx) + c$

We can sketch these graphs by using two methods:

(i) Using transformations of graphs

(ii) Using key points on the graph

Using transformations

We start with the graph of $y = \sin x$. We stretch it along the x-axis by factor $\frac{1}{b}$, and then stretch it along the y-axis by factor a. Finally, we move this graph up the y-axis by c units if c is positive or down the y-axis by c units if c is negative. The resulting graph is $y = a\sin(bx) + c$. We use a similar procedure for sketching $y = a\cos(bx) + c$, but starting with the graph of $y = \cos x$.

Using key points

We start first sketch the graph $y = a\sin bx$ or $y = a\cos bx$, and then shift it along the y-axis by c units. We use the amplitude of the function to identify the maximum and minimum y-values. Use the period $\frac{2\pi}{b}$ and divide the interval $\left[0, \frac{2\pi}{b}\right]$ into four equal subintervals. Find the end points of the cycle (or the curve) and then draw the sinusoidal graph by connecting the points.

EXAMPLE 73 The current i, in amperes, flowing through an alternating circuit at time t seconds is:

$i = 120\sin(30\pi t)$ for $t \geq 0$.

(a) Identify the following.

(i) The period of the function

(ii) The amplitude of the function

(b) Draw the graph of this function for one period.

SOLUTION

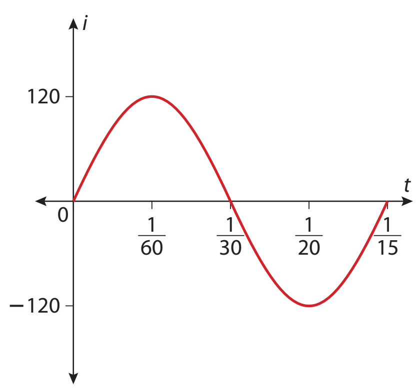
(a) (i) The period $= \frac{2\pi}{30\pi} = \frac{1}{15}$

(ii) The amplitude $= 120$

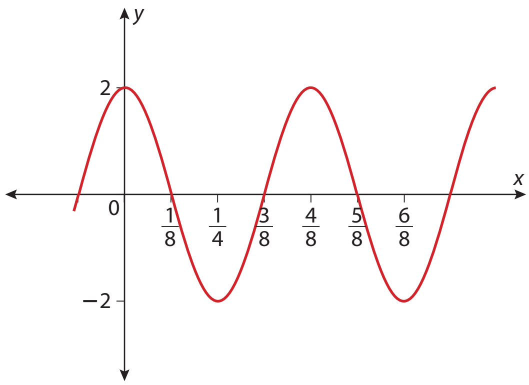
(b) The graph will lie between -120 and 120 along the y-axis. One cycle begins at $t = 0$ and ends at $t = \frac{1}{15}$.

We divide the interval $\left[0, \frac{1}{15}\right]$ into four subintervals of length $\frac{\frac{1}{15}}{4} = \frac{1}{60}$.

Therefore, t:	0	$\frac{1}{60}$	$\frac{1}{30}$	$\frac{1}{20}$	$\frac{1}{15}$
i:	0	120	0	-120	0



EXAMPLE 74 Find an equation for the curve below.



SOLUTION The graph is similar to a cosine function. The amplitude of the function is 2 and the period is $\frac{1}{2}$.

Since period $= \frac{2\pi}{b}$

$\Rightarrow \frac{1}{2} = \frac{2\pi}{b}$

$\Rightarrow b = 4\pi$

The function is of the form $y = a\cos(bx)$, where $a = 2$, $b = 4\pi$.

Hence, the graph represents the function $y = 2\cos(4\pi x)$.

EXAMPLE 75 Write the equation of a function with amplitude 2 and period 3.

SOLUTION Since period $= \frac{2\pi}{b}$

$\Rightarrow 3 = \frac{2\pi}{b}$

$\Rightarrow b = \frac{2\pi}{3}$

$\Rightarrow a = 2$

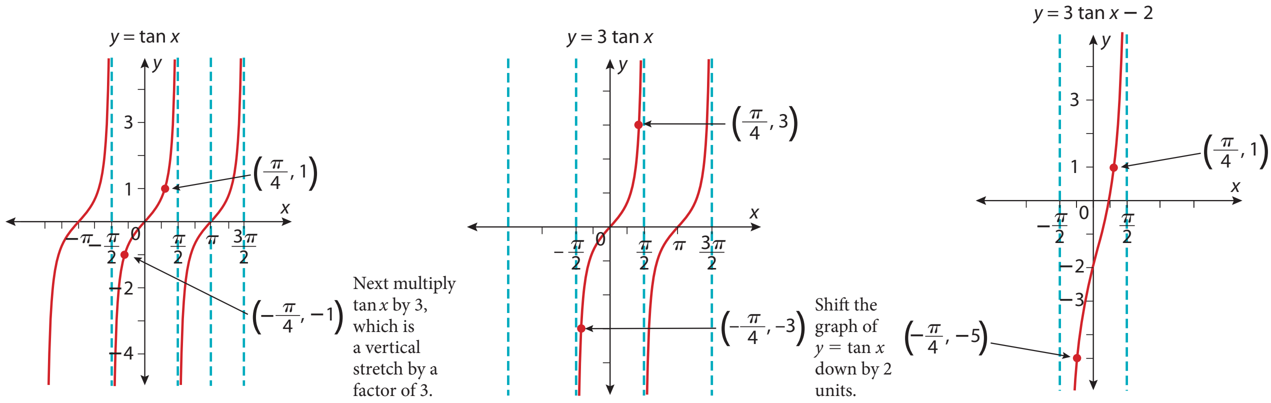
Therefore, $y = 2\cos\left(\frac{2\pi}{3}x\right)$ or $y = 2\sin\left(\frac{2\pi}{3}x\right)$.

$y = a\tan(bx) + c$

Since the range of $\tan x$ is $[-\infty, \infty]$, there is no concept of amplitude. To sketch $y = a\tan(bx) + c$, we can use transformations of graphs. The value of a will identify the vertical stretch, b identifies the horizontal compression $\left(\text{factor } \frac{1}{b}\right)$ and the period is $\frac{\pi}{b}$. The value of c indicates the vertical shift.

EXAMPLE 76 Sketch the graph of $f(x) = 3\tan x - 2$ for $0 \leq x \leq \frac{\pi}{2}$.

SOLUTION Let us look at the transformations.



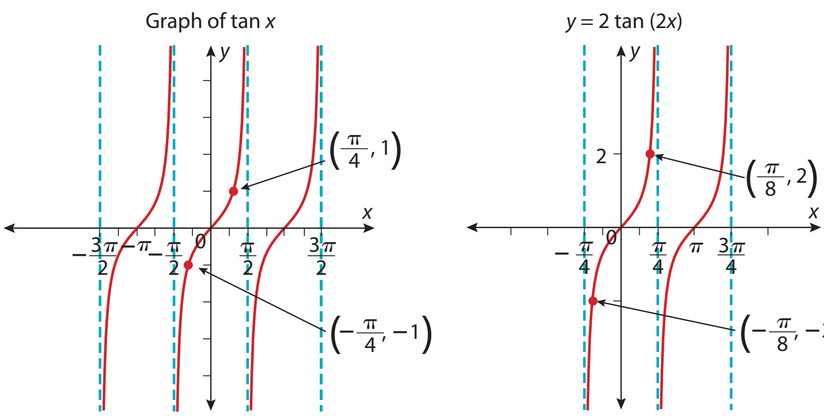
Note the movement of the points $\left(-\frac{\pi}{4}, -1\right)$ and $\left(\frac{\pi}{4}, 1\right)$ throughout.

EXAMPLE 77 Sketch the graph of $y = 2\tan(2x)$.

SOLUTION We start with the graph of $y = \tan x$, indicating $\left(\frac{\pi}{4}, 1\right)$ and $\left(-\frac{\pi}{4}, -1\right)$ on the curve. We can use these points as a guide for each transformation. For $y = \tan(2x)$, we have a horizontal stretch by factor $\frac{1}{2}$. Therefore, $\left(\frac{\pi}{4}, 1\right)$ maps onto $\left(\frac{\pi}{8}, 1\right)$ and $\left(-\frac{\pi}{4}, -1\right)$ onto $\left(-\frac{\pi}{8}, -1\right)$.

When $\tan(2x)$ is multiplied by 2, we have a vertical stretch by factor 2 units.

Therefore, $\left(\frac{\pi}{8}, 1\right)$ maps onto $\left(\frac{\pi}{8}, 2\right)$ and $\left(-\frac{\pi}{8}, -1\right)$ onto $\left(-\frac{\pi}{8}, -2\right)$.



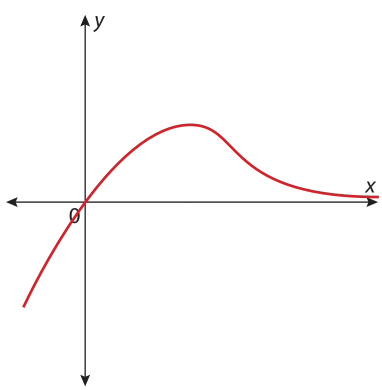
Graphs of rational functions

When sketching graphs of rational functions we first identify the following as far as possible.

(i) Intercepts on the x-axis and y-axis

(ii) Points on the graph classifying as maximum and minimum

(iii) Asymptotes (horizontal or vertical asymptotes)



DEFINITION

An asymptote is a line which approaches a curve, becomes a tangent to the line as x or y tends to infinity but does not touch the curve as x or y approaches infinity.

In the graph, the x-axis is an asymptote to the curve. The graph cuts the x-axis at (0, 0) but as x approaches infinity the curve moves along the x-axis.

Vertical asymptotes

For a rational function $f(x) = \frac{P(x)}{Q(x)}$, the vertical asymptotes of the function occur when the denominator is zero. Set $Q(x) = 0$.

EXAMPLE 78 Identify the vertical asymptote of $y = \frac{x+1}{x+2}$.

SOLUTION Vertical asymptotes occur when the denominator is zero.

$\Rightarrow x + 2 = 0$

$\Rightarrow x = -2$

Therefore, $x = -2$ is a vertical asymptote.

EXAMPLE 79 Find the vertical asymptote of $y = \frac{x}{x^2 - 3x + 2}$.

SOLUTION The denominator becomes zero at $x^2 - 3x + 2 = 0$.

$(x - 1)(x - 2) = 0$

$\Rightarrow x = 1, 2$

Therefore, $x = 1$ and $x = 2$ are vertical asymptotes.

Horizontal asymptotes

Let $y = f(x)$. If $\lim_{x \to \infty} y = a$, then $y = a$ is a horizontal asymptote to the curve $y = f(x)$.

EXAMPLE 80 Find the horizontal asymptote of $y = \frac{2x+1}{x-2}$.

SOLUTION We need to find $\lim_{x \to \infty} y$.

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \left(\frac{2x+1}{x-2}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{2}{x}}$$ (Dividing the numerator and denominator by the highest term in x.)

$$= \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{1 - \frac{2}{x}}$$

$$= \frac{2+0}{1-0}$$

$$= 2$$

Therefore, $y = 2$ is a horizontal asymptote.

EXAMPLE 81 Find the horizontal asymptote of $y = \dfrac{x^2 - 3x - 4}{x^2 - 7x + 10}$.

SOLUTION We need to find $\lim\limits_{x \to \infty} y$.

$$\begin{aligned}
\lim_{x \to \infty} y &= \lim_{x \to \infty} \frac{x^2 - 3x - 4}{x^2 - 7x + 10} \\
&= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{3x}{x^2} - \frac{4}{x^2}}{\frac{x^2}{x^2} - \frac{7x}{x^2} + \frac{10}{x^2}} \qquad \text{(Dividing the numerator and denominator by } x^2\text{)} \\
&= \lim_{x \to \infty} \frac{1 - \frac{3}{x} - \frac{4}{x^2}}{1 - \frac{7}{x} + \frac{10}{x^2}} \\
&= \frac{1 - 0 - 0}{1 - 0 + 0} \\
&= 1
\end{aligned}$$

Therefore, $y = 1$ is the horizontal asymptote.

Try these 14.7 Find the horizontal and vertical asymptotes of

(a) $y = \dfrac{2x + 3}{4x - 1}$ (b) $y = \dfrac{x + 1}{x - 3}$ (c) $y = \dfrac{x^2 + 2x}{x^2 - 7x + 12}$

Sketching graphs of rational functions

EXAMPLE 82 Sketch the graph of $y = \dfrac{x + 1}{x + 2}$.

SOLUTION We find the intercepts first.

When $x = 0$, $y = \dfrac{0 + 1}{0 + 2} = \dfrac{1}{2}$

When $y = 0$, $0 = \dfrac{x + 1}{x + 2} \Rightarrow x + 1 = 0 \Rightarrow x = -1$

Therefore, $\left(0, \frac{1}{2}\right)$ and $(-1, 0)$ are on the curve.

Now we find the asymptotes.

Vertical asymptote:

$x + 2 = 0$

$x = -2$

This would make the denominator zero.

Horizontal asymptote:

$$\begin{aligned}
\lim_{x \to \infty} y &= \lim_{x \to \infty} \frac{x + 1}{x + 2} \\
&= \lim_{x \to \infty} \frac{\frac{x}{x} + \frac{1}{x}}{\frac{x}{x} + \frac{2}{x}} \\
&= \lim_{x \to \infty} \frac{1 + \frac{1}{x}}{1 + \frac{2}{x}} \\
&= \frac{1 + 0}{1 + 0} = 1
\end{aligned}$$

Therefore, $y = 1$ is the horizontal asymptote.

Now we find the stationary points.

$$\frac{dy}{dx} = \frac{(x+2)(1) - (x+1)(1)}{(x+2)^2}$$

$$= \frac{x+2-x-1}{(x+2)^2} = \frac{1}{(x+2)^2}$$

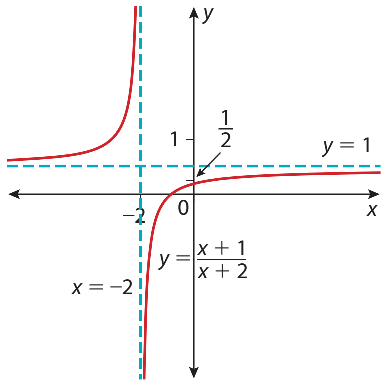
At stationary points $\frac{dy}{dx} = 0$.

$$\Rightarrow \frac{1}{(x+2)^2} = 0$$

$$\Rightarrow 1 = 0$$

This is inconsistent. Therefore, there are no stationary points.

This is the information we gathered and will use to sketch the graph:



$\left(0, \frac{1}{2}\right)$, $(-1, 0)$ are on the curve.

No stationary points

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = -2$

$$y = \frac{x+1}{x+2}$$

When $x = -3$, $y = \frac{-3+1}{-3+2} = \frac{-2}{-1} = 2$.

The curve must pass through $(-3, 2)$ and move along the two asymptotes. We have spanned the whole x-axis when drawing the graph, since the domain of this function is $x \in \mathbb{R}, x \neq -2$.

EXAMPLE 83 Given that $y = \frac{x^2 - 4x + 3}{x^2 - 4}$.

(a) Find the horizontal asymptote and vertical asymptote of y.

(b) Find the stationary points of y.

(c) Sketch the graph of y.

SOLUTION

(a) For horizontal asymptote:

$$\lim_{x \to \infty} y = \lim_{x \to \infty} \frac{x^2 - 4x + 3}{x^2 - 4}$$

$$= \lim_{x \to \infty} \frac{\frac{x^2}{x^2} - \frac{4x}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} - \frac{4}{x^2}}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{4}{x} + \frac{3}{x^2}}{1 - \frac{4}{x^2}}$$

$$= \frac{1 - 0 + 0}{1 - 0}$$

$$= 1$$

Therefore, $y = 1$ is a horizontal asymptote.

For vertical asymptotes:

$$x^2 - 4 = 0$$

$$\Rightarrow (x-2)(x+2) = 0$$

$$\Rightarrow x = 2, -2$$

Therefore, the vertical asymptotes are $x = 2$ and $x = -2$.

(b) $y = \dfrac{x^2 - 4x + 3}{x^2 - 4}$

$$\frac{dy}{dx} = \frac{(x^2 - 4)(2x - 4) - (x^2 - 4x + 3)(2x)}{(x^2 - 4)^2}$$

$$= \frac{2x^3 - 4x^2 - 8x + 16 - 2x^3 + 8x^2 - 6x}{(x^2 - 4)^2}$$

$$= \frac{4x^2 - 14x + 16}{(x^2 - 4)^2}$$

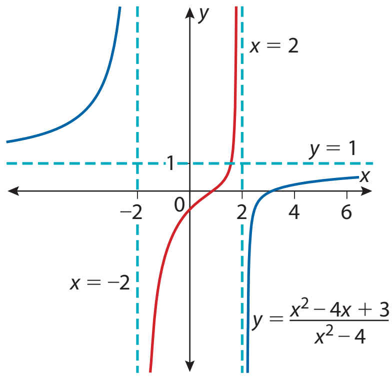
At stationary points $\dfrac{dy}{dx} = 0$.

$\Rightarrow \dfrac{4x^2 - 14x + 16}{(x^2 - 4)^2} = 0$

$\Rightarrow 4x^2 - 14x + 16 = 0$

Since $b^2 - 4ac = (-14)^2 - (4)4(16) = -60$, there are no real roots. Hence, there are no stationary points.

(c)



EXAMPLE 84 The equation of a curve C is given by $y = \dfrac{x}{x^2 - 1}$.

(a) Find the equations of the asymptotes of C.

(b) Show that C has no stationary points.

(c) Sketch C.

SOLUTION

(a) Vertical asymptotes exist when $x^2 - 1 = 0$.

$\Rightarrow x^2 = 1$

$\Rightarrow x = 1, -1$

There are vertical asymptotes at $x = 1$ and $x = -1$.

Horizontal asymptotes exist at $\lim\limits_{x \to \infty} y = \lim\limits_{x \to \infty} \dfrac{x}{x^2 - 1}$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1 - \frac{1}{x^2}}$$

$$= \frac{0}{1}$$

$$= 0$$

Therefore, $y = 0$ is a horizontal asymptote.

(b) $y = \dfrac{x}{x^2 - 1}$

$$\frac{dy}{dx} = \frac{(x^2-1)(1) - x(2x)}{(x^2-1)^2}$$

$$= \frac{x^2 - 1 - 2x^2}{(x^2-1)^2}$$

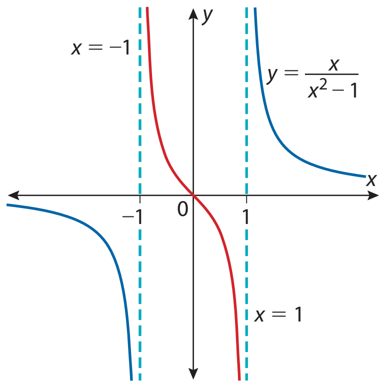
$$= \frac{-x^2 - 1}{(x^2-1)^2}$$

For stationary points $\dfrac{dy}{dx} = 0$.

$\Rightarrow -x^2 - 1 = 0$

$\Rightarrow x^2 = -1$

Therefore, there are no stationary points.



(c) We now find the intercepts.

When $x = 0$, $y = 0$ $(0, 0)$

We find other points.

When $x = \frac{1}{2}$, $y = \dfrac{\frac{1}{2}}{\frac{1}{4} - 1} = -\frac{2}{3}$.

When $x = -\frac{1}{2}$, $y = \dfrac{-\frac{1}{2}}{\frac{1}{4} - 1} = \frac{2}{3}$.

When $x = 2$, $y = \dfrac{2}{4-1} = \frac{2}{3}$.

When $x = -2$, $y = \dfrac{-2}{4-1} = -\frac{2}{3}$.

These values give an indication of where the graph lies.

Shape of a curve for large values of the independent variable

In Module 1 we looked at the shape of polynomials for large values of x and we can summarise the results like this.

(i) For a polynomial with a degree that is odd, the behavior of the end points of the polynomial can be deduced as follows.

- With a positive leading coefficient the graph falls to the left and rises to the right.
- With a negative leading coefficient the graph rises to the right and falls to the left.

(ii) For a polynomial with an even degree.

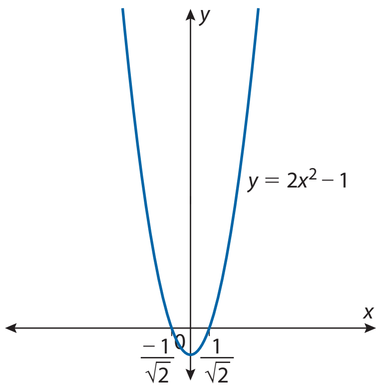
- With a positive leading coefficient the graphs will rise on both ends.
- With a negative leading coefficient the graph falls on both ends.

For a rational function we can see that if $y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$

then $a_n x^n \left(1 + a_{n-1}x^{n-1} + \cdots + \frac{a_0}{a_n}\frac{1}{x_n}\right)$ so that as $|x| \to \infty$, $\frac{y}{a_n x_n} \to 1$.

$\therefore\ y \approx a_n x_n$ and we say that y behaves like $a_n x_n$ as $|x| \to \infty$.

EXAMPLE 85 Discuss the shape at infinity of the curve $y = \frac{4x^4}{4x^2 + 1}$.



SOLUTION

$$y = \frac{4x^4}{2x^2 + 1}$$

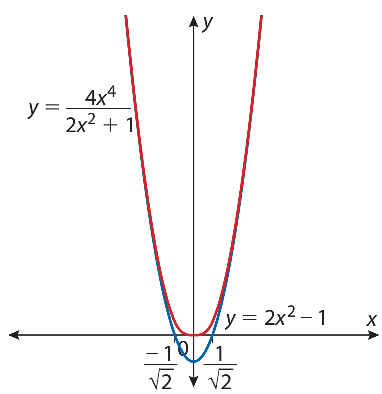
When $x = 0$, $y = 0$.

$\therefore$ (0, 0) is on the graph.

By long division, $y = 2x^2 - 1 + \frac{1}{2x^2 + 1}$.

We have $y = 2x^2 - 1$ as $|x| \to \infty$.

The curve is therefore asymptotic to $y = 2x^2 - 1$ and can be drawn as shown.



EXERCISE 14D

1 Given that $y = x^2 + 2x + 1$. Find and classify the points of y. Sketch the graph of the function.

2 Find the maximum and minimum points on the curve $y = 12x - x^3$. Hence, sketch the graph of $y = 12x - x^3$.

3 Find and classify the turning points of $f(x) = x^4 - 6x^2$.

Sketch the graph of $y = f(x)$.

4 A cricketer hits a ball that follows the path given by $y = -\frac{1}{40}x^2 + x$, where the distances are measured in metres. Sketch the graph of the path of the ball.

5 The angle θ that a robot arm makes with the horizontal as a function of time is given by $\theta = -2t^3 + 12t^2 + 10$. θ is measured in radians and t in seconds. Sketch the graph of θ against t for $0 \leq t \leq 6$.

6 A curve C has equation $y = \dfrac{2x + 1}{x - 3}$.

(a) Show C has no points.

(b) Find the equation of the asymptotes of C.

(c) Sketch C.

7 Sketch the curve of $y = \dfrac{2}{x^2 + 4}$.

8 The equation of a curve C is given by $y = x + \frac{4}{x}$.

(a) Show that C has two stationary points.

(b) Find the equation of the asymptotes of C.

(c) Sketch C.

9 Given that $y = \dfrac{2x + 1}{x - 2}$, sketch the graph of y.

10 The equation of a curve c is given by $y = \dfrac{x^2 + 2x + 5}{x + 2}$, sketch the graph of C.

SUMMARY

Applications of differentiation

Tangents and normals

a Gradient of a tangent $= \dfrac{dy}{dx}$

b Gradient of a normal $= \dfrac{-1}{\frac{dy}{dx}}$

Rate of change

$\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$

Curve sketching

Look for:
- intercepts
- turning points
- asymptotes

Asymptotes occur on rational functions $f(x) = \dfrac{P(x)}{Q(x)}$

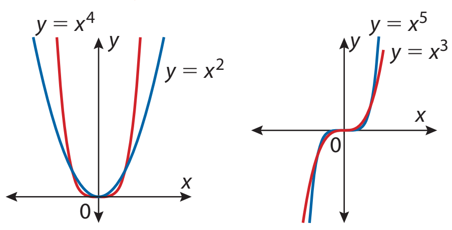
Vertical asymptotes occur at $Q(x) = 0$

Horizontal asymptotes occur at $\lim_{x \to \infty} f(x) = l$

$\Rightarrow y = l$ is a horizontal asymptote.

Graph of a polynomial

(i) Find where the curve cuts the x- and y-axis if possible.
(ii) Find values of y for values of x between the points where the graph cut the x-axis.
(iii) Draw a smooth, continuous curve.



Increasing and decreasing functions

A function is increasing over an interval if $\dfrac{dy}{dx} > 0$ over the interval

A function is decreasing over an interval if $\dfrac{dy}{dx} < 0$ over the interval

Stationary points

Stationary points exist at $\dfrac{dy}{dx} = 0$.

First derivative test for classification of $x = a$

If $\dfrac{dy}{dx} < 0$ for $x < a$ or $\dfrac{dy}{dx} > 0$ for $x > a$, then there is a minimum point when $x = a$.

If $\dfrac{dy}{dx} > 0$ for $x < a$ or $\dfrac{dy}{dx} < 0$ for $x > a$, then there is a maximum point when $x = a$.

If $\dfrac{dy}{dx}$ does not change sign as it passes through a, then there is a point of inflexion when $x = a$.

Second derivative test for classification of $x = a$

If $\dfrac{d^2y}{dx^2} > 0$ when $x = a$, then there is a minimum point at $x = a$.

If $\dfrac{d^2y}{dx^2} < 0$ when $x = a$, then there is a maximum point at $x = a$.

If $\dfrac{d^2y}{dx^2} = 0$, you must test further (in this case you can go back to the first derivative test).

Points of inflexion occur at $\dfrac{d^2y}{dx^2} = 0$. (You must check concavity since this is not a necessary condition.)

Checklist

Can you do these?

- Find the gradient of the tangent to a curve.
- Find the gradient of the normal to a curve.
- Find the equation of a tangent.
- Find the equation of a normal.
- Find the region for which a function is increasing or decreasing.
- Identify points.
- Identify maximum points, minimum points, points of inflexion.
- Use the first derivative to classify maximum points, minimum points and points of inflexion.
- Use the second derivative test to identify maximum, minimum points.
- Solve practical problems involving maximum and minimum.
- Sketch the graph of polynomials.
- Solve graphically $f(x) = g(x)$, $f(x) \leq g(x)$, $f(x) \geq g(x)$
- Sketch the graphs of sec x, cosec x, cot x.
- Sketch the graphs of sin kx, cos kx, tan kx.
- Identify the periodicity, symmetry and amplitude of sec x, cosec x, cot x, sin kx, cos kx, tan kx.
- Solve rate of change problems.
- Identify the properties of a curve and sketch the curve.

Review Exercise 14

1 The equation of a curve is $y = \dfrac{x}{1 - x^3}$. Find the equation of the normal to the curve at the point where $x = 2$.

2 Oil is poured into a container at a rate of $40\,\text{cm}^3\,\text{s}^{-1}$. The volume, $V\,\text{cm}^3\,\text{s}^{-1}$, of the oil in the container, when the depth of the oil is h cm, is given by $V = 0.02h^3 + 0.4h^2 + 400h$. Find the following.

(a) The rate of increase in the depth of oil when $h = 10\,\text{cm}$

(b) The depth of oil when the rate of increase in the depth is $0.04\,\text{cm}^3\,\text{s}^{-1}$

3 Find the coordinates of the point at which the tangent to the curve $y = \cos x + \sin x$, where, $0 \leq x \leq \dfrac{\pi}{2}$ is perpendicular to the line $y + x = 3$.

4 A curve has equation $y = \frac{x+1}{2x-7}$. Find the equation of the following.

(a) The tangent to the curve at the point P(3, −4)

(b) The normal to the curve at the point P(3, −4)

The tangent to the curve at the point P meets the x-axis at A. The normal to the curve meets the y-axis at B.

(c) Find the area of triangle APB.

5 A spherical balloon is being inflated and, at the same instant when its radius is 4 m, its surface area is increasing at a rate of $3\,\mathrm{m^2\,s^{-1}}$. Find the rate of increase, at the same instant, of the radius and the volume.

6 A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is h cm and the area of the cross-section is $20h\,\mathrm{cm^2}$. Given that, when $h = 4$ cm, the area of the cross-section is increasing at a rate of $.05\,\mathrm{cm^2\,s^{-1}}$, find the rate of increase of the volume at this instant.

7 (a) Find the coordinates of the stationary points on the curve.

$y = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x + 1.$

(b) Determine the nature of each of these points.

(c) Find the coordinates of the points of inflexion on the curve.

(d) Sketch the curve.

8 (a) A curve has the equation $y = x^3 + 3x^2 - 24x + 10$. Find the x-coordinates of the turning points on the curve and determine the nature of these turning points. Find also the coordinates of the point of inflexion on the curve.

(b) The normal to the curve $y = \frac{\cos 2x}{1 + \sin 2x}$ at the point $x = \frac{\pi}{2}$ meets the x-axis at the point P. Find the exact coordinates of P.

9 An open metal container, with a square base of side x m and a height of y m, is to be made from $96\,\mathrm{m^2}$ of a thin sheet of metal. The volume of the container is $V\,\mathrm{m^3}$.

(a) Show that $V = 24x - \frac{1}{4}x^3$.

(b) Given that x can vary, show that the maximum volume is $64\sqrt{2}\,\mathrm{m^3}$.

10 A curve has the equation $y = 4x^3 - 24x^2 + 36x$.

(a) Find the coordinates of the stationary points on the curve.

(b) Find the coordinates of any points of inflexion on the curve.

(c) Sketch the curve.

11 A circular cylinder, open at one end, has radius r cm and external surface area $243\pi\,\mathrm{cm^2}$.

(a) Show that the volume of the cylinder, $V\,\mathrm{cm^3}$, is given by $V = \frac{\pi}{2}(243r - r^3)$.

(b) Given that r can vary, find the value of V for which $\frac{\mathrm{d}v}{\mathrm{d}r} = 0$ and determine whether this value is a maximum or a minimum.

12 The variables x and y are related by the equation $y = \frac{4}{(5x - 2)^2}$. If x increases at a rate of 0.25 units per second, find the rate of change of y when

(a) $x = 1$

(b) $y = 9$.

13 A curve has the equation $y = x^3 + 3x^2 + 3x + 2$.

(a) Find the coordinates of the turning points on the curve.

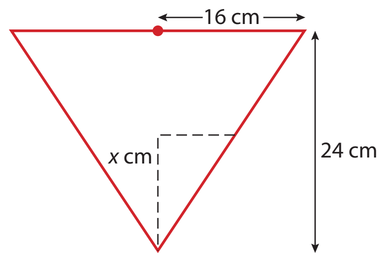
(b) Sketch the curve.

14 The power P produced by a source is given by $P = \frac{36R}{R^2 + 2R + 1}$, where R is the resistance in the circuit. Sketch the graph of P against R.

15 Find the equation of the tangent to the curve $y = \cos^3 2x + \sin^4 2x$ at the point $x = \frac{\pi}{2}$.

16 Sketch the curve $y = \frac{9x}{x^2 + 9}$. Show all turning points and asymptotes.

17 A vessel is in the shape of an inverted cone. The radius of the top is 16 cm and the height is 24 cm. If the height of water in the vessel is x cm, show that $V = \frac{4}{27}\pi x^3$.



Given that water is poured into the container at a rate of 0.05 cm s^{-1}, find the rate at which the volume is increasing when the height is 12 cm.

18 If 2 400 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

19 A cylindrical can is to be made to hold 2 000 cm^2 of water. Find the dimensions that will minimise the cost of the metal to manufacture the can.

CHAPTER 15

Integration

At the end of this chapter you should be able to:

- Define integration as the reverse of differentiation
- Understand and use $\int f(x)\,dx$
- Show that the indefinite integral represents a family of functions
- Know and use the integration theorems:
  - $\int c\,f(x)\,dx = c\int f(x)\,dx$, where c is a constant
  - $\int [f(x) \mp g(x)]\,dx = \int f(x)\,dx \mp \int g(x)\,dx$
- Find integrals using the integration theorems
- Find integrals of polynomials
- Find integrals of trigonometric functions
- Find integrals by substitution
- Find definite integrals
- Know and use $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$
- Know and use $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, $a > 0$

KEYWORDS/TERMS

integration • anti-derivative • indefinite integral • definite integral • polynomials • trigonometric functions • substitution

Anti-derivatives (integrations)

Anti-derivative or integration is the name given to the process of reversing differentiation. Given the derivative of a function we can work backwards to find the function from which it is derived.

In differentiation, if $f(x) = x^3$, then $f'(x) = 3x^2$. It follows that the integral of $3x^2$ is x^3.

The symbol used for integration is an elongated s that is $\int$, which represents the 'integral of'. Together with the integration symbol we use dx to represent that we are integrating with respect to x. Therefore, for the integral of $3x^2$ we write:

$$\int 3x^2\,dx = x^3$$

The constant of integration

Look at these examples of functions and their derivatives.

$y = x^2$

$\Rightarrow \frac{dy}{dx} = 2x$

$y = x^2 + 6$

$\Rightarrow \frac{dy}{dx} = 2x$

$y = x^2 + 200$

$\Rightarrow \frac{dy}{dx} = 2x$

Whenever we add a constant to x^2 and differentiate we get the same result of $2x$. If we differentiate $y = x^2 + c$, where c is any constant, we get $2x$.

Hence, for $y = x^2 + c$

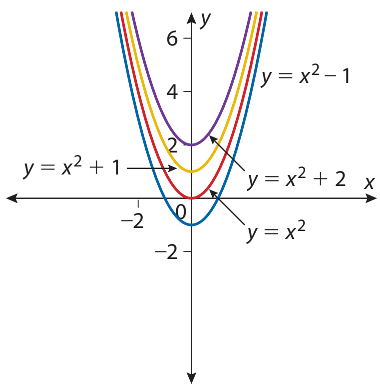
$\frac{dy}{dx} = 2x$

Therefore, $\int 2x\,dx = x^2 + c$, where c is called the constant of integration.

DEFINITION

The indefinite integral of $g(x)$ is defined as $\int g(x)\,dx = G(x) + c$ where $G(x)$ is the anti-derivative of $g(x)$ and c is an arbitrary constant.

If $\frac{d}{dx}F(x) = f(x)$, then $F(x) + c = \int f(x)\,dx$, where c is the constant of integration and $f(x)$ is called the integrand. This integral is called an indefinite integral since c has an indefinite value. The indefinite integral of a function is the family of all antiderivatives of the function. The family of curves representing the antiderivative of $2x$, which is $x^2 + c$, has an infinite number of curves. We can represent the family of curves like this.



Integrals of the form ax^n

Recall $\frac{d}{dx}(x^n) = nx^{n-1}$.

For integration we reverse the process as follows:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, \text{ where } n \neq -1.$$

Note

(i) We increase the power of x by 1. That is, x^n is raised to x^{n+1}.

(ii) Divide the new power of x by $n + 1$. That is, $\frac{x^{n+1}}{n+1}$.

EXAMPLE 1 Find $\int x^4\,dx$.

SOLUTION

$$\int x^4\,dx = \frac{x^{4+1}}{4+1} + c \text{ (raise the power by 1 and divide by the new power)}$$
$$= \frac{x^5}{5} + c$$

EXAMPLE 2 Find $\int x^7\,dx$.

SOLUTION

$$\int x^7\,dx = \frac{x^{7+1}}{7+1} + c \text{ (raise the power by 1 and divide by the new power)}$$
$$= \frac{x^8}{8} + c$$

EXAMPLE 3 Find $\int \sqrt{x}\,dx$.

SOLUTION

We write $\sqrt{x}$ in index form.

Therefore, $\sqrt{x} = x^{\frac{1}{2}}$.

$$\int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$$
$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{2}{3}x^{\frac{3}{2}} + c$$

EXAMPLE 4 Find $\int \frac{1}{x^3}\,dx$.

SOLUTION

We write $\frac{1}{x^3}$ as x^{-3}.

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx$$
$$= \frac{x^{-3}+1}{-3+1} + c$$
$$= \frac{x^{-2}}{-2} + c$$
$$= -\frac{1}{2x^2} + c$$

Integration theorems

Theorem 1

Since $\frac{d}{dx}c = 0$, where c is a constant

$\Rightarrow \int 0\,dx = c$

Theorem 2

Since $\frac{d}{dx}(x + c) = 1$

$\Rightarrow \int 1\,dx = x + c$

Theorem 3

Since $\frac{\mathrm{d}}{\mathrm{d}x}(ax + c) = a$

$\Rightarrow \int a\,\mathrm{d}x = ax + c$, where a is a constant and c is the constant of integration.

EXAMPLE 5 Find $\int 4\,\mathrm{d}x$.

SOLUTION $\int 4\,\mathrm{d}x = 4x + c$

Theorem 4

$\int a\,f(x)\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x$, where a is a constant.

EXAMPLE 6 Find $\int 7x^2\,\mathrm{d}x$.

SOLUTION We can take the 7 out of the integral and then integrate each term.

$$\begin{aligned}\int 7x^2\,\mathrm{d}x &= 7\int x^2\,\mathrm{d}x\\ &= 7\left(\frac{x^{2+1}}{2+1}\right) + c\\ &= \frac{7x^3}{3} + c\end{aligned}$$

Theorem 5

$$\int (f(x) + g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x$$

The integral of a sum is equal to the sum of the integrals.

EXAMPLE 7 Given that $f(x) = 3x^5 + \frac{2}{x^2}$, find $\int f(x)\,\mathrm{d}x$.

SOLUTION

$$\begin{aligned}\int 3x^5 + \frac{2}{x^2} &= \int 3x^5\,\mathrm{d}x + \int \frac{2}{x^2}\,\mathrm{d}x\\ &= 3\int x^5\,\mathrm{d}x + 2\int x^{-2}\,\mathrm{d}x\\ &= 3\left(\frac{x^{5+1}}{5+1}\right) + 2\left(\frac{x^{-2+1}}{-2+1}\right) + c\\ &= \frac{3}{6}x^6 - 2x^{-1} + c\\ &= \frac{1}{2}x^6 - \frac{2}{x} + c\end{aligned}$$

Theorem 6

$$\int (f(x) - g(x))\,\mathrm{d}x = \int f(x)\,\mathrm{d}x - \int g(x)\,\mathrm{d}x$$

The integral of a difference is equal to the difference of the integrals.

EXAMPLE 8 Find $\int (3\sqrt{x} - 8x^3)\,dx$.

SOLUTION

$$\begin{aligned}\int (3\sqrt{x} - 8x^3)\,dx &= 3\int \sqrt{x}\,dx - 8\int x^3\,dx\\ &= 3\int x^{\frac{1}{2}}\,dx - 8\int x^3\,dx\\ &= 3\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right) - 8\left(\frac{x^{3+1}}{3+1}\right) + c\\ &= 2x^{\frac{3}{2}} - 2x^4 + c\end{aligned}$$

Integration of polynomial functions

EXAMPLE 9 Find $\int 2x^7\,dx$.

SOLUTION

$$\begin{aligned}\int 2x^7\,dx &= 2\int x^7\,dx\\ &= 2\left(\frac{x^{7+1}}{7+1}\right) + c\\ &= \frac{2x^8}{8} + c\\ &= \frac{x^8}{4} + c\end{aligned}$$

EXAMPLE 10 Find $\int 6x^6\,dx$.

SOLUTION

$$\begin{aligned}\int 6x^6\,dx &= 6\int x^6\,dx\\ &= 6\left(\frac{x^{6+1}}{6+1}\right) + c\\ &= \frac{6}{7}x^7 + c\end{aligned}$$

EXAMPLE 11 Find $\int (6x^2 - 4x^3 + x)\,dx$.

SOLUTION

$\int (6x^2 - 4x^3 + x)\,dx = \int 6x^2\,dx - \int 4x^3\,dx + \int x\,dx$, using $\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$.

$$\begin{aligned}&= 6\left(\frac{x^{2+1}}{2+1}\right) - 4\left(\frac{x^{3+1}}{3+1}\right) + \frac{x^{1+1}}{1+1} + c\\ &= \frac{6x^3}{3} - \frac{4x^4}{4} + \frac{x^2}{2} + c\\ &= 2x^3 - x^4 + \frac{1}{2}x^2 + c\end{aligned}$$

EXAMPLE 12 Find the integral of $\int (3x^3 + 4x^2 - 2x + 1)\,dx$.

SOLUTION

$$\begin{aligned}\int (3x^3 + 4x^2 - 2x + 1)\,dx &= \frac{3}{4}x^4 + \frac{4}{3}x^3 - \frac{2}{2}x^2 + x + c\\ &= \frac{3}{4}x^4 + \frac{4}{3}x^3 - x^2 + x + c\end{aligned}$$

EXAMPLE 13 Given that $f(x) = 6x^5 - 3x^2 + 4x + 12$, find $\int f(x)\,dx$.

SOLUTION

$$\int f(x)\,dx = \int (6x^5 - 3x^2 + 4x + 12)\,dx$$
$$= \frac{6}{6}x^6 - \frac{3}{3}x^3 + \frac{4}{2}x^2 + 12x + c$$
$$= x^6 - x^3 + 2x^2 + 12x + c$$

EXAMPLE 14 Find $\int (2x - 3)^2\,dx$.

SOLUTION

Expanding $(2x - 3)^2$ gives:

$$\int (2x - 3)^2\,dx = \int 4x^2 - 12x + 9\,dx$$
$$= \frac{4}{3}x^3 - \frac{12}{2}x^2 + 9x + c$$
$$= \frac{4}{3}x^3 - 6x^2 + 9x + c$$

EXAMPLE 15 Given that a, b and c are constants, integrate with respect to x

(a) $ax + bx^7 - cx^{10}$

(b) $4ax^4 + bx^2 + 2c$

SOLUTION

(a) $\int (ax + bx^7 - cx^{10})\,dx = \frac{a}{2}x^2 + \frac{b}{8}x^8 - \frac{c}{11}x^{11} + d$ (where d is a constant)

(b) $\int (4ax^4 + bx^2 + 2c)\,dx = \frac{4}{5}ax^5 + \frac{b}{3}x^3 + 2cx + d$ (where d is a constant)

EXAMPLE 16 Find $\int (3x^2 - 2)(x^3 + 5)\,dx$.

SOLUTION

At this stage, we have no rules for integrating a product. We expand the brackets and integrate. Expanding the brackets gives:

$$(3x^2 - 2)(x^3 + 5) = 3x^5 + 15x^2 - 2x^3 - 10$$
$$\Rightarrow \int (3x^2 - 2)(x^3 + 5)\,dx = \int (3x^5 + 15x^2 - 2x^3 - 10)\,dx$$
$$= \frac{3}{6}x^6 + \frac{15}{3}x^3 - \frac{2}{4}x^4 - 10x + c$$
$$= \frac{1}{2}x^6 + 5x^3 - \frac{1}{2}x^4 - 10x + c$$

EXAMPLE 17 Integrate the expression $\int (6x^4 + \sqrt{x} - 3x^{\frac{2}{3}})\,dx$.

SOLUTION

$$\int (6x^4 + \sqrt{x} - 3x^{\frac{2}{3}})\,dx = \int (6x^4 + x^{\frac{1}{2}} - 3x^{\frac{2}{3}})\,dx$$
$$= 6\left(\frac{x^{4+1}}{4+1}\right) - 4\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} - 3\left(\frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1}\right) + c$$
$$= \frac{6}{5}x^5 + \frac{8}{3}x^{\frac{3}{2}} - \frac{9}{5}x^{\frac{5}{3}} + c$$

EXAMPLE 18 Integrate the expression $\int \frac{t^2+1}{t^4}\,dt$.

SOLUTION We have no rules at this stage for integrating quotients. We rewrite the function and then integrate. We can write this function as the sum of two terms as follows:

$$\begin{aligned}\frac{t^2+1}{t^4} &= \frac{t^2}{t^4} + \frac{1}{t^4}\\ &= \frac{1}{t^2} + \frac{1}{t^4}\\ &= t^{-2} + t^{-4}\end{aligned}$$

Therefore, $\int \frac{t^2+1}{t^4}\,dt = \int t^{-2} + t^{-4}\,dt.$

$$\begin{aligned}&= \frac{t^{-2+1}}{-2+1} + \frac{t^{-4+1}}{-4+1} + c\\ &= \frac{t^{-1}}{-1} + \frac{t^{-3}}{-3} + c\\ &= \frac{-1}{t} - \frac{1}{3t^3} + c\end{aligned}$$

EXAMPLE 19 Find $\int \frac{t^3+t}{\sqrt{t}}\,dt$.

SOLUTION We need to rewrite the equation before integrating, as follows:

$$\begin{aligned}\frac{t^3+t}{\sqrt{t}} &= \frac{t^3+t}{t^{\frac{1}{2}}}\\ &= \frac{t^3}{t^{\frac{1}{2}}} + \frac{t}{t^{\frac{1}{2}}}\\ &= t^{3-\frac{1}{2}} + t^{1-\frac{1}{2}} \qquad \text{(Using rules of indices)}\\ &= t^{2\frac{1}{2}} + t^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}\int \frac{t^3+t}{\sqrt{t}}\,dt &= \int t^{2\frac{1}{2}} + t^{\frac{1}{2}}\,dt\\ &= \frac{t^{\frac{5}{2}+1}}{\frac{5}{2}+1} + \frac{t^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c\\ &= \frac{t^{\frac{7}{2}}}{\frac{7}{2}} + \frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c\\ &= \frac{2}{7}t^{\frac{7}{2}} + \frac{2}{3}t^{\frac{3}{2}} + c\end{aligned}$$

EXAMPLE 20 Integrate the expression $\int x^3(4-\sqrt{x})\,dx$.

SOLUTION Since we have no laws for integrating products at this point, we expand the brackets and then integrate.

Multiplying out the bracket, we get:

$$\begin{aligned}x^3(4-\sqrt{x}) &= x^3(4-x^{\frac{1}{2}})\\ &= 4x^3 - x^{3+\frac{1}{2}}\\ &= 4x^3 - x^{\frac{7}{2}}\end{aligned}$$

$$\text{Now} \int x^3(4 - \sqrt{x})\,dx = \int 4x^3 - x^{\frac{7}{2}}\,dx$$
$$= \frac{4x^{3+1}}{3+1} - \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1} + c$$
$$= \frac{4}{4}x^4 - \frac{x^{\frac{9}{2}}}{\frac{9}{2}} + c$$
$$= x^4 - \frac{2}{9}x^{\frac{9}{2}} + c$$

EXAMPLE 21 Integrate the expression $\int \frac{x^8 - 4x^6}{x^3}\,dx$.

SOLUTION Rewriting the function:

$$\frac{x^8 - 4x^6}{x^3} = \frac{x^8}{x^3} - \frac{4x^6}{x^3}$$
$$= x^5 - 4x^3$$
$$\int \frac{x^8 - 4x^6}{x^3}\,dx = \int x^5 - 4x^3\,dx$$
$$= \frac{x^{5+1}}{5+1} + \frac{4x^{3+1}}{3+1} + c$$
$$= \frac{1}{6}x^6 - x^4 + c$$

EXAMPLE 22 Find $\int x(x+2)^2\,dx$.

SOLUTION Expanding the brackets, we get:

$$x(x+2)^2 = x(x^2 + 4x + 4)$$
$$= x^3 + 4x^2 + 4x$$

Therefore, $\int x(x+2)^2\,dx = \int (x^3 + 4x^2 + 4x)\,dx$

$$= \frac{x^4}{4} + \frac{4x^3}{3} + \frac{4x^2}{2} + c$$
$$= \frac{1}{4}x^4 + \frac{4}{3}x^3 + 2x^2 + c$$

EXAMPLE 23 Given that the rate of change of y with respect to t is $\frac{dy}{dt} = 2t^2 + 3t - 2$ and that $y = 4$ when $t = 0$, find y in terms of t.

SOLUTION

$$\frac{dy}{dt} = 2t^2 + 3t - 2$$

Integrating both sides with respect to t:

$$y = \int (2t^2 + 3t - 2)\,dt$$
$$y = \frac{2t^3}{3} + \frac{3t^2}{2} - 2t + c$$
$$y = \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t + c$$

When $t = 0$, $y = 4$.

Therefore, $4 = \frac{2}{3}(0)^3 + \frac{3}{2}(0)^2 - 2(0) + c$

$$4 = c$$

Hence, $y = \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t + 4$.

EXAMPLE 24 Given that $\frac{dy}{dx} = 4x^3 - 2x + 5$. Find y in terms of x given that $y = 1$ when $x = 1$.

SOLUTION

$$\frac{dy}{dx} = 4x^3 - 2x + 5$$

Integrating both sides with respect to x, we get:

$$y = \int (4x^3 - 2x + 5)\,dx$$

$$y = \frac{4x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 5x + c$$

$$= x^4 - x^2 + 5x + c$$

When $x = 1$, $y = 1$.

Therefore, $1 = 1^4 - 1^2 + 5(1) + c$

$$1 = 1 - 1 + 5 + c$$

$$c = 1 - 5$$

$$= -4$$

Therefore, $y = x^4 - x^2 + 5x - 4$.

EXERCISE 15A

In questions **1** to **15**, integrate each function with respect to x.

1 x^9

2 $5x^6$

3 $\frac{1}{2}\sqrt{x}$

4 $8x^6$

5 $\frac{x+1}{x^3}$

6 $(4x + 1)(x^2 + 2)$

7 $\frac{\sqrt{x} - 2}{\sqrt{x}}$

8 $\frac{x^4 - 6x^2}{x^6}$

9 $(2x^2 + 1)^2$

10 $\frac{(\sqrt{x} + 1)^2}{\sqrt{x}}$

11 $(x - 3\sqrt{x})^2$

12 $\frac{x^5 + 6x^2}{x^7}$

13 $\frac{x^3 + 4}{x^3}$

14 $4x^3 - \frac{5}{x^2}$

15 $(x^5 + 2)^2$

In questions **16** to **25**, find the following integrals.

16 $\int 4x^5\,dx$

17 $\int 7x^3 - \frac{3}{x^5}\,dx$

18 $\int (4 + 2x)^3\,dx$

19 $\int (x - 1)^3\,dx$

20 $\int \left(\frac{x+1}{x^2}\right)^2 dx$

21 $\int (1 - 3x + x^2)^2\,dx$

22 $\int \frac{4}{x^2} + 6x^3 + 10\sqrt{x}\,dx$

23 $\int 4x^7 + 3x^3 - \frac{6}{x^3}\,dx$

24 $\int \frac{x^2 + 4x}{\sqrt{x}}\,dx$

25 $\int \frac{3 + 2t}{t^3}\,dt$

26 Given that the gradient of a curve is $4x^3 - 8x + 2$ and that the curve passes through $(1, 0)$, determine the equation of the curve.

27 Find the equation of the curve which has a gradient of $4x + 3$ and passes through the point (2, 1).

28 The rate of change of y with respect to t is given by $\frac{dy}{dx} = \frac{4t - 3}{t^3}$. Find y in terms of t, given that $y = 1$ when $t = 2$.

29 The gradient of a curve is given by $\frac{dy}{dx} = x^2 - 6 + \frac{x}{2}$. Given that the curve passes through (1, 1), find the equation of the curve.

30 The rate of change of u with respect to t is given by $\frac{dy}{dx} = 4t^2 - \sqrt{t}$. Find u in terms of t, given that $u = 1$ when $t = 1$.

Integration of a function involving a linear factor

Let a and b be constants.

$$\frac{d}{dx}\frac{(ax + b)^{n+1}}{a(n + 1)} = \frac{(n + 1) \times a \times (ax + b)^n}{a(n + 1)}$$
$$= (ax + b)^n$$
$$\Rightarrow \frac{(ax + b)^{n+1}}{a(n + 1)} + c = \int (ax + b)^n\,dx,\ n \neq -1 \text{ and a} \neq 0.$$

EXAMPLE 25 Integrate $\int (2x + 1)^6\,dx$.

SOLUTION Comparing $(2x + 1)^6$ with $(ax + b)^n$, we have $a = 2$, $b = 1$ and $n = 6$.

Using $\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$, we get:

$$\int (2x + 1)^6\,dx = \frac{(2x + 1)^{6+1}}{2(6 + 1)} + c$$
$$= \frac{(2x + 1)^7}{14} + c$$

EXAMPLE 26 Integrate the expression $\int \left(\frac{4}{6x + 3}\right)^2 dx$.

SOLUTION

$$\left(\frac{4}{6x + 3}\right)^2 = \frac{16}{(6x + 3)^2}$$
$$= 16(6x + 3)^{-2}$$

Therefore $\int \left(\frac{4}{6x + 3}\right)^2 dx = \int 16(6x + 3)^{-2}\,dx$

$$= 16\int (6x + 3)^{-2}\,dx$$

Using $a = 6$, $b = 3$ and $n = -2$ in $\int (ax + b)^n\,dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$, we get:

$$\int 16(6x + 3)^{-2}\,dx = 16\frac{((6x + 3)^{-2+1})}{6(-2 + 1)} + c$$
$$= 16\frac{(6x + 3)^{-1}}{-6} + c$$
$$= -\frac{8}{3}\left(\frac{1}{6x + 3}\right) + c$$

EXAMPLE 27 Find $\int \sqrt{3-2t}\,dt$.

SOLUTION

$$\int \sqrt{3-2t}\,dt = \int (3-2t)^{\frac{1}{2}}\,dt$$

Using $a = -2$, $b = 3$ and $n = \frac{1}{2}$ into $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)}$, we get:

$$\int (3-2t)^{\frac{1}{2}}\,dt = \frac{(3-2t)^{\frac{1}{2}+1}}{-2\left(\frac{1}{2}+1\right)} + c$$

$$= \frac{(3-2t)^{\frac{3}{2}}}{-2\left(\frac{3}{2}\right)} + c$$

$$= -\frac{1}{3}(3-2t)^{\frac{3}{2}} + c$$

EXAMPLE 28 Can we find $\int (x^2+1)^{\frac{3}{2}}\,dx$ by using the result $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$.

SOLUTION In the function $(ax+b)^n$, $ax+b$ is a linear function, while x^2+1 is a quadratic function. Hence, we cannot integrate $(x^2+1)^{\frac{3}{2}}$ by using this result.

Try these 15.1

(a) Find the integral of the following.

(i) $\int (4x+2)^8\,dx$

(ii) $\int \sqrt{3x+5}\,dx$

(iii) $\int \sqrt{2x+1}\,dx$

(b) Given that $\frac{dy}{dx} = (3x-1)^6$, find y as a function of x given that $y = 1$ when $x = 1$.

Integration of trigonometric functions

Since $\frac{d}{dx}(\cos x) = -\sin x$, integrating both sides with respect to x, we get:

$$\cos x + c = \int -\sin x\,dx$$

$$\int \sin x\,dx = -\cos x + c$$

Since $\frac{d}{dx}(\sin x) = \cos x$, integrating both sides with respect to x:

$$\sin x + c = \int \cos x\,dx$$

Since $\frac{d}{dx}(\tan x) = \sec^2 x$, integrating both sides with respect to x:

$$\tan x + c = \int \sec^2 x\,dx$$

> $\int \sin x\,dx = -\cos x + c$
> $\int \cos x\,dx = \sin x + c$
> $\int \sec^2 x\,dx = \tan x + c$

EXAMPLE 29 Find $\int (\sin x + 3\cos x)\,dx$.

SOLUTION

$$\int (\sin x + 3\cos x)\,dx = \int \sin x\,dx + 3\int \cos x\,dx$$

$$= -\cos x + 3\sin x + c$$

EXAMPLE 30 Find $\int(4\sin x - 2\cos x)\,dx$.

SOLUTION $\int(4\sin x - 2\cos x)\,dx = -4\cos x - 2\sin x + c$

EXAMPLE 31 Find $\int(x^2 + 6\cos x - 2\sin x)\,dx$.

SOLUTION
$$\begin{aligned}\int(x^2 + 6\cos x - 2\sin x)\,dx &= \int x^2\,dx + 6\int\cos x\,dx - 2\int\sin x\,dx\\ &= \frac{x^3}{3} + 6\sin x - 2(-\cos x) + c\\ &= \frac{1}{3}x^3 + 6\sin x + 2\cos x + c\end{aligned}$$

Let us move onto trigonometric functions of multiple angles.

$\frac{d}{dx}\left(\frac{1}{a}\sin ax\right) = \cos ax$, where a is a constant.

$\Rightarrow \int\cos ax\,dx = \frac{1}{a}\sin ax + c$

$\frac{d}{dx}\left(\frac{1}{a}\cos ax\right) = -\sin ax$

$\Rightarrow -\frac{1}{a}\cos ax + c = \int\sin ax\,dx$

$\frac{d}{dx}\left(\frac{1}{a}\tan ax\right) = \sec^2 ax$

$\Rightarrow \frac{1}{a}\tan ax + c = \int\sec^2 ax\,dx$

> $\int\sin ax\,dx = -\frac{1}{a}\cos ax + c$
>
> $\int\cos ax\,dx = \frac{1}{a}\sin ax + c$
>
> $\int\sec^2 ax\,dx = \frac{1}{a}\tan ax + c$

EXAMPLE 32 Find $\int\sin 4x\,dx$.

SOLUTION $\int\sin 4x\,dx = -\frac{1}{4}\cos 4x + c$ $\quad\left(\text{Using} \int\sin ax\,dx = \frac{-1}{a}\cos ax + c\right)$

EXAMPLE 33 Find $\int 2\cos 3x\,dx$.

SOLUTION $\int 2\cos 3x\,dx = \frac{2}{3}\sin 3x + c$ $\quad\left(\text{Using} \int\cos ax\,dx = \frac{1}{a}\sin ax + c\right)$

EXAMPLE 34 Integrate $\sec^2 6x$.

SOLUTION $\int\sec^2 6x\,dx = \frac{1}{6}\tan 6x + c$

EXAMPLE 35 Find $\int\sec^2\frac{x}{8}\,dx$.

SOLUTION
$$\begin{aligned}\int\sec^2\frac{x}{8}\,dx &= \frac{1}{\frac{1}{8}}\tan\left(\frac{x}{8}\right) + c\\ &= 8\tan\left(\frac{x}{8}\right) + c\end{aligned}$$

Let us move on to trigonometric functions of sums of angles.

Since $\frac{d}{dx}\left(\frac{1}{a}\cos(ax + b)\right) = -\sin(ax + b)$ where a, b and c are constants.

$\Rightarrow \int\sin(ax + b)\,dx = -\frac{1}{a}\cos(ax + b) + c$

$$\frac{d}{dx}\left(\frac{1}{a}\sin(ax+b)\right) = \cos(ax+b)$$

$$\Rightarrow \int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$$

$$\frac{d}{dx}\left(\frac{1}{a}\tan(ax+b)\right) = \sec^2(ax+b)$$

$$\Rightarrow \int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + c$$

EXAMPLE 36 Integrate the expression $\int \cos\left(3x + \frac{\pi}{2}\right)dx$.

SOLUTION Using $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$, where $a = 3$, $b = \frac{\pi}{2}$:

$$\int \cos\left(3x + \frac{\pi}{2}\right)dx = \frac{1}{3}\sin\left(3x + \frac{\pi}{2}\right) + c$$

EXAMPLE 37 Find $\int \sin\left(4x - \frac{\pi}{3}\right)dx$.

SOLUTION $\int \sin\left(4x - \frac{\pi}{3}\right)dx = -\frac{1}{4}\cos\left(4x - \frac{\pi}{3}\right) + c$

EXAMPLE 38 Find $\int 5\sec^2\left(6x + \frac{\pi}{2}\right)dx$.

SOLUTION $\int 5\sec^2\left(6x + \frac{\pi}{2}\right)dx = \frac{5}{6}\tan\left(6x + \frac{\pi}{2}\right) + c$

Note

a, b and c are constants.

Table of integrals for trigonometric functions

Function	Integral
$\sin x$	$-\cos x + c$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$
$\cos x$	$\sin x + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\tan x$	$\ln\lvert\sec x\rvert + c$
$\tan ax$	$\frac{1}{a}\ln\lvert\sec ax\rvert + c$
$\sec x$	$\ln\lvert\sec x + \tan x\rvert + c$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\left(\frac{x}{2}\right)\right\rvert + c$
$\cot x$	$\ln\lvert\sin x\rvert + c$
$\sec^2 x$	$\tan x + c$
$\operatorname{cosec}^2 x$	$-\cot x + c$

Let us use the table above to find some integrals.

EXAMPLE 39 Find $\int \tan 4x\,dx$.

SOLUTION $\int \tan ax\,dx = \frac{1}{a}\ln|\sec ax| + c$

Using $a = 4$, we have:

$\int \tan(4x)\,dx = \frac{1}{4}\ln|\sec 4x| + c$

EXAMPLE 40 Find $\int 3\cot\left(4x + \frac{\pi}{2}\right)dx$.

SOLUTION $\int 3\cot\left(4x + \frac{\pi}{2}\right)dx = \frac{3}{4}\ln\left|\sin\left(4x + \frac{\pi}{2}\right)\right| + c$

Try these 15.2 Find the integrals of the following functions with respect to x.

(a) $\int 4\sin\left(x + \frac{\pi}{2}\right)dx$ **(b)** $\int \cos\left(\frac{\pi}{2} - 3x\right)dx$

(c) $\int 4\tan\left(3x + \frac{\pi}{2}\right)dx$ **(d)** $\int 3\cot(5x)\,dx$

EXERCISE 15B

1 Integrate the following with respect to x:

(a) $\sin 6x$ (b) $\cos 3x$

(c) $4\cos 2x$ (d) $2\sec^2 3x$

(e) $2\cos 6x - \sin 4x$ (f) $\cos 4x + \frac{1}{\cos^2 3x}$

(g) $\sin 4x + \cos 5x + 3\cos x$ (h) $x^2 + 6\tan 2x$

(i) $\frac{1}{x^3} - \sec^2(\sqrt{2}x)$ (j) $4\sec^2(6x + 9)$

2 Find the following indefinite integrals.

(a) $\int \sin(3x - 1)\,dx$ (b) $\int \cos(2x + 3)\,dx$

(c) $\int \tan\left(\frac{\pi}{3} - 4x\right)dx$ (d) $\int \sec^2\left(x - \frac{3\pi}{4}\right)dx$

(e) $\int 7\cos(6x + 9)\,dx$ (f) $\int 6\sin(4x - 6)\,dx$

(g) $\int 8\sec^2(8x - \pi)\,dx$ (h) $\int \cos qx\,dx$

(i) $\int \sin(px + \pi)\,dx$ (j) $\int \tan(4 - rx)\,dx$

3 Integrate the following with respect to t.

(a) $(3t + 1)^6$ (b) $(1 - 4t)^3$ (c) $(2t + 7)^{-4}$

(d) $\sqrt{6t - 1}$ (e) $\frac{4}{\sqrt{2 - 3t}}$ (f) $\frac{4}{\sqrt{7 - 6t}}$

(g) $\frac{6}{7(3x - 1)^3}$ (h) $\frac{2}{(4t - 3)^5}$ (i) $3(6t - 4)^8$

4 Find the equation of the curve which passes through the point (0, 2) and for which $\frac{dy}{dx} = \frac{1}{(3x + 2)^4}$.

5 Find x as a function of t when $t = \frac{\pi}{4}$, given that $\frac{dx}{dt} = \cos\left(2t - \frac{\pi}{4}\right)$ and that $x = 1$.

6 Given that the gradient of a curve is $x(3 + 4x)$ and that the curve passes through $(1, 2)$ and $(-2, a)$, find the value of a.

Integration of more trigonometric functions

Integration of trigonometric functions makes use of the trigonometric identities to convert the more complicated functions to standard integrals. At this stage you should review trigonometric identities. The standard integrals are:

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

$$\int \tan x \, dx + \ln|\sec x| + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

EXAMPLE 41 Determine $\int \sin\left(6x - \frac{\pi}{2}\right) dx$.

SOLUTION

$$\int \sin\left(6x - \frac{\pi}{2}\right) dx = -\frac{1}{6}\cos\left(6x - \frac{\pi}{2}\right) + c$$

EXAMPLE 42 Determine $\int \sec^2\left(2x - \frac{\pi}{4}\right) dx$.

SOLUTION

$$\int \sec^2\left(2x - \frac{\pi}{4}\right) dx = \frac{1}{2}\tan\left(2x - \frac{\pi}{4}\right) + c$$

EXAMPLE 43 Find $\int \cos\left(8x + \frac{\pi}{6}\right) dx$.

SOLUTION

$$\int \cos\left(8x + \frac{\pi}{6}\right) dx = \frac{1}{8}\sin\left(8x + \frac{\pi}{6}\right) + c$$

If an integral is of the form $\int f'(x)(f(x))^n \, dx$ the integral becomes

$$\int f'(x)(f(x))^n \, dx = \frac{(f(x))^{n+1}}{n+1} + c, \text{ where } n \neq -1$$

EXAMPLE 44 Find $\int \sin x \cos^2 x \, dx$.

SOLUTION

$\sin x \cos^2 x$ is of the form $f'(x)(f(x))^n$.

Let $f(x) = \cos x$.

$\Rightarrow f'(x) = -\sin x$

$$\int f'(x)(f(x))^n \, dx = -\int -\sin x \cos^2 x \, dx$$
$$= \frac{-\cos^3 x}{3} + c$$

EXAMPLE 45 Find $\int \sin x \cos^6 x \, dx$.

SOLUTION The integral is of the form $\int f'(x)(f(x))^n \, dx$.

$f(x) = \cos x$

$f'(x) = -\sin x,\ n = 6$

$$\int f'(x)(f(x))^n \, dx = -\int -\sin x \cos^6 x \, dx$$

$$= -\frac{\cos^7 x}{7} + c$$

EXAMPLE 46 Find $\int \sin x \cos^n x \, dx,\ n \neq -1$.

SOLUTION The integral is of the form $\int f'(x)(f(x))^n \, dx$.

$f(x) = \cos x$

$f^1(x) = -\sin x, \quad n \equiv n$

$$\int f'(x)(f(x))^n \, dx = -\int -\sin x \cos^n x \, dx$$

$$= -\frac{\cos^{n+1} x}{n+1} + c, \quad n \neq -1.$$

$$\int \sin x \cos^n x \, dx = -\frac{\cos^{n+1} x}{n+1} + c, \quad n \neq -1.$$

EXAMPLE 47 $\int \cos x \sin^n x \, dx,\ n \neq -1$

SOLUTION The integral is of the form $\int f'(x)(f(x))^n \, dx$.

$f(x) = \sin x$

$f'(x) = \cos x,$

$$\int f'(x)(f(x))^n \, dx = \int \cos x \ \sin^n x \, dx = \frac{\sin^{n+1} x}{n+1} + c, \quad n \neq -1$$

EXAMPLE 48 Determine $\int \tan^n x \sec^2 x \, dx,\ n \neq -1$.

SOLUTION

$$\int \tan^n x \sec^2 x \, dx = \frac{\tan^{n+1} x}{n+1} + c, \quad n \neq -1$$

$\left(\text{Since } \frac{d}{dx} \tan x = \sec^2 x.\right)$

Integrating $\sin^2 x$ and $\cos^2 x$

EXAMPLE 49 Find $\int \sin^2 x \, dx$.

SOLUTION Recall: $\cos 2x = 1 - 2\sin^2 x$

$$2\sin^2 x = 1 - \cos 2x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

Use $\sin^2 x = \dfrac{1 - \cos 2x}{2}$.

Therefore, $\displaystyle\int \sin^2 x \, dx = \int \frac{1}{2} - \frac{1}{2}\cos 2x \, dx$

$= \dfrac{1}{2}x - \dfrac{1}{4}\sin 2x + c.$

Did you consider using $\displaystyle\int \sin^2 x \, dx = \frac{\sin^3 x}{3} + c$? This is incorrect. If you differentiate $\dfrac{\sin^3 x}{3}$, you will not get $\sin^2 x$.

EXAMPLE 50 Find $\displaystyle\int \cos^2 x \, dx$.

SOLUTION

Recall: $\cos 2x = 2\cos^2 x - 1$

$\cos^2 x = \dfrac{\cos 2x + 1}{2}$

Use $\cos^2 x = \dfrac{1}{2} + \dfrac{1}{2}\cos 2x$.

Therefore, $\displaystyle\int \cos^2 x \, dx = \int \frac{1}{2} + \frac{1}{2}\cos 2x \, dx$

$= \dfrac{1}{2}x + \dfrac{1}{4}\sin 2x + c.$

EXAMPLE 51 Find $\displaystyle\int \sin^3 x \, dx$.

SOLUTION

$\sin^3 x = \sin x \sin^2 x$

$\displaystyle\int \sin^3 x \, dx = \int \sin x \sin^2 x \, dx$

Use $\sin^2 x = 1 - \cos^2 x$.

$\displaystyle\int \sin^3 x \, dx = \int \sin x (1 - \cos^2 x) \, dx$

$\displaystyle= \int (\sin x - \sin x \cos^2 x) \, dx$

$= -\cos x + \dfrac{\cos^3 x}{3} + c$

EXAMPLE 52 Find $\displaystyle\int \cos^3 x \, dx$.

SOLUTION

Recall that $\displaystyle\int \cos x \sin^n x \, dx$ can be used to integrate $\displaystyle\int \sin^2 x \cos x \, dx = \frac{\sin^3 x}{3} + c$

The procedure is the same as that of integrating $\sin^3 x$.

Write $\cos^3 x = \cos x \cos^2 x$.

Use $\cos^2 x = 1 - \sin^2 x$.

Therefore $\cos^3 x = \cos x (1 - \sin^2 x)$

$\displaystyle\int \cos^3 x \, dx = \int \cos x (1 - \sin^2 x) \, dx$

$\displaystyle= \int (\cos x - \cos x \sin^2 x) \, dx = \sin x - \frac{\sin^3 x}{3} + c$

We can integrate $\sin^5 x$, $\cos^5 x$, $\sin^7 x$, $\cos^7 x$, etc. using the same procedure as for $\sin^3 x$ and $\cos^3 x$. We split them into $\sin x \sin^{n-1} x$ and replace $\sin^2 x$ with $1 - \cos^2 x$.

EXAMPLE 53 Find $\int \sin^4 x\,dx$.

SOLUTION We use $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$, and write:

$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)^2$

Therefore, $\int \sin^4 x\,dx = \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x\right) dx.$ $\left(\text{Since } \cos^2 x = \frac{1}{2} + \frac{1}{2}\cos 2x\right)$

Use the double angle formula $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 2(2x)$

$= \frac{1}{2} + \frac{1}{2}\cos 4x.$

Therefore, $\int \sin^4 x\,dx = \int \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2}\cos 4x\right) dx$

$$= \int \left(\frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x\right) dx$$

$$= \int \left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx$$

$$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$$

We can integrate $\cos^4 x$, $\sin^6 x$, $\cos^6 x$ etc. in a similar manner.

Try these 15.3

(a) $\int \sin^5 x\,dx$

(b) $\int \cos^5 x\,dx$

(c) $\int \cos^4 x\,dx$

Let us look at $\tan x$.

EXAMPLE 54 Find $\int \tan^2 x\,dx$.

SOLUTION $\tan^2 x = \sec^2 x - 1.$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$

$$= \tan x - x + c.$$

EXAMPLE 55 Determine $\int \tan^3 x\,dx$.

SOLUTION $\tan^3 x = \tan x \tan^2 x$

$= \tan x\,(\sec^2 x - 1)$

$= \tan x \sec^2 x - \tan x$

Therefore, $\int \tan^3 x\,dx = \int (\tan x \sec^2 x - \tan x)\,dx$

Recall that $\int \tan^n x \sec^2 x\,dx = \frac{\tan^{n+1} x}{n+1} + c,\ n \neq -1.$

Therefore, $\therefore \int \tan x \sec^2 x\,dx = \frac{\tan^2 x}{2} + c.$

Therefore, $\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x| + c$

Try these 15.4

Find these.

(a) $\int \tan^4 x\,dx$

(b) $\int \tan^5 x\,dx$

Integration of products of sines and cosines

Recall these.

$2 \sin P \cos Q = \sin(P + Q) + \sin(P - Q)$

$2 \cos P \sin Q = \sin(P + Q) - \sin(P - Q)$

$2 \cos P \cos Q = \cos(P + Q) + \cos(P - Q)$

$-2 \sin P \sin Q = \cos(P + Q) - \cos(P - Q)$

Let us use these to integrate the following.

EXAMPLE 56

Find $\int (\cos 4x \sin 2x)\,dx$.

SOLUTION

Use $2 \cos P \sin Q = \sin(P + Q) - \sin(P - Q)$.

$2 \cos 4x \sin 2x = \sin(4x + 2x) - \sin(4x - 2x)$

$2 \cos 4x \sin 2x = \sin 6x - \sin 2x$

$\therefore \cos 4x \sin 2x = \frac{1}{2}\sin 6x - \frac{1}{2}\sin 2x$

$$\int \cos 4x \sin 2x\,dx = \int \left(\frac{1}{2}\sin 6x - \frac{1}{2}\sin 2x\right) dx$$

$$= -\frac{1}{12}\cos 6x + \frac{1}{4}\cos 2x + c$$

EXAMPLE 57

Find $\int (\cos 5x \cos 3x)\,dx$.

SOLUTION

Use $2 \cos P \cos Q = \cos(P + Q) + \cos(P - Q)$.

$2 \cos 5x \cos 3x = \cos(5x + 3x) + \cos(5x - 3x)$

$2 \cos 5x \cos 3x = \cos 8x + \cos 2x$

$\cos 5x \cos 3x = \frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x$

$$\int \cos 5x \cos 3x\,dx = \int \left(\frac{1}{2}\cos 8x + \frac{1}{2}\cos 2x\right) dx$$

$$= \frac{1}{16}\sin 8x + \frac{1}{4}\sin 2x + c$$

Try these 15.5

Find the following.

(a) $\int (\cos 6x \sin 3x)\,dx$

(b) $\int (\cos 8x \cos 2x)\,dx$

(c) $\int (\sin 10x \sin x)\,dx$

The definite integral

If $\int f(x)\,dx = F(x) + c$, then the definite integral of $f(x)$ between the limits $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx = [F(x)]_a^b$$
$$= F(b) - F(a)$$

We evaluate $F(x)$ when $x = b$ and $F(x)$ when $x = a$ and subtract.

Some results of integration

1 $\int_a^a f(x)\,dx = 0$

2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ We can switch the limits and change the sign of $f(x)$.

3 $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

4 $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

5 $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ The integral of the sum or difference is the sum or difference of the integral.

6 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

7 $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx,\ a > 0.$

EXAMPLE 58 Find the value of $\int_0^1 (x^2 + 2x + 1)\,dx$.

SOLUTION

$$\int_0^1 (x^2 + 2x + 1)\,dx = \left[\frac{x^3}{3} + x^2 + x\right]_0^1$$
$$= \left(\frac{1^3}{3} + (1)^2 + (1)\right) - \left(\left(\frac{0^3}{3}\right) + (0)^2 + (0)\right)$$ (Substitute $x = 1$ and $x = 0$, and subtract.)
$$= \frac{1}{3} + 1 + 1$$
$$= 2\frac{1}{3}$$

EXAMPLE 59 Evaluate $\int_1^2 (4x^3 - 2x^2)\,dx$.

SOLUTION

$$\int_1^2 4x^3 - 2x^2\,dx = \left[x^4 - \frac{2}{3}x^3\right]_1^2$$
$$= \left((2)^4 - \frac{2}{3}(2)^3\right) - \left((1)^4 - \frac{2}{3}(1)^3\right)$$ (Substituting $x = 2$, $x = 1$, and subtracting)
$$= 16 - \frac{16}{3} - \left(1 - \frac{2}{3}\right)$$
$$= 16 - \frac{16}{3} - \frac{1}{3}$$
$$= \frac{48}{3} - \frac{17}{3}$$
$$= \frac{31}{3}$$
$$= 10\frac{1}{3}$$

EXAMPLE 60 Find the value of $\int_1^4 \left(\sqrt{x} + \frac{3}{x^2}\right) \mathrm{d}x$.

SOLUTION

$$\begin{aligned}
\int_1^4 \left(\sqrt{x} + \frac{3}{x^2}\right) \mathrm{d}x &= \int_1^4 \left(x^{\frac{1}{2}} + 3x^{-2}\right) \mathrm{d}x \\
&= \left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{3x^{-2+1}}{-2+1}\right]_1^4 \\
&= \left[\frac{2}{3}x^{\frac{3}{2}} - 3x^{-1}\right]_1^4 \\
&= \left(\frac{2}{3}(4)^{\frac{3}{2}} - \frac{3}{4}\right) - \left(\frac{2}{3}(1)^{\frac{3}{2}} - \frac{3}{1}\right) \\
&= \left(\frac{16}{3} - \frac{3}{4}\right) - \left(\frac{2}{3} - 3\right) \\
&= \frac{64}{12} - \frac{9}{12} + \frac{7}{3} \\
&= \frac{55}{12} + \frac{28}{12} \\
&= \frac{83}{12}
\end{aligned}$$

EXAMPLE 61 Find the exact value of $\int_0^{\frac{\pi}{3}} (\sin x + 4)\, \mathrm{d}x$.

SOLUTION

$$\begin{aligned}
\int_0^{\frac{\pi}{3}} (\sin x + 4)\, \mathrm{d}x &= [-\cos x + 4x]_0^{\frac{\pi}{3}} \\
&= \left(-\cos\frac{\pi}{3} + 4\left(\frac{\pi}{3}\right)\right) - (-\cos 0 + 4(0)) \\
&= -\frac{1}{2} + \frac{4\pi}{3} + 1 \\
&= \frac{1}{2} + \frac{4\pi}{3} \\
&= \frac{1}{6}(3 + 8\pi)
\end{aligned}$$

EXAMPLE 62 Find $\int_0^{\frac{\pi}{2}} 4 \sin 6x \sin 2x\, \mathrm{d}x$.

SOLUTION

Use $2 \sin \mathrm{P} \sin \mathrm{Q} = \cos(\mathrm{P} - \mathrm{Q}) - \cos(\mathrm{P} + \mathrm{Q})$.

$2 \sin 6x \sin 2x = \cos(6x - 2x) - \cos(6x + 2x)$

$2 \sin 6x \sin 2x = \cos 4x - \cos 8x$

$4 \sin 6x \sin 2x = 2 \cos 4x - 2 \cos 8x$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} 4 \sin 6x \sin 2x\, \mathrm{d}x &= \int_0^{\frac{\pi}{2}} (2 \cos 4x - 2 \cos 8x)\, \mathrm{d}x \\
&= \left[\frac{1}{2}\sin 4x - \frac{1}{4}\sin 8x\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{2}\sin 4\left(\frac{\pi}{2}\right) - \frac{1}{4}\sin 8\left(\frac{\pi}{2}\right) - \left[\frac{1}{2}\sin 4(0) - \frac{1}{4}\sin 8(0)\right] \\
&= 0
\end{aligned}$$

EXAMPLE 63 Evaluate $\int_0^{\frac{\pi}{4}} \sin 2x \cos 2x\, \mathrm{d}x$.

SOLUTION

$\sin 4x = 2 \sin 2x \cos 2x$

$\frac{1}{2}\sin 4x = \sin 2x \cos 2x$

$$\int_0^{\frac{\pi}{4}} \sin 2x \cos 2x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 4x \, dx$$
$$= \frac{1}{2}\left[-\frac{1}{4}\cos 4x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(-\frac{1}{4}\cos 4\left(\frac{\pi}{4}\right) + \frac{1}{4}\cos 4(0)\right)$$
$$= \frac{1}{2}\left(-\frac{1}{4}\cos \pi + \frac{1}{4}\cos 0\right)$$
$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right)$$
$$= \frac{1}{2}\left(\frac{1}{2}\right)$$
$$= \frac{1}{4}$$

EXAMPLE 64 Find the value of $\displaystyle\int_1^2 \frac{4}{(3x+2)^2}\,dx$.

SOLUTION

$$\int_1^2 \frac{4}{(3x+2)^2}\,dx = \int_1^2 4(3x+2)^{-2}\,dx$$
$$= \left[\frac{4(3x+2)^{-1}}{3(-1)}\right]_1^2$$
$$= \left[-\frac{4}{3(3x+2)}\right]_1^2$$
$$= -\frac{4}{3(6+2)} - \left(-\frac{4}{3(3+2)}\right)$$
$$= -\frac{4}{24} + \frac{4}{15}$$
$$= -\frac{1}{6} + \frac{4}{15}$$
$$= -\frac{15}{90} + \frac{24}{90}$$
$$= \frac{9}{90}$$
$$= \frac{1}{10}$$

EXAMPLE 65 Given that $\displaystyle\int_0^3 g(x)\,dx = 12$, evaluate:

(a) $\displaystyle\int_0^3 2g(x)\,dx$

(b) $\displaystyle\int_0^3 (g(x)+1)\,dx$

(c) $\displaystyle\int_3^0 g(x)\,dx.$

SOLUTION

(a) $\displaystyle\int_0^3 2g(x)\,dx = 2\int_0^3 g(x)\,dx = 2(12) = 24$

(b) $\displaystyle\int_0^3 (g(x)+1)\,dx = \int_0^3 g(x)\,dx + \int_0^3 1\,dx = 12 + [x]_0^3 = 12 + [3-0] = 12 + 3 = 15$

(c) $\displaystyle\int_3^0 g(x)\,dx = -\int_0^3 g(x)\,dx = -12 \qquad \left(\text{Since } \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$

EXAMPLE 66 Given that $y = \frac{x+1}{2-3x}$, show that $\frac{dy}{dx} = \frac{5}{(2-3x)^2}$. Hence, find $\int_0^{\frac{1}{2}} \frac{1}{(2-3x)^2}dx$

SOLUTION

$y = \frac{x+1}{2-3x}$

Let $u = x + 1$, $v = 2 - 3x$

$\frac{du}{dx} = 1, \frac{dv}{dx} = -3$

Use the quotient rule:

$$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(2-3x)(1) - (x+1)(-3)}{(2-3x)^2}$$

$$= \frac{2 - 3x + 3x + 3}{(2-3x)^2}$$

$$= \frac{5}{(2-3x)^2}$$

Since $\frac{d}{dx}\left[\frac{x+1}{2-3x}\right] = \frac{5}{(2-3x)^2}$, integrating both sides with respect to x from 0 to $\frac{1}{2}$, we get:

$$\left[\frac{x+1}{2-3x}\right]_0^{\frac{1}{2}} = \int_0^{\frac{1}{2}} \frac{5}{(2-3x)^2}dx$$

$$\Rightarrow \frac{\frac{1}{2}+1}{2-3\left(\frac{1}{2}\right)} - \frac{0+1}{2-3(0)} = 5\int_0^{\frac{1}{2}} \frac{1}{(2-3x)^2}dx$$

$$\Rightarrow 3 - \frac{1}{2} = 5\int_0^{\frac{1}{2}} \frac{1}{(2-3x)^2}dx$$

$$\frac{5}{2} \times \frac{1}{5} = \int_0^{\frac{1}{2}} \frac{1}{(2-3x)^2}\, dx$$

$$\frac{1}{2} = \int_0^{\frac{1}{2}} \frac{1}{(2-3x)^2}\, dx$$

EXAMPLE 67 Given that $y = \frac{\sin 2x}{1 + \cos 2x}$, show that $\frac{dy}{dx} = \frac{2}{1 + \cos 2x}$. Hence find $\int_0^{\frac{\pi}{4}} \frac{1}{1+\cos 2x}dx$.

SOLUTION

$$\frac{dy}{dx} = \frac{(1+\cos 2x)(2\cos 2x) - (\sin 2x)(-2\sin 2x)}{(1+\cos 2x)^2}$$

$$= \frac{2\cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(1+\cos 2x)^2}$$

$$= \frac{2\cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(1+\cos 2x)^2}$$

$$= \frac{2\cos 2x + 2}{(1+\cos 2x)^2}$$

$$= \frac{2(1+\cos 2x)}{(1+\cos 2x)^2}$$

$$= \frac{2}{(1+\cos 2x)}$$

Since $\frac{d}{dx}\left[\frac{\sin 2x}{1+\cos 2x}\right] = \frac{2}{1+\cos 2x}$, integrating both sides with respect to x from 0 to $\frac{\pi}{4}$ gives:

$$\left[\frac{\sin 2x}{1+\cos 2x}\right]_0^{\frac{\pi}{4}} = \int_0^{\frac{\pi}{4}}\left(\frac{2}{1+\cos 2x}\right)dx$$

$$\Rightarrow \frac{\sin\frac{\pi}{2}}{1+\cos\frac{\pi}{2}} - \frac{\sin 0}{1+\cos 0} = 2\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2x}\right)dx$$

$$\Rightarrow \frac{1}{1+0} - \frac{0}{1+1} = 2\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2x}\right)dx$$

$$\Rightarrow \frac{1}{2} = \int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2x}\right)dx$$

EXAMPLE 68 Given that $y = x\sqrt{2x+3}$, find $\frac{dy}{dx}$. Hence, find $\int_0^1 \frac{x+1}{\sqrt{2x+3}}\,dx$.

SOLUTION We use the product rule and chain rule to differentiate this function.

$$\left(\frac{d}{dx}(2x+3)^{\frac{1}{2}} = \frac{1}{2}(2x+3)^{-\frac{1}{2}} \times 2\right)$$

$$\frac{d}{dx}x\sqrt{2x+3} = \sqrt{2x+3} + x\left(\frac{1}{2}\right)(2)(2x+3)^{-\frac{1}{2}}$$

$$= \sqrt{2x+3} + \frac{x}{\sqrt{2x+3}}$$

$$= \frac{2x+3+x}{\sqrt{2x+3}}$$

$$= \frac{3x+3}{\sqrt{2x+3}}$$

Since $\frac{d}{dx}x\sqrt{2x+3} = \frac{3(x+1)}{\sqrt{2x+3}}$, integrating both sides with respect to x, gives:

$$[x\sqrt{2x+3}]_0^1 = 3\int_0^1 \frac{x+1}{\sqrt{2x+3}}\,dx$$

$$\left[\frac{1}{3}x\sqrt{2x+3}\right]_0^1 = \int_0^1 \frac{x+1}{\sqrt{2x+3}}\,dx$$

$$\Rightarrow \frac{1}{3}\sqrt{5} = \int_0^1 \frac{x+1}{\sqrt{2x+3}}\,dx$$

Hence, $\int_0^1 \frac{x+1}{\sqrt{2x+3}}\,dx = \frac{\sqrt{5}}{3}$

EXERCISE 15C

1 Given that $\int_1^4 f(x)\,dx = 8$, calculate:

(a) $\int_4^1 f(x)\,dx$ (b) $\int_1^4 5f(x)\,dx$ (c) $\int_1^3 \{f(x) + 4x\}\,dx + \int_3^4 f(x)\,dx$

2 Given that $\int_1^6 g(x)\,dx = 12$, calculate:

(a) $2\int_1^6 g(x)\,dx$ (b) $\int_1^6 \{3g(x) + 5\}\,dx$ (c) $\int_6^1 g(x)\,dx$

(d) Find the value of k for which $\int_1^6 \{g(x) + kx\} = 47$.

3 Given that $\int_0^4 f(x)\,dx = 10$ and $\int_0^4 g(x)\,dx = 6$, state which of the following integrals cannot be evaluated, and evaluate the others.

(a) $\int_0^4 \{f(x) + 3g(x)\}\,dx$

(b) $\int_0^4 f(x)\,f(x)\,dx$

(c) $\int_0^4 f(x)\,dx + \int_4^0 g(x)\,dx$

(d) $\int_0^4 2g(x) + 3\,dx$

(e) $\int_0^5 g(x)\,dx$

4 Given that $y = x(1 + x^2)^{\frac{1}{2}}$, show that $\frac{dy}{dx} = \frac{1 + 2x^2}{(1 + x^2)^{\frac{1}{2}}}$. Hence, find $\int_0^1 \frac{3 + 6x^2}{(1 + x^2)^{\frac{1}{2}}}\,dx$.

5 Given that $y = (1 + 4x)^{\frac{3}{2}}$, show that $\frac{dy}{dx} = 6(1 + 4x)^{\frac{1}{2}}$. Hence, find $\int_0^1 (1 + 4x)^{\frac{1}{2}}\,dx$.

6 Given that $y = \frac{\sin x}{1 + \cos x}$, show that $\frac{dy}{dx} = \frac{1}{1 + \cos x}$. Hence, show that $\int_0^{\frac{\pi}{4}} \frac{1}{1 + \cos x}\,dx = \sqrt{2} - 1$.

7 Given that $y = \frac{x + 1}{2 - 3x}$, show that $\frac{dy}{dx} = \frac{5}{(2 - 3x)^2}$. Hence, find $\int_0^{\frac{1}{2}} \frac{1}{(2 - 3x)^2}\,dx$.

8 Evaluate $\int_0^{\frac{\pi}{4}} 2\sin 5\theta \cos\theta\,d\theta$.

9 Find the value of $\int_0^{\frac{\pi}{6}} 2\sin 7\theta \cos 3\theta\,d\theta$.

10 Evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 6\theta \cos 2\theta\,d\theta$.

11 Evaluate $\int_0^{\frac{\pi}{2}} 2\sin 5\theta \sin\theta\,d\theta$.

12 Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\sin 3\theta \sin 5\theta\,d\theta$.

13 Evaluate $\int_0^{\frac{\pi}{12}} \cos 7\theta \cos 3\theta\,d\theta$.

In questions **14** to **19**, evaluate the definite integrals.

14 $\int_0^{\frac{\pi}{2}} \sin x\,dx$

15 $\int_0^{\frac{3\pi}{2}} \sin 3x\,dx$

16 $\int_0^{\frac{\pi}{4}} (1 + \cos x)\,dx$

17 $\int_0^{\frac{\pi}{4}} \cos 2x\,dx$

18 $\int_0^{\frac{\pi}{2}} \cos 4x\,dx$

19 $\int_0^{\frac{\pi}{2}} \cos\left(7x + \frac{\pi}{2}\right)dx$

In questions **20** to **30**, find the exact value of the definite integral.

20 $\int_0^{\frac{\pi}{2}} \sin^2 2x\,dx$

21 $\int_0^{\frac{\pi}{4}} \cos^2 4x\,dx$

22 $\int_0^{\frac{\pi}{2}} \cos^2 6x\,dx$

23 $\int_0^{\frac{\pi}{2}} \sin^2 \frac{x}{2}\,dx$

24 $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^2 2x\,dx$

25 Evaluate these.

(a) $\int_1^4 4x - 6\sqrt{x}\,dx$

(b) $\int_1^4 2\sqrt{x} - \frac{4}{\sqrt{x}}\,dx$

26 Evaluate these.

(a) $\int_{-1}^0 \frac{4}{\sqrt{1-3x}}\,dx$

(b) $\int_0^2 \sqrt{1+4x}\,dx$

27 Show that $\cos 3x = 4\cos^3 x - 3\cos x$. Hence, find the value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^3 x\,dx$.

28 Evaluate these.

(a) $\int_1^5 \frac{1}{\sqrt{3x+1}}\,dx$

(b) $\int_0^{\frac{\pi}{2}} \sin 3x - \cos 2x\,dx$

29 Evaluate these.

(a) $\int_1^9 \frac{1}{\sqrt[3]{1+7x}}\,dx$

(b) $\int_4^{15} (x-3)^{\frac{-3}{2}}\,dx$

30 Show that $\sin 3x = 3\sin x - 4\sin^3 x$. Hence, evaluate $\int_0^{\frac{\pi}{4}} 4\sin^3 x\,dx$.

Integration by substitution

The method of substitution is used to reduce an integral to a standard form. When using substitution we change from one variable (x say) to a new variable (u) in the following manner:

$\int (fg(x))\,dx$

(i) Let $u = g(x)$.

(ii) Find $\frac{du}{dx}$ and replace dx in the integral by a function of u.

(iii) Change $fg(x)$ to $f(u)$.

EXAMPLE 69 Find $\int x\sqrt{2x+1}\,dx$, using the substitution $u = 2x + 1$.

SOLUTION Let $u = 2x + 1$.

Differentiating with respect to x:

$\frac{du}{dx} = 2$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

After finding dx in terms of u, we next replace the function $x\sqrt{2x+1}$ by a function of u.

Now $u = 2x + 1$

$\Rightarrow \sqrt{u} = \sqrt{2x + 1}$

Since $u = 2x + 1$, making x the subject of the formula:

$2x = u - 1$

$x = \dfrac{u - 1}{2}$

We now have:

$\mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$

$x = \dfrac{u - 1}{2} = \dfrac{1}{2}u - \dfrac{1}{2}$

$\sqrt{2x + 1} = \sqrt{u}$

$$\text{We have} \int x\sqrt{2x + 1}\,\mathrm{d}x = \int \left(\frac{1}{2}u - \frac{1}{2}\right)\sqrt{u}\left(\frac{1}{2}\right)\mathrm{d}u$$

$$= \frac{1}{2}\int \left(\frac{1}{2}u - \frac{1}{2}\right)u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= \frac{1}{2}\int \left(\frac{1}{2}u^{\frac{5}{2}} - \frac{1}{2}u^{\frac{1}{2}}\right)\mathrm{d}u$$

$$= \frac{1}{2}\left[\frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{5}{2}} - \frac{1}{2}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$

$$= \frac{1}{2}\left[\frac{1}{5}u^{\frac{5}{2}} - \frac{1}{3}u^{\frac{3}{2}}\right] + c$$

We now use $u = (2x + 1)$ and get:

$$\int x\sqrt{2x + 1}\,\mathrm{d}x = \frac{1}{2}\left[\frac{1}{5}(2x + 1)^{\frac{5}{2}} - \frac{1}{3}(2x + 1)^{\frac{3}{2}}\right] + c$$

$$= \frac{1}{10}(2x + 1)^{\frac{5}{2}} - \frac{1}{6}(2x + 1)^{\frac{3}{2}} + c$$

EXAMPLE 70 Find $\int 3x\sqrt{1 - 2x^2}\,\mathrm{d}x$, using the substitution $u = 1 - 2x^2$.

SOLUTION $u = 1 - 2x^2$

Differentiating with respect to x:

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = -4x$

$\mathrm{d}u = -4x\,\mathrm{d}x$

$-\dfrac{1}{4}\,\mathrm{d}u = x\,\mathrm{d}x$

Notice the integral contains $x\,\mathrm{d}x$ so we can use:

$-\dfrac{1}{4}\mathrm{d}u = x\,\mathrm{d}x$

Since $u = 1 - 2x^2$, we have $\sqrt{1 - 2x^2} = \sqrt{u}$.

Now that we have converted all our function of x to a function of u, we replace in the integral:

$x\,\mathrm{d}x = -\dfrac{1}{4}\mathrm{d}u$

$\sqrt{1 - 2x^2} = \sqrt{u}$

We get:

$$\int 3x\sqrt{1-2x^2}\,\mathrm{d}x = \int 3\sqrt{u}\left(-\frac{1}{4}\right)\mathrm{d}u$$

$$= -\frac{3}{4}\int u^{\frac{1}{2}}\,\mathrm{d}u$$

$$= -\frac{3}{4}\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$$

$$= -\frac{1}{2}u^{\frac{3}{2}} + c$$

Substituting $u = 1 - 2x^2$ we have:

$$\int 3x\sqrt{1-2x^2\mathrm{d}x} = -\frac{1}{2}(1-2x^2)^{\frac{3}{2}} + c$$

EXAMPLE 71 Find $\int x\sin(x^2)\,\mathrm{d}x$ by using $u = x^2$.

SOLUTION $u = x^2$

Differentiating with respect to x:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$\mathrm{d}u = 2x\,\mathrm{d}x$$

$$\frac{1}{2}\mathrm{d}u = x\,\mathrm{d}x$$

We have $u = x^2$

Therefore, $\sin x^2 = \sin u$

$$\Rightarrow \int x\sin(x^2)\,\mathrm{d}x = \int \frac{1}{2}\sin u\,\mathrm{d}u$$

$$= -\frac{1}{2}\cos u + c$$

Substituting $u = x^2$ gives:

$$\int x\sin(x^2)\,\mathrm{d}x = -\frac{1}{2}\cos x^2 + c$$

EXAMPLE 72 Show that $\int \sin^3\theta\cos\theta\,\mathrm{d}\theta = \frac{\sin^4\theta}{4} + c$ by using the substitution $u = \sin\theta$.

SOLUTION $\int \sin^3\theta\cos\theta\,\mathrm{d}\theta$

Let $u = \sin\theta$

Differentiating with respect to θ.

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = \cos\theta$$

$$\Rightarrow \mathrm{d}u = \cos\theta\,\mathrm{d}\theta$$

Since $u = \sin\theta$

$$\Rightarrow u^3 = \sin^3\theta$$

Use $\mathrm{d}u = \cos\theta\,\mathrm{d}\theta$

$u^3 = \sin^3\theta$

We have $\int \sin^3\theta \cos\theta \, d\theta = \int u^3 \, du$

$$= \frac{u^4}{4} + c$$

Substituting $u = \sin\theta$

$\Rightarrow \int \sin^3\theta \cos\theta \, d\theta = \frac{1}{4}(\sin^4\theta) + c$

EXAMPLE 73 Show that $\int \tan^4\theta \sec^2\theta \, d\theta = \frac{\tan^5\theta}{5} + c$ by using $t = \tan\theta$.

SOLUTION $t = \tan\theta$

Differentiating with respect to θ:

$\frac{dt}{d\theta} = \sec^2\theta$

$dt = \sec^2\theta \, d\theta$

Since $t = \tan\theta$

$\Rightarrow t^4 = \tan^4\theta$

Substituting $dt = \sec^2\theta \, d\theta$ and $t^4 = \tan^4\theta$

We have $\int \tan^4\theta \sec^2\theta \, d\theta = \int t^4 \, dt$

$$= \frac{t^5}{5} + c$$

Replace $t = \tan\theta$.

Therefore, $\int \tan^4\theta \sec^2\theta \, d\theta = \frac{1}{5}\tan^5\theta + c$

Substituting with limits

When the integral is a definite integral, we change our limits from limits of x to limits of u using the given substitution.

EXAMPLE 74 Evaluate $\int_0^1 \frac{x}{(2x + 1)^3} \, dx$ by using the substitution $u = 2x + 1$.

SOLUTION $u = 2x + 1$

Differentiate with respect to x:

$\frac{du}{dx} = 2$

$du = 2\,dx$

$\frac{1}{2}\,du = dx$

Next change our limits, by using $u = 2x + 1$.

When $x = 0$, $\quad u = 2(0) + 1 = 1$

When $x = 1$, $\quad u = 2(1) + 1 = 3$

Changing the functions of x:

$(2x + 1)^3 = u^3$

Since $u = 2x + 1 \Rightarrow u - 1 = 2x$

We have:

$\Rightarrow x = \dfrac{u-1}{2}$

$\Rightarrow \mathrm{d}x = \dfrac{1}{2}\,\mathrm{d}u$

When $x = 0$, $u = 1$

When $x = 1$, $u = 3$

$(2x + 1)^3 = u^3$

$x = \dfrac{u-1}{2} = \dfrac{1}{2}u - \dfrac{1}{2}$

Substituting into our given integral:

$$\int_0^1 \frac{x}{(2x+1)^3}\,\mathrm{d}x = \int_1^3 \frac{\frac{1}{2}u - \frac{1}{2}}{u^3}\left(\frac{1}{2}\right)\mathrm{d}u$$

$$= \frac{1}{2}\int_1^3 \left(\frac{1}{2}u - \frac{1}{2}\right)u^{-3}\,\mathrm{d}u$$

$$= \frac{1}{2}\int_1^3 \left(\frac{1}{2}u^{-2} - \frac{1}{2}u^{-3}\right)\mathrm{d}u$$

$$= \frac{1}{4}\int_1^3 (u^{-2} - u^{-3})\,\mathrm{d}u$$

$$= \frac{1}{4}\left[-u^{-1} - \frac{u^{-2}}{-2}\right]_1^3$$

$$= \frac{1}{4}\left[-\frac{1}{u} + \frac{1}{2u^2}\right]_1^3$$

$$= \frac{1}{4}\left[\left(-\frac{1}{3} + \frac{1}{2(3^2)}\right) - \left(-\frac{1}{1} + \frac{1}{2(1)^2}\right)\right]$$

$$= \frac{1}{4}\left[\left(-\frac{1}{3} + \frac{1}{18}\right) - \left(-1 + \frac{1}{2}\right)\right]$$

$$= \frac{1}{4}\left[-\frac{1}{3} + \frac{1}{18} + 1 - \frac{1}{2}\right]$$

$$= \frac{1}{4}\left(\frac{4}{18}\right) = \frac{1}{18}$$

Hence, $\displaystyle\int_0^1 \frac{x}{(2x+1)^3}\,\mathrm{d}x = \frac{1}{18}$

EXAMPLE 75 Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^3 x \cos x\,\mathrm{d}x$ by using the substitution $u = \sin x$.

SOLUTION Since $u = \sin x$

$\Rightarrow \mathrm{d}u = \cos x\,\mathrm{d}x$

Next we change the limits.

When $x = 0$, $u = \sin 0 = 0$

When $x = \dfrac{\pi}{2}$, $\quad u = \sin\left(\dfrac{\pi}{2}\right) = 1$

$u = \sin x \Rightarrow u^3 = \sin^3 x$

Replace $\mathrm{d}u = \cos x\,\mathrm{d}x$ into our integral:

$x = 0$, $u = 0$

$x = \dfrac{\pi}{2}$, $u = 1$

$u^3 = \sin^3 x$

We have:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos x \, dx = \int_0^1 u^3 \, du$$
$$= \left[\frac{1}{4}u^4\right]_0^1$$
$$= \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4$$
$$= \frac{1}{4}$$

EXAMPLE 76 Using the substitution $u = x + 1$, evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} dx$.

SOLUTION $u = x + 1$

Differentiating with respect to x:

$du = dx$

Next we change the limits.

When $x = 0$, $u = 0 + 1 = 1$

When $x = 3$, $u = 3 + 1 = 4$

Since $u = x + 1 \Rightarrow x = u - 1$

$\sqrt{x+1} = \sqrt{u}$

$$\therefore \int_0^3 \frac{x}{\sqrt{x+1}} dx = \int_1^4 \frac{u-1}{\sqrt{u}} du$$
$$= \int_1^4 (u-1)u^{-\frac{1}{2}} du$$
$$= \int_1^4 u^{\frac{1}{2}} - u^{-\frac{1}{2}} du$$
$$= \left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^4$$
$$= \left[\frac{2}{3}(4)^{\frac{3}{2}} - 2(4)^{\frac{1}{2}}\right] - \left[\frac{2}{3} - 2\right]$$
$$= \left(\frac{16}{3} - 4\right) - \left(\frac{2}{3} - 2\right)$$
$$= \frac{8}{3}$$

EXERCISE 15D

1 Find $\int x^2(x^3 + 1) \, dx$, using the substitution $u = x^3 + 1$.

2 Find $\int \frac{x+1}{(6x^2 + 12x + 5)^4} dx$, using the substitution $u = 6x^2 + 12x + 5$.

3 Using the substitution, $u = \cos x$ find $\int \cos^4 x \sin x \, dx$.

4 Find $\int \cos 4x \sin^3 4x \, dx$, using $u = \sin 4x$.

5 Using the substitution $u = 3x + 1$, find $\int \frac{6x}{\sqrt{3x+1}} dx$.

6 Evaluate $\int_0^1 \frac{x}{(7x+2)^3} dx$, using the substitution $u = 7x + 2$.

7 Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$, using the substitution $u = \tan x$.

8 By using the substitution $u = \sin x$, show that $\int_0^{\frac{\pi}{4}} 2 \sin^4 x \cos x \, dx = \frac{\sqrt{2}}{20}$.

9 Using the substitution $u = x + 1$, show that $\int_0^1 \frac{x+2}{\sqrt{1+x}} \, dx = \frac{10}{3}\sqrt{2} - \frac{8}{3}$.

10 Using the substitution $u = \sin x$, show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 x \cos x \, dx = \frac{19}{24(3\sqrt{3} + 2\sqrt{2})}$.

11 Find these.

(a) $\int x \cos x^2 \, dx$, using $u = x^2$

(b) $\int x^2 \sin (x^3) \, dx$, using $u = x^3$

(c) $\int \frac{2 - 3x}{(4 + x)^4} \, dx$, using $u = 4 + x$

The equation of a curve

Given the gradient function of a curve $\frac{dy}{dx}$, we can find the equation of the curve by integration. The constant of integration will depend on the points on the curve. Given a point on the curve, we can find the constant of integration by substituting the point into the equation found by integration.

EXAMPLE 77 A curve is such that $\frac{dy}{dx} = 4x^3 - 4x + 2$. Given that the curve passes through the point (1, 3), find the equation of the curve.

SOLUTION Since $\frac{dy}{dx} = 4x^3 - 4x + 2$, integrating both sides:

$y = \int 4x^3 - 4x + 2 \, dx$

$y = \frac{4x^4}{4} - \frac{4x^2}{2} + 2x + C$

Therefore, $y = x^4 - 2x^2 + 2x + C$

We can find C by substituting $x = 1$, $y = 3$.

Hence, $3 = (1)^4 - 2(1)^2 + 2(1) + C$

$\Rightarrow 3 = 1 + C$, $C = 2$

Therefore, the equation of the curve is $y = x^4 - 2x^2 + 2x + 2$.

EXAMPLE 78 Given that $\frac{dy}{dx} = 4x - 6$ and that $y = 3$ when $x = 1$, express y in terms of x.

SOLUTION Since $\frac{dy}{dx} = 4x - 6$, integrating both sides of the equation:

$y = \frac{4x^2}{2} - 6x + C = 2x^2 - 6x + C$

Substituting $x = 1, y = 3$:

$3 = 2(1)^2 - 6(1) + C$

$3 = -4 + C$

$C = 7$

Hence, $y = 2x^2 - 6x + 7$.

EXAMPLE 79 A curve, for which $\frac{dy}{dx} = \frac{3}{\sqrt{3x+6}}$, passes through the point (1, 4). Find the equation of the curve.

SOLUTION Since $\frac{dy}{dx} = \frac{3}{\sqrt{3x+6}}$, integrating both sides:

Recall

$\int (ax+b)^n\, dx$

$= \frac{1}{a}\left[\frac{(ax+b)^{n+1}}{n+1}\right] + C$

$n \neq -1$

$$y = \int \frac{3}{\sqrt{3x+6}}\, dx$$

$$\therefore\ y = \int 3(3x+6)^{\frac{-1}{2}}\, dx$$

$$y = \frac{3}{3}\left(\frac{(3x+6)^{\frac{1}{2}}}{\frac{1}{2}}\right) + C$$

$y = 2(3x+6)^{\frac{1}{2}} + C$

When $x = 1, y = 4$

$\therefore\ 4 = 2(9)^{\frac{1}{2}} + C$

$4 = 6 + C, C = -2$

Hence, $y = 2(3x+6)^{\frac{1}{2}} - 2$.

EXERCISE 15E

1 A curve is such that $\frac{dy}{dx} = 6x^2 - 4$. Given that it passes through (1, 2), find its equation.

2 A curve is such that $\frac{dy}{dx} = px - 5$, where p is a constant. Given that the normal at the point (−3, 2) on the curve is $\frac{-1}{4}$.

(a) Find the value of p.

(b) Find the equation of the curve.

3 A curve is such that $\frac{dy}{dx} = \frac{6}{\sqrt{4x+1}}$. Given that the curve passes through (2, 6) and (6, h), find the value of h.

4 A curve is such that $\frac{dy}{dx} = 3x^2 + \frac{2}{x^2}$. Given that the curve passes through (1, 4), find the equation of the curve.

5 The curve for which $\frac{dy}{dx} = \sqrt{1 + 8x}$ passes through the point (3, 8). Find the equation of the curve.

6 A curve is such that $\frac{dy}{dx} = (4x - 6)^3$. Given that the curve passes through (1, 2), find its equation.

7 The gradient of a curve at any point (x, y) on the curve is given by $1 - \frac{4}{x^2}$. Given that the curve passes through the point (4, 6), find its equation.

8 A curve C passes through the point (4, 5) and has gradient $4x^3 - 2x + 1$ at any point (x, y) on C. Find the equation of C.

9 The function $f(x)$ is such that $f'(x) = 3x^3 + 6x^2 - 2x + k$, where k is a constant. Given that $f(0) = 4$ and $f(1) = -2$, find the function $f(x)$.

10 Find the equation of the curve which passes through the point (2, −2) and for which $\frac{dy}{dx} = x^3(2x + 3)$.

SUMMARY

Integration

Integration is the reverse of differentiation

$\int cf(x)\,dx = c\int f(x)\,dx$

$\int f(x) \pm g(x)\,dx = \int f(x)\,dx \pm g(x)\,dx$

$\int_a^b f(x)\,dx = F(b) - F(a)$

$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

$\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx, a > 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Some standard results

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

$\int (ax + b)^n\,dx = \frac{1}{a}\frac{(ax + b)^{n+1}}{n+1} + c, n \neq -1$

$\int \sin x\,dx = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$\int \sin^n x \cos x\,dx = \frac{\sin^{n+1} x}{n+1} + c, n \neq -1$

$\int \cos^n x \sin x\,dx = -\frac{\cos^{n+1} x}{n+1} + c, n \neq -1$

$\int \tan^n x \sec^2 x\,dx = \frac{\tan^{n+1} x}{n+1} + c, n \neq -1$

$\int \sin^2 x\,dx = \frac{1}{2}\int 1 - \cos 2x\,dx$

$\int \cos^2 x\,dx = \frac{1}{2}\int 1 + \cos 2x\,dx$

$\int \tan^2 x\,dx = \int \sec^2 x - 1\,dx$

Checklist

Can you do these?

- Define integration as the reverse of differentiation.
- Understand and use $\int f(x)\,dx$.
- Show that the indefinite integral represents a family of functions.
- Know and use the integration theorems:
  a $\int cf(x)\,dx$
  b $\int [f(x) \mp g(x)]\,dx = \int f(x)\,dx \mp \int g(x)\,dx$.
- Find integrals using the integration theorems.
- Find integrals of polynomials.
- Find integrals of trigonometric functions.
- Find integrals by substitution.
- Find definite integrals.
- Know and use $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.
- Know and use $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, $a > 0$.

Review Exercise 15

1 Given that $\int_1^9 f(x)\,dx = 42$, find these.

(a) $\int_1^9 6f(x)\,dx$

(b) $\int_1^9 \{f(x) - 4\}\,dx$

2 Given that $\int_0^3 g(x)\,dx = 12$, evaluate these.

(a) $\int_0^3 5g(x)\,dx$

(b) $\int_0^3 g(x) + 2\,dx$

(c) $\int_3^0 g(x) + x\,dx$

3 Evaluate $\int_1^2 \frac{t^3 + 4t^2}{t^6}\,dt$.

4 Find $\int_0^1 (x+1)(2x-3)\,dx$.

5 Find these.

(a) $\int \sqrt{4x-1}\,dx$

(b) $\int \frac{4}{\sqrt{2-3t}}\,dt$

6 Find the exact value of $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 2x\,dx$.

7 Find these.

(a) $\int \sin 6x \cos 4x\,dx$ (b) $\int \sin 8x \sin 4x\,dx$

8 Given that $y = \frac{x}{1 + 2x}$, show that $\frac{dy}{dx} = \frac{1}{(1 + 2x)^2}$. Hence, find $\int \frac{1}{(1 + 2x)^2}\,dx$.

9 Show that $\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$. Hence, find $\int \cot^2 2x\,dx$.

10 Show that $\int_0^{\frac{\pi}{2}} (\sin x + \cos x)^2\,dx = \frac{\pi}{2} + 1$.

11 Given that $y = \frac{x}{1 + x^2}$, show that $\frac{dy}{dx} = \frac{1 + x^2}{(1 + x^2)^2}$. Hence, find $\int \frac{4 - 4x^2}{(1 + x^2)^2}\,dx$.

12 Find the derivative of $x \sin x$ with respect to x. Hence, or otherwise, evaluate

$\int_0^{\frac{\pi}{4}} x \cos x\,dx$.

13 (a) Given that $y = \frac{x}{(x^2 + 32)}$, find $\frac{dy}{dx}$.

(b) Hence, find $\int_0^1 \frac{32 - x^2}{(x^2 + 32)^2}\,dx$ to 3 d.p.

14 Given that $f(x) = \frac{2x}{(1 - 3x)}$, find $f'(x)$. Hence, evaluate $\int_0^1 \frac{6}{(1 - 3x)^2}\,dx$.

15 If $y = \frac{x}{(2x^2 + 3)^{\frac{1}{2}}}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{6}{(2x^2 + 3)^{\frac{3}{2}}}\,dx$.

16 If $y = \frac{5x}{3x^2 + 1}$, find $\frac{dy}{dx}$. Hence, evaluate $\int_0^1 \frac{1 - 3x^2}{(3x^2 + 1)^2}\,dx$.

17 The gradient of a curve is $4x^3 - 6x + 2$ and the curve passes through $(1, -5)$.

(a) Determine the equation of the curve.

(b) Find the equation of the tangent to the curve at the point $x = 2$.

18 Evaluate $\int_{\frac{1}{2}}^{\frac{3}{4}} \frac{x}{\sqrt{1 - x^2}}\,dx$, using the substitution $u = 1 - x^2$.

19 Using the substitution $x = 2\sin\theta$, evaluate $\int_0^1 \frac{x^2}{\sqrt{4 - x^2}}\,dx$.

20 A curve is such that $\frac{dy}{dx} = \frac{8}{(4x - 5)^2}$. Given that the curve passes through the point $(2, 6)$, find the coordinates of the point where the curve crosses the x-axis.

21 By using the substitution $u = 1 + \cos x$, show that $\int \frac{\sin^3 x}{(1 + \cos x)^4}\,dx = \int \frac{u - 2}{u^3}\,du$.

Hence, find $\int \frac{\sin^3 x}{(1 + \cos x)^4}\,dx$.

22 Evaluate $\int_0^1 \frac{x - 1}{(3x^2 - 6x + 5)^4}\,dx$, using the substitution $u = 3x^2 - 6x + 5$.

23 Using the substitution $u = \cos\theta$, show that $\int_0^{\frac{\pi}{6}} \frac{\sin\theta}{\cos^3\theta}\,d\theta = \frac{1}{6}$.

24 Using the substitution $u = 1 - x^2$, show that $\int_0^{\frac{1}{2}} \frac{x}{(1 - x^2)^2}\,dx = \frac{1}{6}$.

25 By using the substitution $x = \tan u$ or otherwise, evaluate $\int_0^{\sqrt{3}} \frac{1}{(1 + x^2)^{\frac{3}{2}}}\,dx$.

CHAPTER 16

Applications of Integration

At the end of this chapter you should be able to:

- Estimate the area under a curve using rectangles
- Understand that the limiting sum gives the exact area under the curve
- Use integration to find the area under the curve
- Use integration to find the area between two curves
- Find the volume of a solid formed when a region is rotated about the x-axis
- Find the volume of a solid formed when a region is rotated about the y-axis

KEYWORDS/TERMS

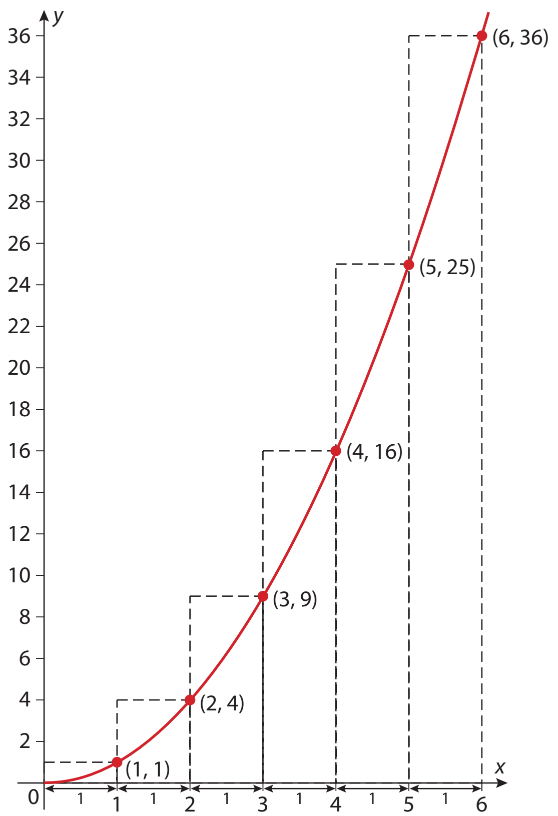
estimate • area • rectangles • limiting sum • integration • volume of solid • rotation • rotation about the x-axis • rotation about the y-axis

Approximating the area under a curve, using rectangles

EXAMPLE 1 Estimate the area under the curve $y = x^2$ bounded by the lines $x = 1$ and $x = 6$ and the x-axis, using rectangles of unit width.

SOLUTION $y = x^2$ is a quadratic curve with a minimum point at (0, 0).

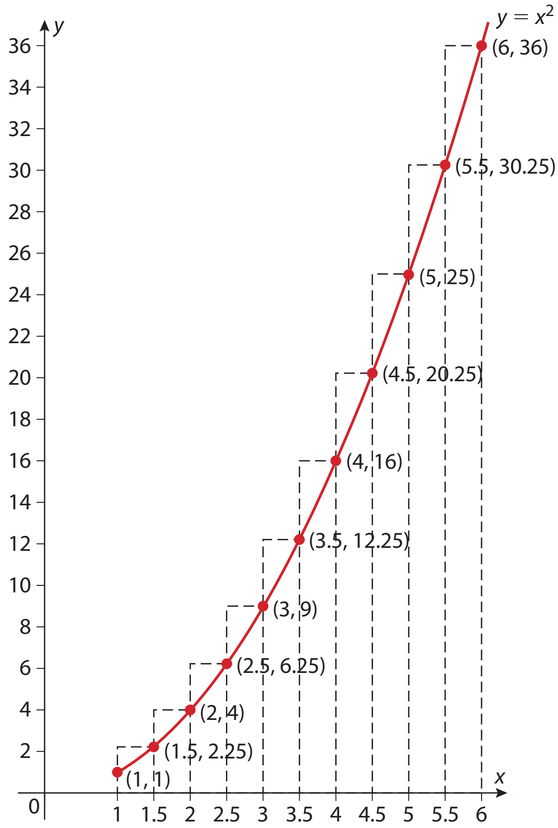
We can approximate this area by forming rectangles of unit width and finding the area of each rectangle and their sum.



$$\text{Area of the rectangles} = (1)(1) + (1)(4) + (1)(9) + (1)(16) + (1)(25) + (1)(36)$$
$$= 1 + 4 + 9 + 16 + 25 + 36 = 91 \text{ square units}$$

Area under the curve $y = x^2$ from $x = 1$ to $x = 6$ is approximately 91 square units.

We can get a better approximation if we increase the number of rectangles as follows: using $\frac{1}{2}$ unit intervals.

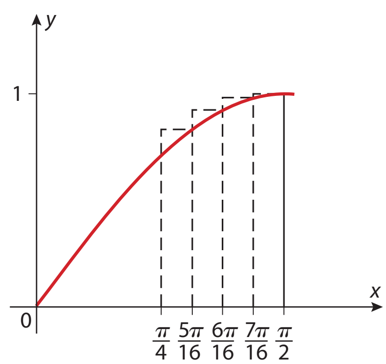


$$\begin{aligned}\text{Area under the curve} &\approx (1)\left(\frac{1}{2}\right) + (1.5)^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{2}\right) + (2.5)^2\left(\frac{1}{2}\right) + 3^2\left(\frac{1}{2}\right)\\ &\quad + (3.5)^2\left(\frac{1}{2}\right) + 4^2\left(\frac{1}{2}\right) + (4.5)^2\left(\frac{1}{2}\right) + 5^2\left(\frac{1}{2}\right) + (5.5)^2\left(\frac{1}{2}\right) + 6^2\left(\frac{1}{2}\right)\\ &= \frac{1}{2} + \frac{9}{8} + 2 + \frac{25}{8} + \frac{9}{2} + \frac{49}{8} + 8 + \frac{81}{8} + \frac{25}{2} + \frac{121}{8} + 18\\ &= 81\frac{1}{8}\end{aligned}$$

As we increase the number of rectangles, the area above the curve will decrease until we get an exact value for the area under the curve. As the number of rectangles increases, the estimate gets better.

EXAMPLE 2

Estimate the area under the curve $y = \sin x$ from $x = \frac{\pi}{4}$ to $x = \frac{\pi}{2}$ by using four rectangles of equal width.

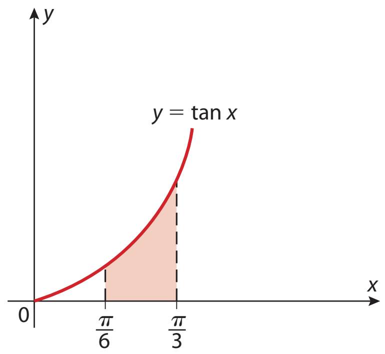


SOLUTION

$$\begin{aligned}\text{Area of the rectangles} &= \left(\sin\frac{5\pi}{16}\right)\left(\frac{\pi}{16}\right) + \left(\sin\frac{6\pi}{16}\right)\left(\frac{\pi}{16}\right)\\ &\quad + \left(\sin\frac{7\pi}{16}\right)\left(\frac{\pi}{16}\right) + \left(\sin\frac{\pi}{2}\right)\left(\frac{\pi}{16}\right)\\ &= \frac{\pi}{16}\left(\sin\frac{5\pi}{16} + \sin\frac{6\pi}{16} + \sin\frac{7\pi}{16} + \sin\frac{\pi}{2}\right)\\ &= \frac{\pi}{16}(3.736\,13)\\ &= 0.733\,59\end{aligned}$$

The area under the curve is approximately 0.73359 square units.

EXAMPLE 3 Use five rectangles of equal width to estimate the shaded area to four decimal places.



SOLUTION

$$\frac{\frac{\pi}{3}-\frac{\pi}{6}}{5}=\frac{\pi}{30}$$

$$\begin{aligned}\text{Area of the rectangles} &= \frac{\pi}{30}\left(\tan\frac{6\pi}{30}+\tan\frac{7\pi}{30}\right.\\ &\quad\left.+\tan\frac{8\pi}{30}+\tan\frac{9\pi}{30}+\tan\frac{\pi}{3}\right)\\ &= 0.6122\end{aligned}$$

The approximate area of the shaded region is 0.6122 square units.

Estimating the area under a curve using *n* rectangles of equal width

We can estimate the area under the curve $y = f(x)$ from $x = a$ to $x = b$ using n rectangles of equal width.

The widths of the rectangles are equal.

Therefore, the width of one rectangle $= \frac{b-a}{n}$.

Let h be the heights of the rectangles.

Sum of the areas of the rectangles

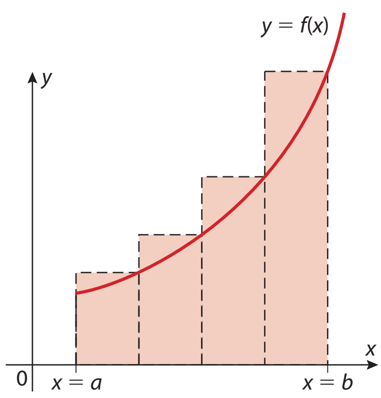
$$\begin{aligned}&= \frac{(b-a)}{n}h_1 + \frac{(b-a)}{n}h_2 + \frac{(b-a)}{n}h_3 + \cdots + \frac{(b-a)}{n}h_a\\ &= \frac{(b-a)}{n}(h_1 + h_2 + h_3 + \cdots + h_a)\\ &= \frac{b-a}{n}\sum_{i=1}^{n} h_i\end{aligned}$$

As the number of rectangles increases the area under the curve can be found exactly.

$$\begin{aligned}\text{Therefore, the area under the curve} &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} h_i\\ &= \lim_{n\to\infty}\left\{\frac{b-a}{n}(f(x_1)+f(x_2)+\ldots+f(x_n))\right\}\end{aligned}$$

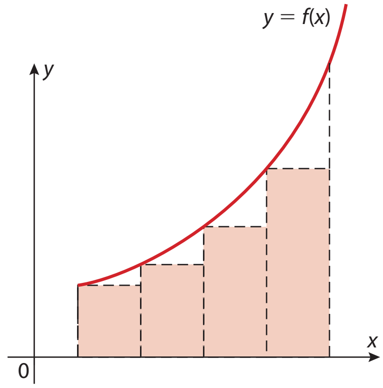
When approximating the area under a curve using rectangles, we can form rectangles with the top left-hand corner of the rectangle on the curve or the top right-hand corner of the rectangle on the curve. We can also use the midpoint of the interval to form the rectangle. The following diagrams will illustrate each case.

Case 1



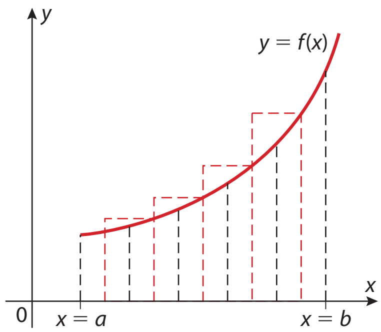
The area under the curve is greater than the area found by using the rectangles. The top right-hand corner of the rectangle is on the curve.

Case 2



In this case the area under the curve is less than the estimate using the area of the rectangles. The height of the rectangle on the top left-hand corner is on the curve.

Case 3



Using the midpoint of the intervals will give the best estimate of the area under the curve.

EXAMPLE 4 Find the exact area of the region bounded by $y = x^2$ and the x-axis and the lines $x = 0$ and $x = 3$.

SOLUTION Dividing the area into n rectangles, the width of each rectangle is $x = \frac{3-0}{n} = \frac{3}{n}$.

x	Y	x	Y
$\frac{3}{n}$	$\left(\frac{3}{n}\right)^2$	$\frac{12}{n}$	$\left(\frac{12}{n}\right)^2$
$\frac{6}{n}$	$\left(\frac{6}{n}\right)^2$	...	...
$\frac{9}{n}$	$\left(\frac{9}{n}\right)^2$	$\frac{3n}{n}$	$\left(\frac{3n}{n}\right)^2$

$$\text{Exact area} = \lim_{n \to \infty} \frac{3}{n}\left(\frac{3^2}{n^2} + \frac{6^2}{n^2} + \frac{9^2}{n^2} + \ldots \frac{9n^2}{n^2}\right)$$

$$= \lim_{n \to \infty} \frac{3}{n}\sum_{r=1}^{n}\left(\frac{3r}{n}\right)^2$$

$$= \lim_{n \to \infty} \frac{3}{n^3} \times \frac{9n(n+1)(2n+1)}{6} \qquad \left(\text{Since the sum of } n \text{ square numbers } = \frac{n(n+1)(2n+1)}{6}\right)$$

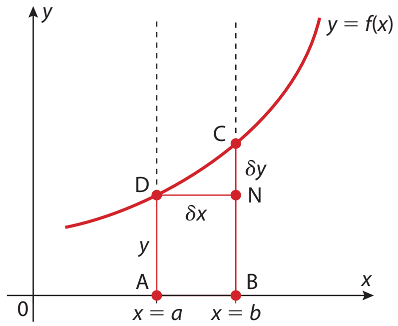
$$= \frac{9}{2}\lim_{n \to \infty} \frac{n(n+1)(2n+1)}{n^3}$$

$$= \frac{9}{2} \lim_{n \to \infty} \frac{2n^2 + 3n + 1}{n^2}$$
$$= \frac{9}{2} \lim_{n \to \infty} 2 + \frac{3}{n} + \frac{1}{n^2}$$
$$= \frac{9}{2} \times 2 = 9$$

Using integration to find the area under a curve

The area of the region bounded by the curve $y = f(x)$, the lines $x = a$, $x = b$ and the x-axis is given by

$$\int_a^b f(x)\,\mathrm{d}x$$



Area of ABCD = area of rectangle ABND + area of triangle CND

Area of rectangle ABND $= \delta x \times y = y\delta x$

Area of triangle CND $= \frac{1}{2} \times \delta x \times \delta y$
$= \frac{1}{2}\,\delta x\,\delta y$

Let area of ABCD $= \delta A$.

$$\delta A = \frac{1}{2}\,\delta x\,\delta y + \delta x y$$

$$\Rightarrow \frac{\delta A}{\delta x} = \frac{1}{2}\,\delta y + y \qquad \text{(Dividing by } \delta x\text{)}$$

$$\lim_{\delta x \to 0} \frac{\delta A}{\delta x} = \frac{\mathrm{d}A}{\mathrm{d}x} = y$$

Hence, $\frac{\mathrm{d}A}{\mathrm{d}x} = y$.

Now the area can be written like this:

$$A = \lim_{\delta x \to 0} \sum_a^b y\,\delta x$$

From $\frac{\mathrm{d}A}{\mathrm{d}x} = y$:

$$A = \int_a^b y\,\mathrm{d}x$$

Therefore, $A = \lim_{\delta x \to 0} y\,\delta x = \int_a^b y\,\mathrm{d}x$.

The limiting value of the sum is equal to the integral between limits.

EXAMPLE 5 Find the area of the region bounded by the curve $y = x^2$ the lines $x = 1$, $x = 2$ and the x-axis.

SOLUTION

Area under the curve $= \int_a^b y\,\mathrm{d}x$

$$= \int_1^2 x^2\,\mathrm{d}x \qquad \text{(Since } y = x^2\text{)}$$
$$= \left[\frac{x^3}{3}\right]_1^2$$
$$= \frac{2^3}{3} - \frac{1^3}{3}$$

$$= \frac{8}{3} - \frac{1}{3}$$
$$= \frac{7}{3} \text{ square units}$$

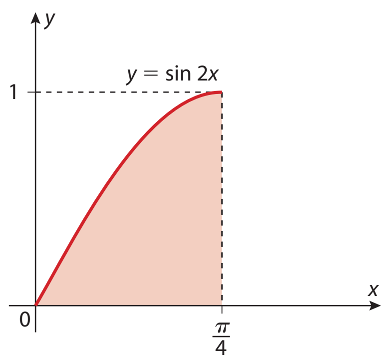
EXAMPLE 6 Find the area of the region bounded by the curve $y = \dfrac{1}{(2x+1)^3}$, the lines $x = 1$ and $x = 3$ and the x-axis.

SOLUTION

$$\text{Area under the curve} = \int_a^b y\,dx$$
$$= \int_1^3 \frac{1}{(2x+1)^3}\,dx$$
$$= \int_1^3 (2x+1)^{-3}\,dx$$
$$= \left[\frac{(2x+1)^{-2}}{2(-2)}\right]_1^3$$
$$= -\frac{1}{4}\left[(2x+1)^{-2}\right]_1^3$$
$$= -\frac{1}{4}\left(\frac{1}{(2(3)+1)^2} - \frac{1}{(2(1)+1)^2}\right)$$
$$= -\frac{1}{4}\left(\frac{1}{49} - \frac{1}{9}\right)$$
$$= \frac{10}{441}$$

Therefore, the area of the region is $\dfrac{10}{441}$ square units.

EXAMPLE 7 The diagram shows part of the curve $y = \sin 2x$. Find the area of the shaded region.



SOLUTION

$$\text{Area} = \int_0^{\frac{\pi}{4}} \sin 2x\,dx$$
$$= \left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{4}}$$
$$= \left(-\frac{1}{2}\cos\frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos 0\right)$$
$$= \frac{1}{2} \text{ square units}$$

Shaded area $= \dfrac{1}{2}$ square units

EXAMPLE 8 Find the area of the region bounded by the curve $y = \cos x - 2\sin x$ and the lines $x = \dfrac{\pi}{6}$ to $x = \dfrac{\pi}{3}$.

SOLUTION

$$\text{Area of the region} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos x - 2\sin x\,dx$$
$$= \left[\sin x + 2\cos x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$
$$= \left(\sin\frac{\pi}{3} + 2\cos\frac{\pi}{3}\right) - \left(\sin\frac{\pi}{6} + 2\cos\frac{\pi}{6}\right)$$
$$= \left(\frac{\sqrt{3}}{2} + 1\right) - \left(\frac{1}{2} + 2\left(\frac{\sqrt{3}}{2}\right)\right)$$

$$= \frac{\sqrt{3}}{2} + 1 - \frac{1}{2} - \sqrt{3}$$

$$= \frac{1}{2} - \frac{\sqrt{3}}{2}$$

$$= \frac{1}{2}(1 - \sqrt{3})$$

Area of the region $= \frac{1}{2}(1 - \sqrt{3})$ square units

Area between two curves

Let $f(x)$ and $g(x)$ be two continuous functions over the interval (a, b). If $f(x)$ is greater than $g(x)$. For any x in (a, b) then the area between $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is give by

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

When finding the area between to curves carry out these steps.

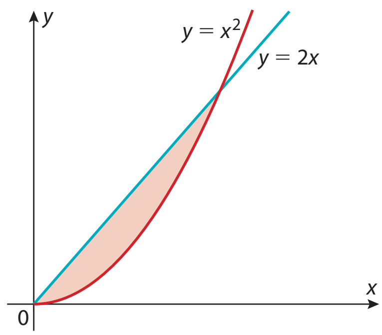
(i) Find the point of intersection of the two curves.

(ii) Find the area under each curve.

(iii) Subtract the two areas. (Larger minus smaller.)

EXAMPLE 9 Calculate the area bounded by the curves $y = 2x$ and $y = x^2$ when $x > 0$.

SOLUTION First solve the equations simultaneously.



$y = 2x$

$y = x^2$

$x^2 = 2x$

$x^2 - 2x = 0$

$x(x - 1) = 0$

$\Rightarrow x = 0, x = 2$

Since the line is above the curve the shaded area:

$$\int_0^2 2x\,dx - \int_0^2 x^2\,dx = [x^2]_0^2 - \left[\frac{x^3}{3}\right]_0^2$$

$$= (2^2 - 0^2) - \left(\frac{2^3}{3} - \frac{0^3}{3}\right)$$

$$= 4 - \frac{8}{3}$$

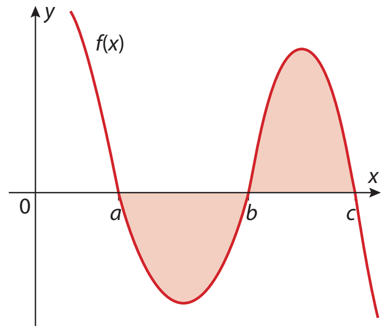
$$= \frac{12}{3} - \frac{8}{3}$$

$$= \frac{4}{3}$$

Hence, the shaded area $= \frac{4}{3}$ square units.

Area below the x-axis

When using integration to find area, the integral of the region below the x-axis is found as negative. The actual area is then given as the positive of the value found by integration. We can find the area below the x-axis as $\left|\int_a^b f(x)\,dx\right|$. To find the area of the shaded region in the graph above you have to divide the region into two parts, from $x = a$ to $x = b$ and $x = b$ to $x = c$, and integrate over each region separately. Therefore, the area under the curve $= \left|\int_a^b f(x)\,dx\right| + \int_b^c f(x)\,dx$.



EXAMPLE 10 Find the area between the curve $y = x^3 - 2x^2 - x + 2$ and the x-axis, for $-2 < x < 3$.

SOLUTION

First we sketch the graph.

We find the point(s) of intersection with the x-axis, when $y = 0$.

$x^3 - 2x^2 - x + 2 = 0$

When $x = 1$, $1^3 - 2(1)^2 - 1 + 2 = 1 - 2 - 1 + 2 = 0$

Therefore, $x - 1$ is a factor.

$$
\begin{array}{r}
x^2 - x - 2 \\
x - 1 \overline{)\, x^3 - 2x^2 - x + 2} \\
\underline{x^3 - x^2 } \\
-x^2 - x \\
\underline{-x^2 + x } \\
-2x + 2 \\
\underline{-2x + 2} \\
0
\end{array}
$$

$\therefore x^3 - 2x^2 - x + 2 = (x - 1)(x^2 - x - 2)$

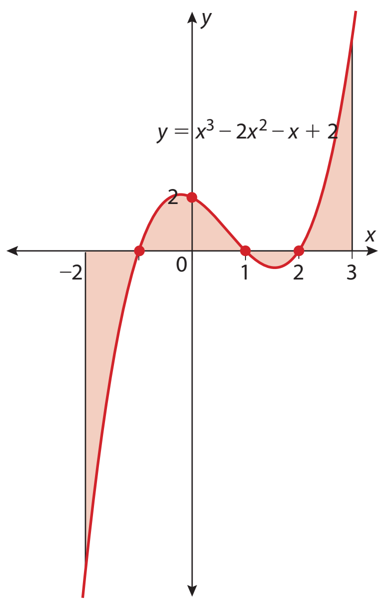
$= (x - 1)(x - 2)(x + 1)$

$\Rightarrow (x - 1)(x - 2)(x + 1) = 0$

$\Rightarrow x = 1, 2, -1$ when $y = 0$

When $x = 0$, $y = 2$

We can sketch the graph.



From the graph we can divide the area into four parts.

$$\text{Area} = \left|\int_{-2}^{-1} f(x)\,dx\right| + \int_{-1}^{1} f(x)\,dx + \left|\int_{1}^{2} f(x)\,dx\right| + \int_{2}^{3} f(x)\,dx$$

$$\int x^3 - 2x^2 - x + 2\,dx = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x + c$$

$$\left|\int_{-2}^{-1} x^3 - 2x^2 - x + 2\,dx\right| = \left|\left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{-1}\right|$$

$$= \left|\left(\frac{(-1)^4}{4} - 2\frac{(-1)^3}{3} - \frac{1}{2}(-1)^2 + 2(-1)\right) - \left(\frac{(-2)^4}{4} - 2\frac{(-2)^3}{3} - \frac{1}{2}(-2)^2 + 2(-2)\right)\right|$$

$$= \left|\left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right) - \left(4 + \frac{16}{3} - 2 - 4\right)\right|$$

$$= \frac{59}{12}$$

$$\int_{-1}^{1}(x^3 - 2x^2 - x + 2)\,\mathrm{d}x = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{-1}^{1}$$

$$= \left(\frac{(1)^4}{4} - 2\frac{(1)^3}{3} - \frac{1}{2}(1)^2 + 2(1)\right) - \left(\frac{(-1)^4}{4} - 2\frac{(-1)^3}{3} - \frac{1}{2}(-1)^2 + 2(-1)\right)$$

$$= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right)$$

$$= \frac{8}{3}$$

$$\left|\int_{1}^{2}(x^3 - 2x^2 - x + 2)\,\mathrm{d}x\right| = \left|\left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{1}^{2}\right|$$

$$= \left|\left(\frac{(2)^4}{4} - 2\frac{(2)^3}{3} - \frac{1}{2}(2)^2 + 2(2)\right) - \left(\frac{(1)^4}{4} - 2\frac{(1)^3}{3} - \frac{1}{2}(1)^2 + 2(1)\right)\right|$$

$$= \left|\left(4 - \frac{16}{3} - 2 + 4\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)\right|$$

$$= \left|-\frac{5}{12}\right| = \frac{5}{12}$$

$$\int_{2}^{3}(x^3 - 2x^2 - x + 2)\,\mathrm{d}x = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right]_{2}^{3}$$

$$= \left(\frac{(3)^4}{4} - 2\frac{(3)^3}{3} - \frac{1}{2}(3)^2 + 2(3)\right) - \left(\frac{(2)^4}{4} - 2\frac{(2)^3}{3} - \frac{1}{2}(2)^2 + 2(2)\right)$$

$$= \left(\frac{81}{4} - 18 - \frac{9}{2} + 6\right) - \left(4 - \frac{16}{3} - 2 + 4\right)$$

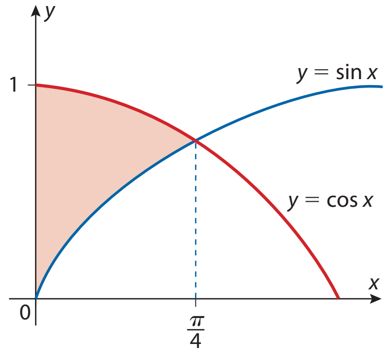
$$= \frac{37}{12}$$

$$\text{Total area} = \frac{59}{12} + \frac{8}{3} + \frac{5}{12} + \frac{37}{12}$$

$$= \frac{133}{12}$$

$$= 11\frac{1}{12} \text{ square units}$$

EXAMPLE 11 Find the area of the shaded region bounded by the curve $y = \sin x$, $y = \cos x$ and the line $x = 0$.



SOLUTION The curves intersect when $\sin x = \cos x$.

$\Rightarrow \tan x = 1$

$\Rightarrow x = \frac{\pi}{4}$

Therefore, the curves intersect at $\frac{\pi}{4}$.

Shaded area = (area under the curve $y = \cos x$) − (area under the curve $y = \sin x$) from 0 to $\frac{\pi}{4}$.

Area under the curve $y = \cos x$ is given by $\int_0^{\frac{\pi}{4}} \cos x \, dx$

$$\int_0^{\frac{\pi}{4}} \cos x \, dx = [\sin x]_0^{\frac{\pi}{4}}$$
$$= \sin \frac{\pi}{4} - \sin 0$$
$$= \frac{\sqrt{2}}{2}$$

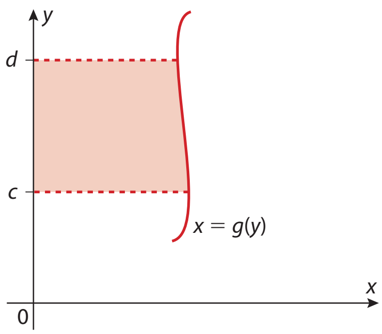
Area under the curve $y = \sin x$ is given by $\int_0^{\frac{\pi}{4}} \sin x \, dx$

$$\int_0^{\frac{\pi}{4}} \sin x \, dx = [-\cos x]_0^{\frac{\pi}{4}}$$
$$= \left(-\cos \frac{\pi}{4}\right) + (\cos 0)$$
$$= 1 - \frac{\sqrt{2}}{2}$$

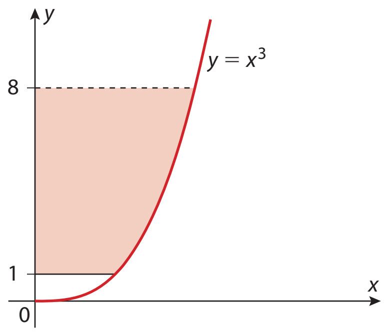
$$\text{Shaded area} = \frac{\sqrt{2}}{2} - \left(1 - \frac{\sqrt{2}}{2}\right)$$
$$= (\sqrt{2} - 1) \text{ square unit}$$

Area between the curve and the y-axis

The area A bounded by the curve $x = g(y)$, the y-axis and the lines $y = c$ and $y = d$ is given by $\int_c^d x \, dy$.



EXAMPLE 12 Find the area of the shaded region.



SOLUTION

$$A = \int_c^d x\,dy$$

Since $y = x^3$, making x the subject of the formula gives:

$$x = \sqrt[3]{y} = y^{\frac{1}{3}}$$

$$\therefore A = \int_1^8 y^{\frac{1}{3}}\,dy$$

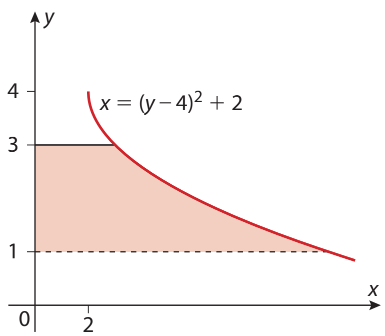
$$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_1^8$$

$$= \frac{3}{4}\left(8^{\frac{4}{3}} - 1\right)$$

$$= \frac{3}{4}(16 - 1)$$

$$= \frac{45}{4} \text{ square units}$$

EXAMPLE 13 Find the area of the shaded region shown in the diagram.



SOLUTION

$$A = \int_c^d x\,dy$$

Therefore, $A = \int_1^3 [(y-4)^2 + 2]\,dy$

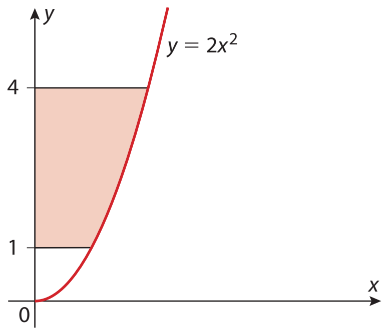
$$= \left[\frac{1}{3}(y-4)^3 + 2y\right]_1^3$$

$$= \left(\frac{1}{3}(3-4)^3 + 2(3)\right) - \left(\frac{1}{3}(1-4)^3 + 2(1)\right)$$

$$= -\frac{1}{3} + 6 + 9 - 2$$

$$= \frac{38}{3} \text{ square units}$$

EXAMPLE 14 Find the area of the shaded region shown in the diagram.



SOLUTION

$$A = \int_c^d x\,\mathrm{d}y$$

Since $y = 2x^2$,

$$x^2 = \frac{y}{2}$$

$$x = \sqrt{\frac{y}{2}} = \frac{1}{\sqrt{2}}y^{\frac{1}{2}}$$

$$A = \int_1^4 \frac{1}{\sqrt{2}}y^{\frac{1}{2}}\,\mathrm{d}y$$

$$= \frac{1}{\sqrt{2}}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_1^4$$

$$= \frac{2}{3\sqrt{2}}\left(4^{\frac{3}{2}} - 1\right)$$

$$= \frac{2}{3\sqrt{2}} \times 7$$

$$= \frac{14}{3\sqrt{2}}$$

$$= \frac{14\sqrt{2}}{6}$$

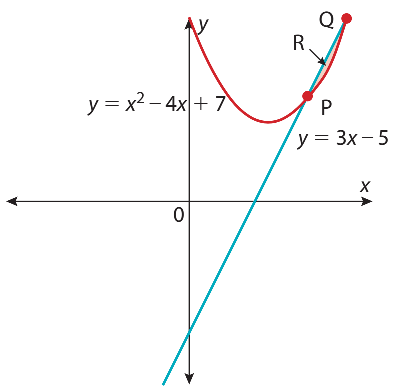
$$= \frac{7\sqrt{2}}{3} \text{ square units}$$

EXERCISE 16A

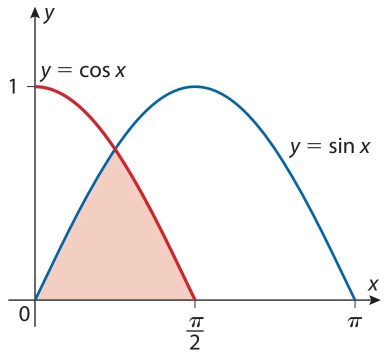
1 The diagram shows part of the curve $y = x^2 - 4x + 7$ and part of the line $y = 3x - 5$. Find the following.

(a) The coordinates of P and Q

(b) The area of the region R



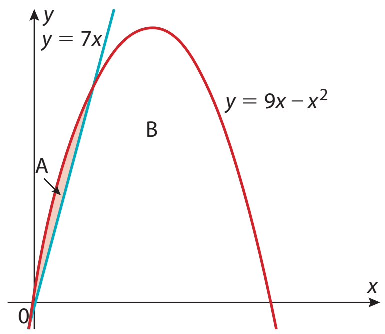
2 The diagram shows part of the curve $y = \sin x$ and $y = \cos x$. Calculate the area of the shaded region.



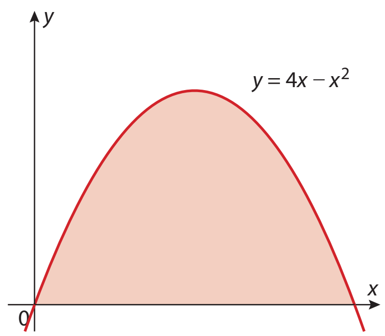
3 Find the area of the region enclosed by $y = 4 - x^2$ and the x-axis from $x = -1$ to $x = 2$.

4 Sketch the curve $y = (x + 2)(3 - x)$. Find the area of the region enclosed by the curve and the x-axis.

5 The diagram shows part of the line $y = 7x$ and part of the curve $y = 9x - x^2$. Calculate the ratio of the areas of the regions of A and B.



6 The diagram shows part of the curve $y = 4x - x^2$. Calculate the area of the shaded region.



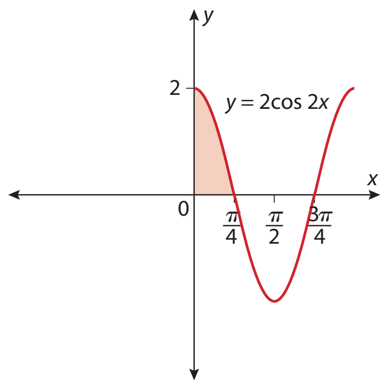
7 The equation of a curve is $y = x + \dfrac{4}{x^2}$. Find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 3$.

8 Calculate the area of the region bounded by the curve $y = x^2 + 2x - 8$ and the line $y = x + 4$.

9 (a) Find the x-coordinates of the turning points on the curve $y = 2 + \sin 3x$ over the interval $0 \leq x \leq \pi$.

(b) Calculate the area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

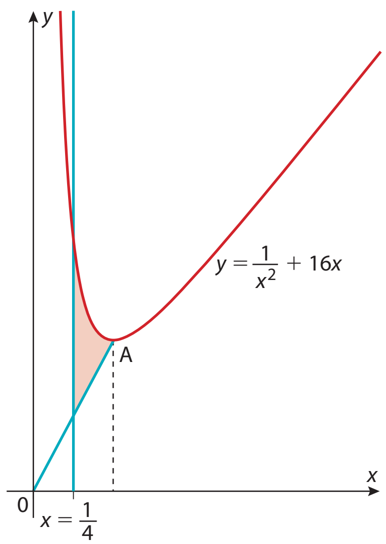
10 Calculate the area of the shaded region.



11 The diagram shows part of the curve $y = \frac{1}{x^2} + 16x$, part of the line $x = \frac{1}{4}$ and the line OA joining the origin O to the minimum point of the curve A.

(a) Show that the x-coordinate of A is $\frac{1}{2}$.

(b) Find the area of the shaded region.



12 Find the area of the region enclosed by the curve $y = 6\cos^2 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{8}$.

13 Sketch the curve $y = x^3 - 6x^2 + 11x - 6$. Find the area of the region enclosed by the curve and the x-axis.

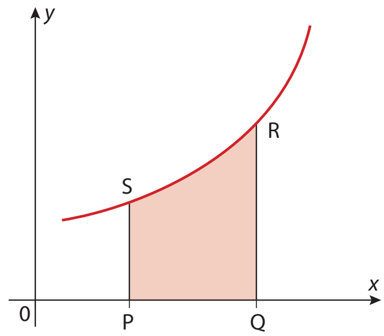
14 Sketch the curve $y = 2x^3 + 3x^2 - 23x - 12$. Find the area of the region enclosed by the curve and the x-axis.

15 Find the area of the region enclosed by the curve $y = x(x + 1)(x + 2)$ and the x-axis.

Volume of solids of revolution

Rotation about the *x*-axis

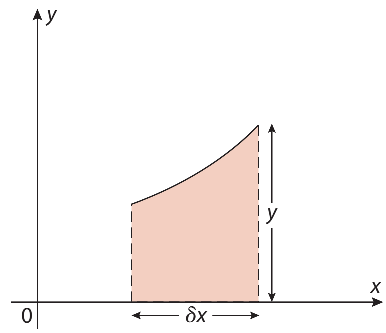
When the area PQRS is rotated 360° about the x-axis, a volume called the **solid of revolution** is formed. If we divide the area into a large number of strips each of width δx, when the area is rotated about the x-axis each strip produces a solid of revolution with thickness δx.



The solid of revolution formed is approximately equal to a circular disc.

Volume of 1 disc = area of cross-section × thickness = $(\pi y^2) \times \delta x$

Total volume $= \lim\limits_{\delta x \to 0} \sum\limits_{x=a}^{b} \pi y^2 \, \delta x$



The limiting sum is equivalent to the integral from $x = a$ to $x = b$.

$$\lim_{\delta x \to 0} \sum_{x=a}^{b} \pi y^2 \, \delta x = \pi \int_a^b y^2 \, dx$$

Therefore, the volume, V, of the solid formed when the curve $y = f(x)$ is rotated one revolution about the x-axis between $x = a$ and $x = b$ is

$$V = \pi \int_a^b y^2 \, dx$$

EXAMPLE 15 Find the volume of the solid formed when the region bounded by the curve $y = \sqrt{x}$ and the lines $x = 1$, $x = 4$ and the x-axis is rotated 2π radians about the x-axis.

SOLUTION

$$v = \pi \int_a^b y^2 \, dx$$

Since $y = \sqrt{x}$,

$$\begin{aligned} V &= \pi \int_1^4 (\sqrt{x})^2 \, dx \\ &= \pi \int_1^4 x \, dx \\ &= \pi \left[\frac{x^2}{2}\right]_1^4 \\ &= \pi \left(\frac{4^2}{2} - \frac{1^2}{2}\right) \\ &= \pi \left(8 - \frac{1}{2}\right) \\ &= 7\tfrac{1}{2}\pi \text{ cubic units} \end{aligned}$$

EXAMPLE 16 Find the volume of the solid formed when the region bounded by the curve $y = 2 + \sin x$, the lines $x = 0$, $x = 2\pi$ and the x-axis is rotated through 2π radians about the x-axis.

SOLUTION Since rotation is 2π radians about the x-axis,

$$V = \pi\int_a^b y^2\,dx$$

$$= \pi\int_0^{2\pi} (2 + \sin x)^2\,dx$$

Expanding the brackets we have:

$$V = \pi\int_0^{2\pi} (4 + 4\sin x + \sin^2 x)\,dx$$

Replacing $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$V = \pi\int_0^{2\pi} (4 + 4\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x)\,dx$$

$$= \pi\int_0^{2\pi} (4\tfrac{1}{2} + 4\sin x - \frac{1}{2}\cos 2x)\,dx$$

$$= \pi\left[\frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x\right]_0^{2\pi}$$

$$= \pi\left(\frac{9}{2}(2\pi) - 4\cos 2\pi - \frac{1}{4}\sin 2(2\pi)\right) - \pi\left(\frac{9}{2}(0) - 4\cos 0 - \frac{1}{4}\sin 2(0)\right)$$

$$= \pi((9\pi - 4) - (-4))$$

$$= 9\pi^2 \text{ cubic units}$$

EXAMPLE 17 Find the volume of the solid formed when the region bounded by the curve $y = \sqrt{r^2 - x^2}$ and the line $x = -r$, $x = r$ and the x-axis is rotated through 2π radians about the x-axis.

SOLUTION Since we are rotating 2π radians about the x-axis,

$$V = \pi\int_a^b y^2\,dx, \text{ where } a = -r,\ b = r$$

$$y = \sqrt{r^2 - x^2}$$

$$y^2 = r^2 - x^2$$

Therefore, $V = \pi\int_{-r}^{r} r^2 - x^2\,dx$

$$= \pi\left[r^2x - \frac{x^3}{3}\right]_{-r}^{r}$$

$$= \pi\left(r^2(r) - \frac{r^3}{3}\right) - \pi\left(r^2(-r) - \frac{(-r)^3}{3}\right)$$

$$= \pi\left(r^3 - \frac{r^3}{3}\right) - \pi\left(-r^3 + \frac{r^3}{3}\right)$$

$$= \pi\left(2r^3 - \frac{2r^3}{3}\right)$$

$$= \pi\left(\frac{6r^3}{3} - \frac{2r^3}{3}\right)$$

$$= \frac{4}{3}\pi r^3$$

Note

$y = \sqrt{r^2 - x^2}$
$\Rightarrow y^2 = r^2 - x^2$
$x^2 + y^2 = r^2$ is a circle centre $(0, 0)$ radius r. When we rotate the semicircle $y = \sqrt{r^2 - x^2}$, our solid of revolution is a sphere of radius r.

EXAMPLE 18 Determine the volume obtained when the straight line segment $y = 5 - 4x$ lying between $x = 0$ and $x = 1$ is rotated through 2π radians about the x-axis.

SOLUTION

$$V = \pi\int_a^b y^2\,dx, \text{ where } a = 0, b = 1$$

$$y = 5 - 4x$$

$$\Rightarrow y^2 = (5 - 4x)^2$$

$$= 25 - 40x + 16x^2$$

$$V = \pi\int_0^1 25 - 40x + 16x^2\,dx$$

$$= \pi\left[25x - 20x^2 + \frac{16}{3}x^3\right]_0^1$$

$$= \pi\left(25 - 20(1)^2 + \frac{16}{3}(1)^3\right) - \pi\left(25(0) - 20(0)^2 + \frac{16}{3}(0)^3\right)$$

$$= \pi\left(5 + \frac{16}{3}\right) = \frac{31}{3}\pi \text{ cubic units}$$

EXAMPLE 19 Determine the volume obtained when the part of the curve $y = \cos 2x$ lying between $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ is rotated through 2π radians about the x-axis.

SOLUTION

$$V = \pi\int_a^b y^2\,dx, \text{ where } a = \frac{\pi}{6}, b = \frac{\pi}{3}$$

$$y = \cos 2x$$

$$\Rightarrow y^2 = \cos^2 2x$$

$$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 2x\,dx$$

Replacing $\cos^2 2x = \frac{1}{2} + \frac{1}{2}\cos 4x$, we have:

$$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} + \frac{1}{2}\cos 4x\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{1}{4}\sin 4x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{\pi}{2}\left(\left(\frac{\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3}\right) - \left(\frac{\pi}{6} + \frac{1}{4}\sin\frac{4\pi}{6}\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{3} - \frac{\sqrt{3}}{8}\right) - \frac{\pi}{2}\left(\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)$$

EXAMPLE 20 The region bounded by the curve $y = \frac{2}{3x + 1}$, the axes and the line $x = 1$ is rotated through 360° about the x-axis. Find the volume of the solid of revolution formed.

SOLUTION

$$V = \pi\int_a^b y^2\,dx, \text{ where } a = 0, b = 1$$

$$y = \frac{2}{3x + 1}$$

$$y^2 = \left(\frac{2}{3x + 1}\right)^2$$

$$= \frac{4}{(3x + 1)^2}$$

$$= 4(3x + 1)^{-2}$$

$$\text{Therefore, } V = \pi\int_0^1 4(3x+1)^{-2}\,dx$$
$$= 4\pi\left[\frac{(3x+1)^{-1}}{3(-1)}\right]_0^1$$
$$= 4\pi\left(\left(-\frac{1}{3}\right)(3+1)^{-1} - \left(-\frac{1}{3}\right)(3(0)+1)^{-1}\right)$$
$$= 4\pi\left(-\frac{1}{12}+\frac{1}{3}\right)$$
$$= 4\pi\left(\frac{3}{12}\right)$$
$$= \pi \text{ cubic units}$$

EXAMPLE 21 The curve $y = \sec\frac{x}{2}$ is rotated 2π radians about the x-axis between the limits $x = 0$ and $x = \frac{\pi}{3}$. Show that the volume of the solid formed is $\frac{2\sqrt{3}}{3}\pi$.

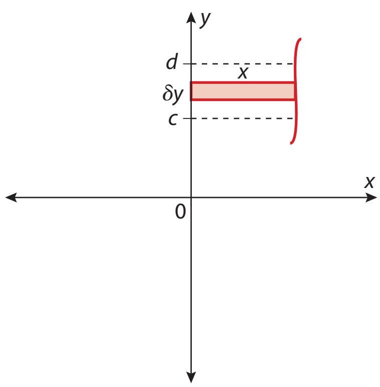
SOLUTION Since we are rotating about the x-axis:

$$V = \pi\int_a^b y^2\,dx$$
$$= \pi\int_0^{\frac{\pi}{3}}\left(\sec\left(\frac{x}{2}\right)\right)^2 dx$$
$$= \pi\int_0^{\frac{\pi}{3}}\sec^2\frac{x}{2}\,dx$$
$$= \pi\left[2\tan\frac{x}{2}\right]_0^{\frac{\pi}{3}} \qquad \left(\text{Since } \int\sec^2(ax)dx = \frac{1}{a}\tan(ax)+c\right)$$
$$= 2\pi\left(\tan\frac{\pi}{6}-\tan 0\right)$$
$$= 2\pi\frac{\sqrt{3}}{3} \text{ cubic units}$$

Rotation about the y-axis

Volume of 1 disc $= \pi(x)^2\,\delta y = \pi x^2\,\delta y$

Total volume $= \lim_{\delta y \to 0}\sum_{y=c}^{d}\pi x^2\,\delta y = \int_c^d \pi x^2\,dy$



The volume, V, of the solid formed when the curve $y = f(x)$ is rotated once about the y-axis between $y = c$ and $y = d$ is $\pi\int_c^d x^2\,dy$.

If a curve $x = f(y)$ is rotated 2π radians about the y-axis between the limits $y = c$ and $y = d$ then the volume generated is given by

$$V = \pi\int_c^d x^2\,dy$$

EXAMPLE 22 Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = 2x^2 - 3$, the y-axis and the lines $y = 0$, $y = 2$ through 360° about the y-axis.

SOLUTION

$$V = \pi\int_c^d x^2\,dy$$

Since $y = 2x^2 - 3$,

$y + 3 = 2x^2$

$x^2 = \dfrac{y+3}{2}$

Therefore, $V = \pi\displaystyle\int_0^2 \frac{y+3}{2}\,dy$

$$= \frac{\pi}{2}\left[\frac{1}{2}y^2 + 3y\right]_0^2$$

$$= \frac{\pi}{2}\left[\frac{1}{2}(2)^2 + 3(2)\right] - \left[\frac{1}{2}(0)^2 + 3(0)\right]$$

$$= \frac{\pi}{2}(2 + 6)$$

$= 4\pi$ cubic units

EXAMPLE 23 Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = \frac{3}{x}$, the y-axis and the lines $y = 2$, $y = 3$ through 360° about the y-axis.

SOLUTION

Since $y = \frac{3}{x}$,

$x = \dfrac{3}{y}$

$\Rightarrow x^2 = \left(\dfrac{3}{y}\right)^2$

$= \dfrac{9}{y^2}$

$$V = \pi\int_2^3 \left(\frac{3}{y}\right)^2 dy$$

$$= \pi\int_2^3 \frac{9}{y^2}\,dy$$

$$= 9\pi\int_2^3 y^{-2}\,dy$$

$$= 9\pi\left[-\frac{1}{y}\right]_2^3$$

$$= 9\pi\left(-\frac{1}{3} + \frac{1}{2}\right)$$

$$= \frac{9\pi}{6}$$

$= \frac{3}{2}\pi$ cubic units

EXAMPLE 24 Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = 3x^2 + 2$, the y-axis and the lines $y = 2$, $y = 5$ through 360° about the y-axis.

SOLUTION

Since $y = 3x^2 + 2$

$3x^2 = y - 2$

$$x^2 = \frac{y-2}{3}$$

$$V = \pi\int_2^5 x^2\,dy$$

$$= \pi\int_2^5 \left(\frac{y-2}{3}\right)dy$$

$$= \frac{\pi}{3}\int_2^5 (y-2)\,dy$$

$$= \frac{\pi}{3}\left[\frac{y^2}{2} - 2y\right]_2^5$$

$$= \frac{\pi}{3}\left(\left(\frac{25}{2} - 10\right) - \left(\frac{4}{2} - 4\right)\right)$$

$$= \frac{\pi}{3}\left(\frac{5}{2} + 2\right)$$

$$= \frac{\pi}{3}\left(\frac{9}{2}\right)$$

$$= \frac{3\pi}{2}\text{ cubic units}$$

EXAMPLE 25 Find the volume of the solid of revolution formed by rotating the area enclosed by the curve $y = 2x + 1$, the y-axis and the lines $y = 1$, $y = 3$ through 360° about the y-axis.

SOLUTION

$$y = 2x + 1$$

$$\Rightarrow y - 1 = 2x$$

$$\Rightarrow x = \frac{y-1}{2}$$

$$\Rightarrow x^2 = \frac{(y-1)^2}{2^2}$$

$$= \frac{1}{4}(y^2 - 2y + 1)$$

$$V = \pi\int_1^3 x^2\,dy$$

$$= \pi\int_1^3 \frac{1}{4}(y^2 - 2y + 1)\,dy$$

$$= \frac{\pi}{4}\left[\frac{y^3}{3} - y^2 + y\right]_1^3$$

$$= \frac{\pi}{4}\left(\left(\frac{27}{3} - 9 + 3\right) - \left(\frac{1}{3} - 1 + 1\right)\right)$$

$$= \frac{\pi}{4}\left(9 - 9 + 3 - \frac{1}{3}\right)$$

$$= \frac{\pi}{4}\left(\frac{8}{3}\right)$$

$$= \frac{2}{3}\pi\text{ cubic units}$$

Alternative solution:

$$V = \pi\int_1^3 x^2\,dy$$

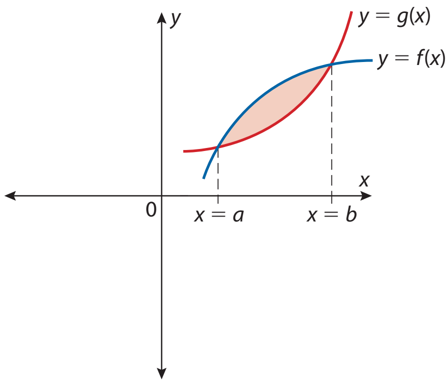
$$= \frac{\pi}{4}\int_1^3 \frac{1}{4}(y-1)^2\,dy$$

$$= \frac{\pi}{4}\left[\frac{(y-1)^3}{3}\right]_1^3$$

$$= \frac{\pi}{4}\left[\frac{8}{3}\right]$$

$$= \frac{2}{3}\pi \text{ cubic units}$$

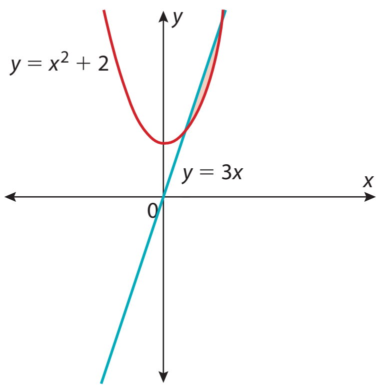
Volume generated by the region bounded by two curves



The volume generated by rotating a region S bounded by the curves $y = f(x)$ and $y = g(x)$ through 360° about the x-axis from $x = a$ to $x = b$ is

$$V = \pi\int_a^b (f(x))^2\,dx - \pi\int_a^b (g(x))^2\,dx$$

EXAMPLE 26 Find the volume generated when the region bounded by the line $y = 3x$ and the curve $y = x^2 + 2$.



SOLUTION

$y = x^2 + 2$

$y = 3x$

Find the points of intersection:

$x^2 + 2 = 3x$

$x^2 - 3x + 2 = 0$

$\Rightarrow (x - 1)(x - 2) = 0$

$\Rightarrow x = 1, x = 2$

Volume formed by rotating $y = 3x$:

$$= \pi\int_1^2 (3x)^2\,dx$$
$$= 9\pi\int_1^2 x^2\,dx$$
$$= 9\pi\left[\frac{x^3}{3}\right]_1^2$$
$$= 9\pi\left(\frac{2^3}{3} - \frac{1^3}{3}\right)$$
$$= 9\pi \times \frac{7}{3}$$
$$= 21\pi \text{ cubic units}$$

Volume formed by rotating $y = x^2 + 2$:

$$V = \pi\int_1^2 (x^2 + 2)^2\,dx$$
$$= \pi\int_1^2 x^4 + 4x^2 + 4\,dx$$
$$= \pi\left[\frac{x^5}{5} + \frac{4}{3}x^3 + 4x\right]_1^2$$
$$= \pi\left(\frac{(2)^5}{5} + \frac{4}{3}(2)^3 + 4(2)\right) - \pi\left(\frac{(1)^5}{5} + \frac{4}{3}(1)^3 + 4(1)\right)$$
$$= \pi\left(\frac{32}{5} + \frac{32}{3} + 8 - \frac{1}{5} - \frac{4}{3} - 4\right) = \frac{293}{15}\pi \text{ cubic units}$$

Required volume $= 21\pi - \frac{293\pi}{15} = \frac{22\pi}{15}$ cubic units

EXAMPLE 27 The region bounded by the curve $y = \frac{2}{x}$ and $y = 3 - x$ is rotated 2π radians about the x-axis. Calculate the volume of the solid formed.

SOLUTION We need to find the points of intersection.

$$y = \frac{2}{x}$$
$$y = 3 - x$$
$$\Rightarrow \frac{2}{x} = 3 - x$$
$$2 = (3 - x)x$$
$$2 = 3x - x^2$$
$$x^2 - 3x + 2 = 0$$
$$\Rightarrow (x - 1)(x - 2) = 0$$
$$\Rightarrow x = 1, x = 2$$

Volume under the line:

$$V = \pi\int_1^2 (3 - x)^2\,dx$$
$$= \pi\int_1^2 (9 - 6x + x^2)\,dx$$
$$= \pi\left[9x - 3x^2 + \frac{x^3}{3}\right]_1^2$$
$$= \pi\left(9(2) - 3(2)^2 + \frac{2^3}{3}\right) - \left(9(1) - 3(1)^2 + \frac{1^3}{3}\right)$$
$$= \pi\left(18 - 12 + \frac{8}{3} - 9 + 3 - \frac{1}{3}\right)$$
$$= \frac{7}{3}\pi \text{ cubic units}$$

Alternatively:

$$V = \pi\int_1^2 (3 - x)^2\,dx$$
$$= \pi\left[\frac{-(3 - x)^3}{3}\right]_1^2$$
$$= \pi\left[\frac{-1}{3} + \frac{8}{3}\right]$$
$$= \frac{7}{3}\pi \text{ cubic units}$$

Volume under the curve $y = \frac{2}{x}$

$$V = \pi\int_1^2 \left(\frac{2}{x}\right)^2 dx$$
$$= \pi\int_1^2 \frac{4}{x^2}\,dx$$
$$= \pi\int_1^2 4x^{-2}\,dx$$
$$= \pi[-4x^{-1}]_1^2$$
$$= \pi\left(\left(-\frac{4}{2}\right) - \left(-\frac{4}{1}\right)\right)$$
$$= \pi(-2 + 4)$$
$$= 2\pi \text{ cubic units}$$

$$\text{Required volume} = \frac{7\pi}{3} - 2\pi$$
$$= \frac{7\pi}{3} - \frac{6\pi}{3}$$
$$= \frac{\pi}{3} \text{ cubic units}$$

EXAMPLE 28 Calculate the volume generated when the region bounded by the curves $y^2 = 27x$ and $y = x^2$ is rotated through 360° about the y-axis.

SOLUTION First find the points of intersection.

$y^2 = 27x$

$y = x^2$

$\Rightarrow (x^2)^2 = 27x$

$\Rightarrow x^4 - 27x = 0$

$\Rightarrow x(x^3 - 27) = 0$

$\Rightarrow x = 0, x^3 = 27 \Rightarrow x = 3$

When $x = 0, y = 0$

When $x = 3, y^2 = 81 \Rightarrow y = 9$

Since we are rotating about the y-axis

$$V = \pi\int_0^3 x^2\,dy$$

For $y^2 = 27x$ (Since we need to find x^2)

$$x = \frac{y^2}{27}$$

$$\Rightarrow x^2 = \frac{y^4}{27^2}$$

$$\Rightarrow V = \pi\int_0^9 \frac{y^4}{27^2}\,dy$$
$$= \pi\left[\frac{y^5}{27^2} \times \frac{1}{5}\right]_0^9$$
$$= \frac{\pi}{27^2}\left[\frac{y^5}{5}\right]_0^9$$
$$= \frac{\pi}{27^2} \times \frac{9^5}{5}$$
$$= \frac{81}{5}\pi \text{ cubic units}$$

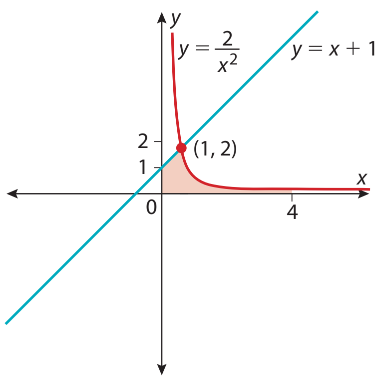
For $y = x^2$

$$V = \pi\int_0^9 y\,dy = \pi\left[\frac{y^2}{2}\right]_0^9$$
$$= \frac{81}{2}\pi \text{ cubic units}$$

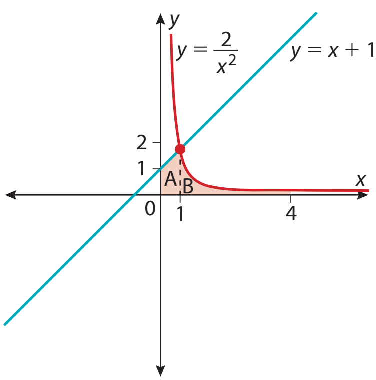
Required volume $= \frac{81}{2}\pi - \frac{81}{5}\pi$

$= \frac{243}{10}\pi$ cubic units

EXAMPLE 29 The diagram shows part of the curve $y = \frac{2}{x^2}$. The straight line $y = x + 1$ cuts the curve at (1, 2). Calculate the volume generated when the shaded region is rotated through 360° about the x-axis.



SOLUTION We can split the region in two as follows:



The volume of the solid formed when region A is rotated about the x-axis:

$$\begin{aligned} V &= \pi\int_0^1 (x+1)^2\,\mathrm{d}x \\ &= \pi\int_0^1 x^2 + 2x + 1\,\mathrm{d}x \\ &= \pi\left[\frac{x^3}{3} + x^2 + x\right]_0^1 \\ &= \pi\left(\frac{1}{3} + 1 + 1\right) \\ &= \frac{7}{3}\pi \text{ cubic units} \end{aligned}$$

Alternatively:

$$\begin{aligned} V &= \pi\int_0^1 (x+1)^2\,\mathrm{d}x \\ &= \frac{\pi}{3}\left[(x+1)^3\right]_0^1 \\ &= \frac{\pi}{3}[8-1] \\ &= \frac{7}{3}\pi \text{ cubic units} \end{aligned}$$

The volume of the solid formed when region B is rotated about the x-axis:

$$\begin{aligned} V &= \pi\int_1^4 \left(\frac{2}{x^2}\right)^2\mathrm{d}x \\ &= \pi\int_1^4 \frac{4}{x^4}\,\mathrm{d}x \\ &= 4\pi\int_1^4 x^{-4}\,\mathrm{d}x \\ &= 4\pi\left[-\frac{1}{3}x^{-3}\right]_1^4 \\ &= \frac{4\pi}{3}\left(-\frac{1}{4^3} - (-1)^3\right) \\ &= \frac{4\pi}{3}\left(\frac{-1}{64} + 1\right) \\ &= \frac{4\pi}{3} \times \frac{63}{64} \\ &= \frac{21}{16}\pi \text{ cubic units} \end{aligned}$$

$$\begin{aligned} \text{Total volume} &= \frac{21}{16}\pi + \frac{7}{3}\pi \\ &= \frac{175}{48}\pi \text{ cubic units} \end{aligned}$$

EXAMPLE 30 Find the volume of the solid generated when the region bounded by the curve $y = 2\sin x$, the lines $y = 1$, $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$ is rotated through 2π radians about the x-axis.

SOLUTION When $y = 2\sin x$ is rotated about the x-axis.

$$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (2\sin x)^2\,\mathrm{d}x$$

Remember

$\cos 2x = 1 - \sin^2 2x$
$\Rightarrow 2\sin^2 x = 1 - \cos 2x$
$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$

$$= \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 4\sin^2 x\,dx$$

$$= 4\pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{4\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= 2\pi\left(\left(\frac{\pi}{4} - \frac{1}{2}\sin\frac{\pi}{2}\right) - \left(\frac{\pi}{6} - \frac{1}{2}\sin\frac{\pi}{3}\right)\right)$$

$$= 2\pi\left(\frac{\pi}{4} - \frac{\pi}{6} - \frac{1}{2} + \frac{\sqrt{3}}{4}\right)$$

$$= 2\pi\left(\frac{\pi}{12} - \frac{1}{2} + \frac{\sqrt{3}}{4}\right) \text{ cubic units}$$

When $y = 1$ is rotated about the x-axis:

$$V = \pi\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} 1^2\,dx$$

$$= \pi[x]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \pi\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$= \pi\left(\frac{\pi}{12}\right)$$

$$= \frac{\pi^2}{12} \text{ cubic units}$$

$$\text{Required volume} = 2\pi\left(\frac{\pi}{12} - \frac{1}{2} + \frac{\sqrt{3}}{4}\right) - \pi\left(\frac{\pi}{12}\right)$$

$$= 2\pi\left(\frac{\pi}{12} - \frac{1}{2} + \frac{\sqrt{3}}{4} - \frac{\pi}{24}\right)$$

$$= 2\pi\left(\frac{\pi}{24} - \frac{1}{2} + \frac{\sqrt{3}}{4}\right) \text{ cubic units}$$

EXAMPLE 31 The region bounded by the curves $y = \cos x$, $y = x^2$ and the lines $x = -\frac{\pi}{6}$ $x = \frac{\pi}{6}$ is rotated 2π radians about the x-axis. Find the volume of the solid of revolution formed.

SOLUTION For the curve $y = x^2$:

$$V = \pi\int_a^b y^2\,dx$$

$$= \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (x^2)^2\,dx$$

$$= \pi\left[\frac{x^5}{5}\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{5}\left(\frac{\pi^5}{6^5} - \left(-\frac{\pi^5}{6^5}\right)\right)$$

$$= \frac{\pi}{5}\left(\frac{2\pi^5}{6^5}\right)$$

For $y = \cos x$:

$$V = \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 x\,dx$$

$$= \pi\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx$$

$$= \frac{\pi}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (1 + \cos 2x)\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{1}{2}\sin 2x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \frac{\pi}{2}\left(\frac{\pi}{6} + \frac{1}{2}\sin\frac{\pi}{3}\right) - \frac{\pi}{2}\left(-\frac{\pi}{6} + \frac{1}{2}\sin\left(-\frac{2\pi}{6}\right)\right)$$

$$= \frac{\pi}{2}\left(\frac{\pi}{6} + \frac{\pi}{6} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4}\right)$$

$$= \frac{\pi}{2}\left(\frac{2\pi}{6} + 2\frac{\sqrt{3}}{4}\right)$$

$$= \pi\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)$$

Remember

$\cos^2 x = \frac{1}{2} + \frac{1}{2} \times \cos 2x$

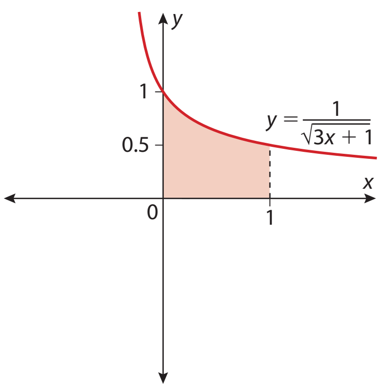
$$\text{Required volume} = \pi\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right) - \frac{\pi}{5}\left(\frac{2\pi^5}{6^5}\right)$$

$$= \pi\left(\frac{\pi}{6} + \frac{\sqrt{3}}{4} - \frac{2\pi^5}{38880}\right)$$

$$= 2.96 \text{ units}^3$$

EXERCISE 16B

1 The diagram shows part of the curve $= \frac{1}{\sqrt{3x+1}}$. Find the volume generated when the shaded region is rotated through 360° about the x-axis.
(Use $\int \frac{1}{x} = \ln|x|$.)



2 (a) Show that $(1 - \sqrt{2}\sin x)^2 = 2 - 2\sqrt{2}\sin x - \cos 2x$.

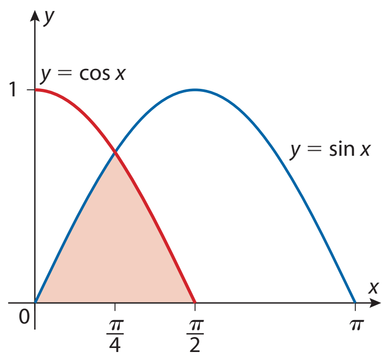
(b) The region bounded by the curve $y = 1 - \sqrt{2}\sin x$, the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$ is rotated through 360° about the x-axis. Find the volume generated.

3 Find the volume of the solid formed when the area bounded by the curve $y = 3x - x^2$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis.

4 The region bounded by the curve $y = \sin x$, the x-axis and the line $x = \frac{\pi}{4}$ is rotated through 360° about the x-axis. Calculate the volume of the solid of revolution thus formed.

5 Find in terms of π the volume of the solid of revolution formed when the region $0 \le x \le \frac{3\pi}{2}$ bounded by the curve $y = 1 + \sin x$, the x-axis and the y-axis is rotated through 360° about the x-axis.

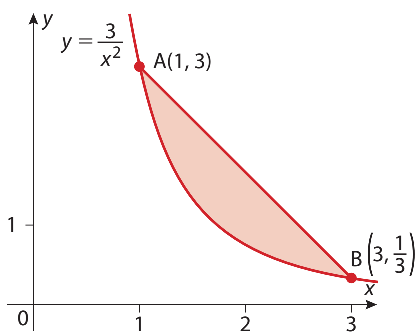
6 The diagram shows part of the graphs of $y = \sin x$ and $y = \cos x$. Find the volume generated when the shaded region is rotated through 360° about the x-axis.



7 The points A(1, 3) and B$\left(3, \frac{1}{3}\right)$ lie on the curve $y = \frac{3}{x^2}$, as shown in the diagram.

(a) Find the equation of the line AB.

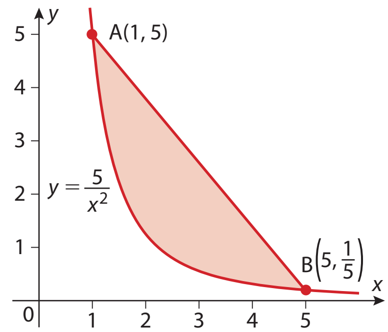
(b) Calculate the volume obtained when the shaded region is rotated through 360° about the x-axis.



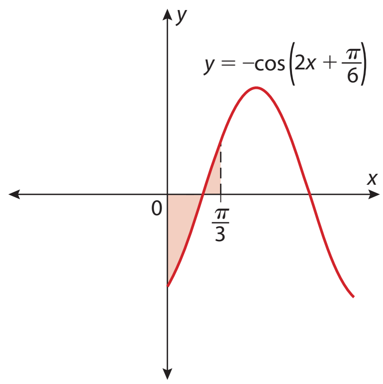
8 The points A(1, 5) and B$\left(5, \frac{1}{5}\right)$ lie on the curve $y = \frac{5}{x^2}$ as shown in the diagram.

(a) Find the equation of the line AB.

(b) Calculate the volume obtained when the shaded region is rotated through 360° about the x-axis.

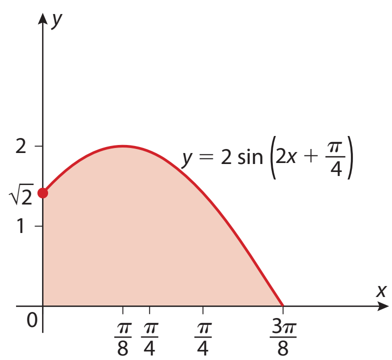


9 Calculate the volume generated when the shaded region is rotated through four right angles about the x-axis.



10 Find the volume generated when the region bounded by the curve $y = \tan x$, the x-axis and the line $x = \frac{\pi}{4}$ is rotated through 2π radians about the x-axis.

11 The diagram shows part of the curve $y = 2\sin\left(2x + \frac{\pi}{4}\right)$. Find the volume generated when the shaded region is rotated through 2π radians about the x-axis.



12 Find the volume generated when the region bounded by the curve $y = 2\sin x + 4\cos x$, the lines $x = 0$ and $x = a$, where a is the x-coordinate of the maximum point on the curve is rotated through 360° about the x-axis.

13 The part of the curve $y = x^3$ from $x = 1$ to $x = 2$ is rotated about the y-axis through 2π radians. Find the volume of the solid formed.

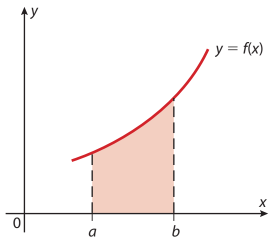
14 Find the volume of the solid generated by rotating completely about the y-axis the area enclosed by the curve $xy = 2$, the lines $x = 0$, $y = 2$ and $y = 5$.

SUMMARY

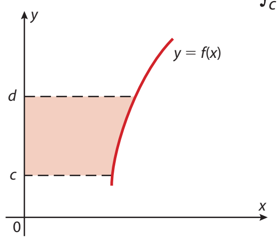
Applications of integration

Area under a curve

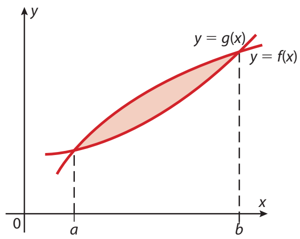
Area between the curve and the x-axis from $x = a$ to $x = a$ is $\int_a^b y\,dx$



Area between the curve and the y-axis from $y = c$ to $y = d$ is $\int_c^d x\,dx$



Area between two curves

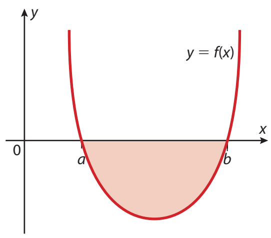


(i) Find the points of intersection of the two curves.

(ii) Shaded area =

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

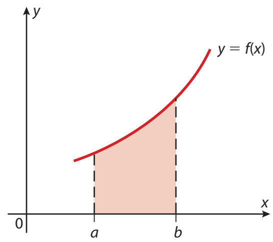
Area below the x-axis



Shaded area $= \left|\int_a^b f(x)\,dx\right|$

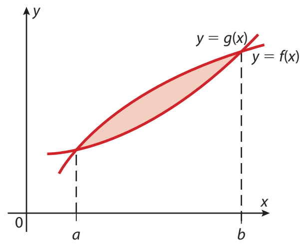
Rotation about the x-axis

Volume of the solid formed when the shaded region is rotated 2π radians about the x-axis is $\pi \int_a^b y^2\,dx$



Volume of the solid formed when the shaded region is rotated 2π radians about the x-axis is

$$\pi \int_a^b (g(x))^2\,dx - \pi \int_a^b (f(x))^2\,dx$$



Rotation about the y-axis

Volume of the solid formed when the shaded region is rotated 2π radians about the y-axis is $\pi \int_c^d x^2\,dy$

Checklist

Can you do these?

- Estimate the area under a curve using rectangles.
- Understand that the limiting sum gives the exact area under the curve.
- Use integration to find the area under the curve.
- Use integration to find the area between two curves.
- Find the volume of a solid formed when a region is rotated about the x-axis.
- Find the volume of a solid formed when a region is rotated about the y-axis.

Review Exercise 16

1 Show that the area of the region bounded by the curves $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$ is $(\sqrt{2} - 1)$ square units.

2 Find the area of the region bounded by the curves $y = 6x - x^2$ and $y = x^2 - 2x$.

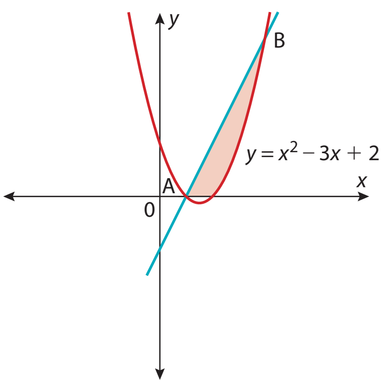
3 Sketch the curve $y = x^3 - 6x^2 + 8x$. Find the area of the region bounded by the curve $y = x^3 - 6x^2 + 8x$ and the x-axis.

4 Find the point of intersection of the curves $y = x^2 - 4$ and $y = -2x^2$. Hence, find the area bounded by the two curves.

5 The points A(1, 0) and B(3, 2) lie on the curve $y = x^2 - 3x + 2$ as shown in the diagram.

(a) Find the equation of the line AB.

(b) Find the volume, when the shaded region is rotated through 360° about the x-axis.



6 (a) Find the volume obtained when the region bounded by the curve $y = \frac{1}{4x - 1}$, the x-axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x-axis.

(b) Find the area of the region bounded by the curve $y = 5x - x^2$ and the line $y = 6$.

7 Sketch the curve $y = x(x - 2)(x - 3)$, and find the ratio of the areas of the two regions bounded by the curve and the x-axis.

8 (a) Sketch the graph of $y = x^3 + 4x^2 + 3x$, from $x = -4$ to $x = 1$.

(b) Find the gradient of the graph at the points where the graph cuts the x-axis.

(c) Calculate the area enclosed by the x-axis and the curve between $x = -3$ and $x = -1$.

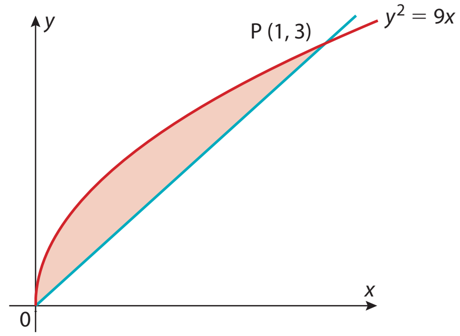
9 Calculate the volume generated when the region bounded by the curve $y = \dfrac{6}{x + 2}$, the x-axis, and the lines $x = 1$ and $x = 4$ is rotated through 360° about the x-axis.

10 Find the volume of the solid generated by rotating completely about the y-axis the area enclosed by the curve $x - y^2 - 4 = 0$, the lines $x = 0$, $y = 0$ and $y = 3$.

11 The area bounded by the curve $y = \sqrt{4x}$ and the line $y = 3$ is rotated completely about the y-axis. Find the volume generated.

12 The area bounded by the curve $y = 9 - x^2$, the x-axis and the lines $x = 2$, and $x = 3$ is rotated about the x-axis through 360°, find the volume generated.

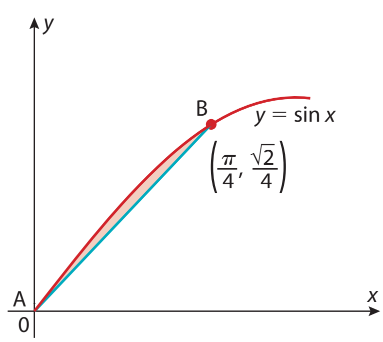
13 The diagram shows part of the curve $y^2 = 9x$, and the line OP where O is (0, 0) and P is (1, 3). Find the volume, in terms of π, when the shaded region is rotated through 2π radians about the x-axis.



14 The points A(0, 0) and $B\left(\dfrac{\pi}{4}, \dfrac{\sqrt{2}}{2}\right)$ lie on the curve $y = \sin x$ as shown in the diagram.

(a) Find the equation of the line AB.

(b) Show that the volume formed when the shaded region is rotated through 360° about the x-axis is given by $V = \dfrac{\pi}{12(\pi - 3)}$.



15 Calculate the area of the region between the curve $y = 9 - x^2$ and $y = x^2 + 1$ from $x = 0$ to $x = 3$.

16 Calculate the volume of the solid formed when the area bounded by the curves $y = x^2$ and $y = \sqrt{x}$ rotated about

(a) the x-axis

(b) the y-axis.

17 (a) Sketch the curves $y = x^2$ and $y = 18 - x^2$.

(b) Find the points of intersections of the curves.

(c) Calculate the area enclosed by the two curves.

18 (a) Find the points of intersections of the curves $y = \sin 2x$ and $y = \cos x$ for $0 \leq x \leq \frac{\pi}{2}$.

(b) Find the area of the shaded region bounded by the curves from $x = a$ to $x = b$, where a and b are the coordinates of the points of intersections found in a.

CHAPTER 17

Differential Equations

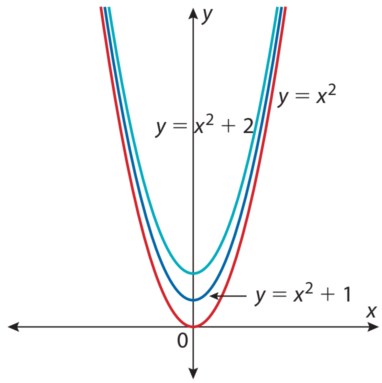
At the end of this chapter you should be able to:

- Identify a first order differential equation
- Separate the variables of a first order differential equation
- Find the general solution of a first order differential equation
- Find the solution of a differential equation given boundary conditions
- Sketch the solution curve for a differential equation
- Form and solve a differential equation for a practical problem
- Solve second order differential equations

KEYWORDS/TERMS

differential equations • first order • general solution • boundary conditions • solution curve

Families of curves



The diagram shows the graphs of:

(a) $y = x^2$

(b) $y = x^2 + 1$

(c) $y = x^2 + 2$

SOLUTION

Notice that $y = x^2$, $y = x^2 + 1$ and $y = x^2 + 2$ are three curves with the same general shape but different positions in the x–y plane. These curves are called a **family of curves**.

An equation of the form $y = x^2 + c$, where c is a constant, represents a family of curves. Given values of x and y we can find a value for c. The value of c corresponds to the boundary conditions in the problem.

When $x = 0$ and $y = 0$, we get $c = 0$ and the curve $y = x^2$ corresponds to a particular solution of the equation. The condition (0, 0) is called a **boundary condition** of the equation. As the boundary condition changes the equation of the curve changes, and we can find the members of the family of curves.

Since $y = x^2$, $\frac{dy}{dx} = 2x$. The equation $\frac{dy}{dx} = 2x$ is called a **differential equation**. A differential equation is an equation containing a derivative.

Classifying differential equations

The order of a differential equation is the order of its highest derivative. For example:

$\frac{dy}{dx} = x$ is a first order differential equation.

$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = x + 2$ is a second order differential equation and so on.

Linear versus non-linear differential equations

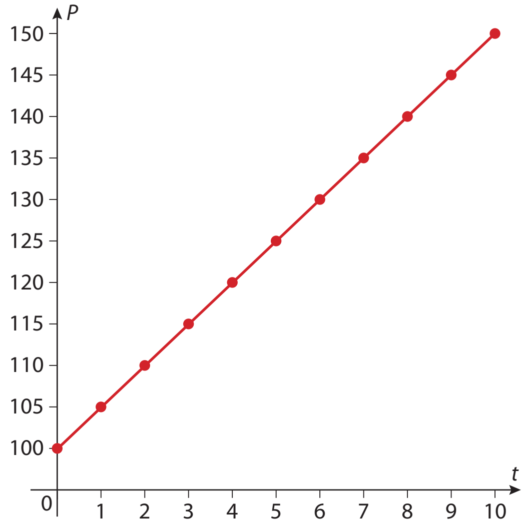
Differential equations can also be classified as linear or non-linear differential equations. A differential equation is linear, if it exclusively involves terms with a power of 1. For example, $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 3y = x$ is a linear differential equation and $\frac{d^2y}{dx^2} + 4\sin y = 5x + 2$ is a non-linear differential equation.

Practical applications of differential equations

Here is an example of a practical application of differential equations.

In the year 2000 Ariel started buying CDs at a shop in Gulf City Mall. over the last ten years Ariel noticed a gradual increase in the price of the CDs and decided to plot a graph of the total cost for each year.

Year	Price
2000	TT$100
2001	TT$105
2002	TT$110
2003	TT$115
2004	TT$120
2005	TT$125
2006	TT$130
2007	TT$135
2008	TT$140
2009	TT$145
2010	TT$150



Ariel found the gradient of the line using $\frac{\Delta P}{\Delta t} = 5$, replacing the rate of change of the price P with time t (in years) Ariel obtained $\frac{dp}{dt} = 5$. This is called a differential equation.

Using integration, Ariel obtained $P = 5t + c$ where P is the price of the CD at time t in years and c is an arbitrary constant.

Substituting $t = 0$, $P =$ TT$100, Ariel got $100 = c$ and her equation connecting price and time is:

$P = 5t + 100$

Ariel could use her model to predict the price of a CD in the year 2011 and beyond.

First order differential equations

$\frac{dy}{dx} = f(x)$ is an example of a first order differential equation. To solve this differential equation we integrate both sides of the equation. The solution that contains the constant of integration is called the **general solution** of the equation. When additional information is given so that a particular curve can be identified, the solution is a **particular solution**.

EXAMPLE 1 Solve the differential equation $\frac{dy}{dx} = 2x + 3$.

SOLUTION

$\frac{dy}{dx} = 2x + 3$

$\int dy = \int 2x + 3\,dx$

$\Rightarrow y = x^2 + 3x + c$

Therefore, the general solution is $y = x^2 + 3x + c$.

EXAMPLE 2 Find the general solution of the equation $\frac{2}{x+1}\frac{dy}{dx} = \sqrt{x}$

SOLUTION

$$\frac{2}{x+1}\frac{dy}{dx} = \sqrt{x}$$

$$\Rightarrow 2\frac{dy}{dx} = \sqrt{x}\,(x+1) \quad \text{(Multiplying both sides by } (x+1))$$

$$\int 2\,dy = \int x^{\frac{1}{2}}(x+1)\,dx$$

$$\Rightarrow 2y = \int x^{\frac{3}{2}} + x^{\frac{1}{2}}\,dx$$

$$2y = \frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}} + c$$

$$y = \frac{1}{5}x^{\frac{5}{2}} + \frac{1}{3}x^{\frac{3}{2}} + \frac{c}{2}$$

EXAMPLE 3 Find the particular solution of the differential equation $x^2\frac{dy}{dx} = 4 - x^4$, when $x = 1$ and $y = 2$.

SOLUTION

$$x^2\frac{dy}{dx} = 4 - x^4$$

$$\Rightarrow \frac{dy}{dx} = \frac{4}{x^2} - \frac{x^4}{x^2} \quad \text{(Dividing both sides by } x^2)$$

$$\frac{dy}{dx} = \frac{4}{x^2} - x^2$$

Integrating both sides with respect to x.

$$y = \int \frac{4}{x^2} - x^2\,dx$$

$$y = \int 4x^{-2} - x^2\,dx$$

$$y = \frac{4x^{-1}}{-1} - \frac{x^3}{3} + c$$

$$y = -\frac{4}{x} - \frac{x^3}{3} + c$$

When $x = 1$, $y = 2$:

$$2 = -\frac{4}{1} - \frac{1}{3} + c$$

$$c = 2 + 4 + \frac{1}{3}$$

$$= 6\tfrac{1}{3}$$

Therefore, the particular solution is $y = -\frac{4}{x} - \frac{x^3}{3} + 6\frac{1}{3}$

EXAMPLE 4 Find the solution of the differential equation $3\frac{dr}{d\theta} + \sin\theta = 0$, given that $r = 5$ when $\theta = \frac{\pi}{2}$.

SOLUTION

$$3\frac{dr}{d\theta} + \sin\theta = 0$$

$$\Rightarrow 3\frac{dr}{d\theta} = -\sin\theta$$

$$\int 3\,dr = \int -\sin\theta\,d\theta$$

$$\Rightarrow 3r = \cos\theta + c$$

When $r = 5$, $\theta = \frac{\pi}{2}$:

$\Rightarrow 15 = \cos\frac{\pi}{2} + c$

$\Rightarrow c = 15$

Hence, $3r = \cos\theta + 15$

$\Rightarrow r = \frac{1}{3}\cos\theta + 5$

The solution of the equation is $r = \frac{1}{3}\cos\frac{\pi}{2} + 5$.

Solutions of variable-separable differential equations

An equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)\,g(y)$, where $f(x)$ is a function of x and $g(y)$ is a function of y, can be rearranged to:

$$\frac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x$$

Integrating both sides gives:

$$\int\frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$$

An equation of this form, where we can separate the functions of x and y, is called a variable-separable equation.

EXAMPLE 5 Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x+1}{y+2}$.

SOLUTION

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4x+1}{y+2}$$

Separating the variables we get:

$$(y+2)\,\mathrm{d}y = (4x+1)\,\mathrm{d}x$$

Integrating both sides gives:

$$\int(y+2)\,\mathrm{d}y = \int(4x+1)\,\mathrm{d}x$$

$$\Rightarrow \frac{y^2}{2} + 2y = 2x^2 + x + c$$

We now have a relationship containing x and y.

Alternatively:

$$\int(y+2)\,\mathrm{d}y - \int(4x+1)\,\mathrm{d}x$$

$$\Rightarrow \frac{(y+2)^2}{2} = \frac{1}{8}(4x+1)^2 + c$$

$$\Rightarrow (y+2)^2 = \frac{1}{4}(4x+1)^2 + 2c$$

$$\Rightarrow y + 2 = \sqrt{\frac{1}{4}(4x+1)^2 + 2c}$$

$$\Rightarrow y = \sqrt{\frac{1}{4}(4x+1)^2 + 2c} - 2$$

EXAMPLE 6 Solve the differential equation $y\frac{dy}{dx} = \frac{\tan x + 2}{\cos^2 x}$, given that $y = 1$, when $x = \frac{\pi}{4}$.

SOLUTION

$$y\frac{dy}{dx} = \frac{\tan x + 2}{\cos^2 x}$$

$$\int y\,dy = \int \tan x \sec^2 x + 2\sec^2 x\,dx \qquad \text{(Separating variables)}$$

$$\frac{y^2}{2} = \frac{1}{2}\tan^2 x + 2\tan x + c$$

When $x = \frac{\pi}{4}, y = 1$:

$$\frac{1}{2} = \frac{1}{2}\tan^2\frac{\pi}{4} + 2\tan\frac{\pi}{4} + c$$

$$\frac{1}{2} = \frac{1}{2} + 2 + c$$

$$\Rightarrow c = -2$$

$$\Rightarrow \frac{y^2}{2} = \frac{1}{2}\tan^2 x + 2\tan x - 2$$

$$y^2 = \tan^2 x + 4\tan x - 4$$

$$y = \sqrt{\tan^2 x + 4\tan x - 4}$$

EXAMPLE 7 Given that $\frac{(3y + 2)}{\cos x}\frac{dy}{dx} = 1$, find y in terms of x, when $x = 0, y = 0$.

SOLUTION

$$\frac{(3y + 2)}{\cos x}\frac{dy}{dx} = 1$$

> **Remember**
>
> To solve an equation of this form, we bring all *x*s to one side, all *y*s on the other and then integrate.

Separating variables we have:

$$3y + 2\,dy = \cos x\,dx$$

$$\int (3y + 2)^1\,dy = \int \cos x\,dx$$

$$\Rightarrow \frac{1}{3}\frac{(3y + 2)^2}{2} = \sin x + c$$

$$\frac{(3y + 2)^2}{6} = \sin x + c$$

When $x = 0, y = 0$

$$\frac{(0 + 2)^2}{6} = c$$

$$\frac{4}{6} = c$$

$$\frac{2}{3} = c$$

Therefore, $\frac{(3y + 2)^2}{6} = \sin x + \frac{2}{3}$

Make y the subject of the formula:

$$(3y + 2)^2 = 6\sin x + 4$$

$$3y + 2 = \sqrt{6\sin x + 4}$$

$$3y = -2 + \sqrt{6\sin x + 4}$$

$$y = \frac{-2 + \sqrt{6\sin x + 4}}{3}$$

EXAMPLE 8 Given that $\frac{dy}{dx} + y^2 \sec^2 x = 0$. Find the equation of the curve which passes through $\left(\frac{\pi}{4}, \frac{1}{2}\right)$.

SOLUTION

$$\frac{dy}{dx} + y^2 \sec^2 x = 0$$

$$\frac{dy}{dx} = -y^2 \sec^2 x \qquad \text{(Separating variables)}$$

$$-\frac{1}{y^2}\,dy = \sec^2 x\,dx$$

$$\Rightarrow \int -\frac{1}{y^2}\,dy = \int \sec^2 x\,dx$$

$$\Rightarrow \frac{1}{y} = \tan x + c \qquad \left(\text{Since} \int \frac{1}{y^2}\,dy = \int y^{-2}\,dy = \frac{y^{-1}}{-1} + A\right)$$

When $x = \frac{\pi}{4}, y = \frac{1}{2}$:

$$\frac{1}{\frac{1}{2}} = \tan\frac{\pi}{4} + c$$

$$2 = 1 + c$$

$$c = 1$$

Therefore, $\frac{1}{y} = \tan x + 1$

$$y = \frac{1}{\tan x + 1}$$

Modelling problems

EXAMPLE 9 During a spell of ice rain, the ice on a windscreen has thickness x mm at time t hours after the start of freezing. At 6:00 p.m., after two hours of freezing, the ice on the windscreen is 3 mm thick. The rate of increase of x is proportional to $\frac{1}{x}$.

(a) Set up a differential equation for x.

(b) Using the model, find the time at which the thickness of ice on the windscreen is 5 mm thick.

SOLUTION

(a) Our differential equation is a rate of change. When forming the equation, look for the rate. In this question, the words 'the rate of increase of x' are replaced by $\frac{dx}{dt}$ and 'is proportional to' by the symbol $\propto$. Therefore, $\frac{dx}{dt} \propto \frac{1}{x}$ is our differential equation.

$\frac{dx}{dt} = \frac{k}{x}$, where k is the constant of proportionality

$$\Rightarrow x\,dx = k\,dt$$

$$\Rightarrow \int x\,dx = \int k\,dt$$

$$\Rightarrow \frac{x^2}{2} = kt + c$$

$$\Rightarrow x^2 = nt + a \qquad (n = 2k \text{ and } a = 2c)$$

When $t = 0, x = 0$:

$$\Rightarrow 0 = (0 \times n) + a$$

$$\Rightarrow a = 0$$

When $t = 2, x = 3$:

$\Rightarrow 9 = 2n + 0$

$\Rightarrow n = \frac{9}{2}$

Therefore, $x^2 = \frac{9}{2}t$

(b) $x^2 = \frac{9}{2}t$

When $x = 5$:

$$25 = \frac{9}{2}t$$
$$t = 25 \times \frac{2}{9}$$
$$= \frac{50}{9} \text{ hours}$$
$$= 5\frac{5}{9} \text{ hours}$$

EXAMPLE 10 A point moves on the y-axis so that its coordinates at time t is given by $\frac{dy}{dt} = 4 + h\cos 2t$ for $h \in \mathbb{R}$.

It is observed that $y = 2$ when $t = 0$, and $y = 0$ when $t = \frac{\pi}{4}$. Find the value of h and the value of y when $t = \frac{\pi}{6}$.

SOLUTION $\frac{dy}{dt} = 4 + h\cos 2t$

Separating variables and integrating both sides gives:

$\int dy = \int (4 + h\cos 2t)\,dt$

$\Rightarrow y = 4t + \frac{h}{2}\sin 2t + c$

When $t = 0, y = 2$:

$\Rightarrow 2 = c$

Therefore, $y = 4t + \frac{1}{2}h\sin 2t + 2$

When $y = 0, t = \frac{\pi}{4}$:

$\Rightarrow 0 = \pi + \left(\frac{h}{2} \times 1\right) + 2$

$\Rightarrow h = -2(\pi + 2)$

$\Rightarrow y = 4t - (\pi + 2)\sin 2t + 2$

When $t = \frac{\pi}{6}$:

$y = \frac{4\pi}{6} - (\pi + 2)\sin\left(\frac{\pi}{3}\right) + 2$

$= -0.358$

EXAMPLE 11 A cylindrical container has a height of 100 cm. The container was initially full of water but there is a leak in the base of the container. Ryan noticed the leak in the container when it was dripping at a rate of $\frac{1}{2}$ cm per minute and the container was half full. The rate at which the container is leaking is proportional to $\sqrt{h}$ where h is the depth of the water remaining.

(a) Show that this leads to the differential equation $\frac{dh}{dt} = -\frac{1}{10\sqrt{2}}\sqrt{h}$, where t is the time in minutes.

(b) Obtain the general solution of the equation.

(c) Find t when $h = 50\,\text{cm}$.

SOLUTION

(a) 'The rate at which the container is leaking' can be replaced by $-\frac{dh}{dt}$. We replace 'is proportional to' by $\propto$. The differential equation then becomes:

$$-\frac{dh}{dt} \propto \sqrt{h}$$

$$\Rightarrow -\frac{dh}{dt} = k\sqrt{h}, \text{ where } k \text{ is the constant of proportionality}$$

When $h = 50\,\text{cm}$, $\frac{dh}{dt} = -0.5\,\text{cm min}^{-1}$:

$$\Rightarrow 0.5 = k\sqrt{50}$$

$$\Rightarrow k = \frac{1}{2\sqrt{50}}$$

$$= \frac{1}{2\sqrt{25 \times 2}}$$

$$= \frac{1}{10\sqrt{2}}$$

$$\Rightarrow \frac{dh}{dt} = -\frac{1}{10\sqrt{2}}\sqrt{h}$$

(b) $\frac{1}{\sqrt{h}}\,dh = -\frac{1}{10\sqrt{2}}\,dt$ (Separating variables)

$$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -\frac{1}{10\sqrt{2}}\,dt$$

$$\Rightarrow 2\sqrt{h} = -\frac{1}{10\sqrt{2}}t + c$$

When $t = 0$, $h = 100$:

$$\Rightarrow 2 \times \sqrt{100} = -\frac{1}{10\sqrt{2}} \times 0 + c$$

$$\Rightarrow 2\sqrt{100} = c$$

$$\Rightarrow c = 20$$

Therefore, $2\sqrt{h} = 20 - \frac{1}{10\sqrt{2}}t$

$$\Rightarrow \sqrt{h} = 10 - \frac{1}{20\sqrt{2}}t$$

$$\Rightarrow h = \left(10 - \frac{1}{20\sqrt{2}}t\right)^2$$

(c) When $h = 50$:

$$50 = \left(10 - \frac{1}{20\sqrt{2}}t\right)^2$$

$$\sqrt{50} = 10 - \frac{1}{20\sqrt{2}}t$$

$$\frac{1}{20\sqrt{2}}t = 10 - \sqrt{50}$$

$$t = 20\sqrt{2}\,(10 - \sqrt{50})$$

$$= 82.84 \text{ minutes}$$

EXAMPLE 12 A rat has mass 30 g at birth. The rat reaches maturity in 4 months. The rate of growth of the rat is modelled by the differential equation $\frac{dx}{dt} = 90(t-4)^2$, where x is the mass of the rat t months after birth. Find the mass of the rat when the rat is 2 months old.

SOLUTION

$$\frac{dx}{dt} = 90(t-4)^2$$

Separating variables gives:

$$\int dx = \int 90(t-4)^2\,dt$$

$$x = \frac{90}{3}(t-4)^3 + c$$

$$x = 30(t-4)^3 + c$$

When $t = 0$, $x = 30$:

$\Rightarrow 30 = 30(-4)^3 + c$

$30 = 30(-64) + c$

$\Rightarrow c = 30(65) = 1950$

$\therefore x = 30(t-4)^3 + 1950$

When $t = 2$:

$$x = 30(-2)^3 + 1950 = 1950 - 240 = 1710\text{ g}$$

Second order differential equations

Consider $y = Ax^2 + Bx + C$.

If we differentiate this equation twice we get

$$\frac{dy}{dx} = 2Ax + B$$

$$\frac{d^2y}{dx^2} = 2A$$

The equation in $\frac{d^2y}{dx^2}$ is a second order differential equation.

We can move in reverse by integrating the second order differential equation twice to obtain the function of y. Let us see how this works:

$$\frac{d^2y}{dx^2} = 2A$$

Integrating both sides with respect to x gives:

$$\int \frac{d^2y}{dx^2}\,dx = \int 2A\,dx$$

$\Rightarrow \frac{dy}{dx} = 2Ax + B$, where B is the constant of integration.

Integrating again with respect to x, we now have:

$$\int \frac{dy}{dx}\,dx = \int (2Ax + B)\,dx$$

$\Rightarrow y = Ax^2 + Bx + C.$

By integrating twice we have two constants of integration introduced into our solution for y.

To solve a general second order differential equation of the form $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = f(x)$, we integrate twice and the general solution involves two arbitrary constants.

To determine a specific solution curve for this differential equation one point is not enough, we need a point and the first derivative at that particular point.

EXAMPLE 13 Find the general solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^3 - 6x + 12$.

SOLUTION

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = x^3 - 6x + 12$$

Integrating both sides with respect to x gives:

$$\int \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\,\mathrm{d}x = \int (x^3 - 6x + 12)\,\mathrm{d}x$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}\,\frac{1}{4}x^4 - 3x^2 + 12x + A$$

Integrating again with respect to x gives:

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \left(\frac{1}{4}x^4 - 3x^2 + 12x + A\right)\mathrm{d}x$$

$$\Rightarrow y = \frac{1}{20}x^5 - x^3 + 6x^2 + Ax + B$$

This is the general solution of the differential equation.

Notice that we have two arbitrary constants in this solution.

EXAMPLE 14 Find the solution of the differential equation $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = 4\sin^2 t$, when $t = 0$, $y = 1$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

SOLUTION

Integrating both sides with respect to t:

$$\int \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\,\mathrm{d}t = \int 4\sin^2 t\,\mathrm{d}t$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = \int 4\left(\frac{1 - \cos 2t}{2}\right)\mathrm{d}t$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2\int (1 - \cos 2t)\,\mathrm{d}t$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2\left[t - \frac{1}{2}\sin 2t\right] + A$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 2t - \sin 2t + A$$

Integrating again with respect to t gives:

$$\int \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t = \int (2t - \sin 2t + A)\,\mathrm{d}t$$

$$\Rightarrow y = t^2 + \frac{1}{2}\cos 2t + At + B$$

We now find the constants A and B as follows.

Substituting $t = 0$, $y = 1$ gives:

$$1 = \frac{1}{2} + B$$

$\Rightarrow B = \frac{1}{2}$

$\frac{dy}{dt} = 2t - \sin 2t + A$

When $t = 0, \frac{dy}{dt} = 2, \Rightarrow 2 = A$

The solution is $y = t^2 + \frac{1}{2}\cos 2t + 2t + \frac{1}{2}$.

EXAMPLE 15 A particle moves in a straight line through a fixed point O. Its acceleration is given by $\frac{d^2x}{dt^2} = 3t - 4$ where t is the time in seconds after passing O, and x is the displacement from O. The particle reaches a point A when $t = 2$ and $\frac{dx}{dt} = 3$.

(a) Find $\frac{dx}{dt}$ when $t = 1$.

(b) Find x as a function of t.

SOLUTION

(a) $\frac{d^2x}{dt^2} = 3t - 4$

Integrating with respect to t gives:

$$\int \frac{d^2x}{dt^2}\,dt = \int (3t - 4)dt$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{6}(3t - 4)^2 + c$$

When $t = 2, \frac{dx}{dt} = 3$

$$\Rightarrow 3 = \frac{1}{6}(6 - 4)^2 + c$$

$$\Rightarrow 3 = \frac{4}{6} + c$$

$$\Rightarrow c = 3 - \frac{4}{6} = \frac{7}{3}$$

Therefore, $\frac{dx}{dt} = \frac{1}{6}(3t - 4)^2 + \frac{7}{3}$

When $t = 1, \frac{dx}{dt} = \frac{1}{6} + \frac{7}{3} = \frac{15}{6} = \frac{5}{2}\text{ms}^{-1}$

(b) Since $\frac{dx}{dt} = \frac{1}{6}(3t - 4)^2 + \frac{7}{3}$, we can find x in terms of t by integrating both sides with respect to t.

$$\int \frac{dx}{dt}\,dt = \int \left(\frac{1}{6}(3t - 4)^2 + \frac{7}{3}\right) dt$$

$$\Rightarrow x = \frac{1}{54}(3t - 4)^3 + \frac{7}{3}t + c$$

When $t = 0, x = 0$

$$0 = \frac{-64}{54} + c$$

$$\Rightarrow c = \frac{64}{54} = \frac{32}{27}$$

$$\Rightarrow x = \frac{1}{54}(3t - 4)^3 + \frac{7}{3}t + \frac{32}{27}$$

EXERCISE 17

In questions 1 to 8, find the general solutions.

1 $\frac{dy}{dx} = 4x^3 - 2x + 1$

2 $\frac{dy}{dx} = \sin \frac{1}{2}x$

3 $\frac{dy}{dx} = \cos^2 x$

4 $\frac{dy}{dx} = x(x^2 + 2)$

5 $\frac{dy}{dx} = \frac{y^2}{\sqrt{3x + 1}}$

6 $x^2 \frac{dy}{dx} = y^2$

7 $\sec^2 x \frac{dy}{dx} = \cos^2 x$

8 $\frac{dy}{dx} = y^2 + 2xy^2$

9 Solve the differential equation $\frac{dx}{dt} = \cos t + \sin t$, subject to the condition $x = 0$ when $t = 0$.

10 Given that $\frac{dy}{dx} = (x + 1)^3$, solve the differential equation when $y = 0$ and $x = 2$.

11 Find the solution of the differential equation $\frac{dy}{dt} = \sin 2t$, given that $y = 1$ when $t = \frac{\pi}{4}$.

12 Find the general solution of the differential equation $\frac{dy}{dx} = 2xy^2 - y^2$.

13 Find the general solution of the differential equation $\frac{dy}{dx} = \frac{(4x + 1)^3}{(y - 1)^2}$. (Make y the subject of the formula.)

14 Solve the differential equation $\frac{dy}{dx} = \frac{3x^2 + 4x - 4}{2y - 4}$ given that when $x = 3$, $y = 1$, write y as a function of x.

15 The angular velocity w of a wheel of moment inertia is given by $I\frac{dw}{dt} + \alpha = 0$, where α is a constant. Determine w in terms of t given that $w = w_0$ when $t = 0$.

16 An equation of motion may be represented by the equation $\frac{dv}{dt} = -kv^2$ where v is the velocity of a body travelling in a restraining medium. Show that $v = \frac{v_1}{1 + ktv_1}$ given $v = v_1$ when $t = 0$.

17 The rate of increase in the height of a tamarind tree while growing, after being planted is proportional to $(9 - x)^{\frac{1}{3}}$, where x is the height of the tree in metres at time t years. It is given that when $t = 0$, $x = 1$ and $\frac{dx}{dt} = 1$. Form a differential equation relating x and t, solve this differential equation, and obtain an expression for x in terms of t.

18 Compressed air is escaping from a balloon. The pressure of the air in the balloon at time t is P, and the constant atmospheric pressure of the air outside the

balloon is P_0. The rate of decrease of P is proportional to the square root of the difference in pressure $(P - P_0)$.

(a) Write down a differential equation for this information.

(b) Find, in any form, the general solution of this differential equation.

(c) Given that $P = 5P_0$ when $t = 0$, and that $P = 2P_0$ when $t = 2$, find k in terms of P_0.

(d) Find also P in terms of P_0 and t.

19 In a certain chemical process a substance is being formed, and t minutes after the start of the process there are y grams of the substance present. The rate of increase of y is proportional to $(40 - y)^2$. When $t = 0$, $y = 0$, and $\dfrac{dy}{dt} = 4$.

(a) Write down a differential equation relating y and t.

(b) Solve this differential equation expressing y in terms of t.

(c) Calculate the time taken for the mass to increase from 0 grams to 35 grams.

20 Waste material is dumped along the Beetham in Port of Spain, Trinidad. The dump heap is conical in shape and continually increases as more waste material is added to the top. In a mathematical model, the rate at which the height x of the dump-heap increases is inversely proportional to x^2.

(a) Express this statement as a differential equation relating x and t.

(b) A new dump-heap was started at time $t = 0$, and after 2 years its height was 18 metres. Find the time by which the dump-heap had grown to 30 metres.

21 Find the general solution of each of the following differential equations.

(a) $\dfrac{d^2y}{dx^2} = (3x + 2)^2$

(b) $\dfrac{d^2y}{dx^2} = \sin 2x \cos 2x$

22 Find the general solution of the differential equation $\dfrac{d^2y}{dt^2} = -4t^2 + 5t + 3$.
Hence, find the solution when $t = 0$, $y = 1$ and $\dfrac{dy}{dt} = 1$.

23 Sketch the solution curve to the differential equation $\dfrac{d^2y}{dt^2} = x + 2$, given that $y = 0$ when $x = 0$ and $\dfrac{dy}{dt} = 1$ when $x = 0$.

24 Find the general solution of the differential equation $\dfrac{dx}{dt} = \dfrac{2}{(1 - 2x)^3}$.

25 A particle moving in a straight line passes a fixed point O on the line. The acceleration of the particle $\dfrac{d^2y}{dt^2}$, t seconds after passing O is given by $\dfrac{d^2y}{dt^2} = 13 - 6t$ and x is the displacement from O. When the particle is at O its velocity is $30\,\text{ms}^{-1}$ that is when $x = 0$ and $\dfrac{dx}{dt} = 30$.

Calculate the following.

(a) $\dfrac{dx}{dt}$ when $t = 3$

(b) the value of t when $\dfrac{dx}{dt} = 0$

(c) the value of x when $\dfrac{dx}{dt} = 0$

SUMMARY

Differential equations

First order differential equations:
$$\frac{dy}{dx} = f(x)\,g(y)$$

Separate the variables:
$$\frac{1}{g(y)}\,dx = f(x)\,dx$$

Integrate both sides:
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Do not forget to add the constant of integration.

If x and y are given, use these values to find the constant.

Worded equations

Identify the variables clearly.

Remember a differential equation represents a rate of change.

One variable will be changing with respect to the other: this gives the rate of change.

For example: x changes proportional to t gives:
$$\frac{dx}{dt} = kt$$

Follow the steps for solving first order differential equations.

$$\frac{d^2y}{d^2x} = f(x)$$

Integrate twice to solve. At each stage introduce a constant of integration.

Checklist

Can you do these?

- Identify a first order differential equation.
- Separate the variables of a first order differential equation.
- Find the general solution of a first order differential equation.
- Find the solution of a differential equation given boundary conditions.
- Sketch the solution curve for a differential equation.
- Form and solve a differential equation for a practical problem.
- Solve second order differential equations.

Module 3 Tests

Module 3 Test 1

1 (a) Differentiate with respect to x.

(i) $(x+1)\sqrt{4x^2+1}$ [4]

(ii) $\cos^3(3x-2)$ [4]

(b) (i) Given that $\int_0^2 f(x)\,dx = 8$, evaluate $\int_0^2 x^2 - f(x)\,dx$. [4]

(ii) Find the area enclosed between the curve $y = x^2 + 2$ and the line $y + x = 14$. [6]

(c) A lidless box with square ends is to be made from a thin sheet of metal. What is the least area of the metal for which the volume of the box is 0.064 m^3? [7]

2 (a) (i) Find $\lim\limits_{x\to\frac{5}{4}} \dfrac{64x^3-125}{12x^2-11x-5}$. [5]

(ii) Evaluate $\lim\limits_{x\to 0} \dfrac{\sin 9x}{x}$. [5]

(b) The curve C passes through the point $\left(1, \frac{5}{2}\right)$ and its gradient at any point (x, y) is given by $\dfrac{dy}{dx} = 3x - \dfrac{4}{x^2}$.

(i) Find the equation of C. [3]

(ii) Find the coordinates of the stationary point of C and determine the nature of the stationary point. [4]

(c) (i) Find the general solution of the equation $\dfrac{d^2y}{dx^2} = 4x^3 + 3x^2$. [4]

(ii) Hence, find the solution when $x = 0$, $y = 1$ and $\dfrac{dy}{dx} = 0$. [4]

3 (a) A model for the height, y metres, of a mango tree at time t years after being planted assumes that, while the mango tree is growing, the rate of increase of the height is proportional to $(9-y)^{\frac{1}{3}}$. It is given that when $t = 0$, $y = 1$ and $\dfrac{dy}{dt} = 0.2$.

(i) Form a differential equation connecting y and t. [3]

(ii) Solve the differential equation, expressing y in terms of t. [8]

(iii) Calculate the time taken for the tree to reach half its maximum height. [4]

(b) The function $f(x)$ is such that $f'(x) = 4x^3 + 6x^2 + 2x + k$, where is a constant. Given that $f(0) = 5$ and $f(1) = 10$, find the function $f(x)$. [5]

(c) Given that $y = A\cos 3x + B\sin 3x$, where A are B constants, show that $\dfrac{d^2y}{dx^2} + 9y = 0$. [5]

4 (a) The point A(1, 4) is a point of inflexion on the curve $y = x^3 + bx^2 + x + c$, where b and c are constants.

(i) Find the values of b and c. [5]

(ii) Find the equation of the tangent to the curve at A. [3]

(b) Evaluate $\int_0^1 \frac{x^2 + 2}{(3x^3 + 18x + 1)^3}\,\mathrm{d}x$, using the substitution $u = 3x^3 + 18x + 1$. [8]

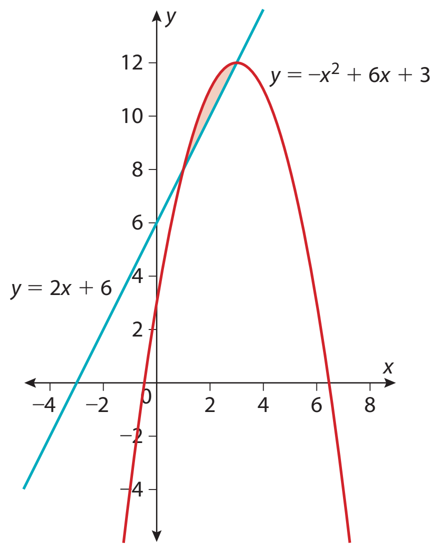
(c) Find the exact value of

(i) $\int_0^{\frac{\pi}{6}} \cos 4\theta \cos 2\theta \,\mathrm{d}\theta$ [5]

(ii) $\int_0^{\frac{\pi}{3}} \cos^2 3x \,\mathrm{d}x$ [4]

Module 3 Test 2

1 (a) The diagram shows the shaded region bounded by the line $y = 2x + 6$ and the curve $y = -x^2 + 6x + 3$. Find the area of the shaded region. [6]



(b) Find the exact volume of the solid formed when the region bounded by the curve $y = \cos 2x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{8}$ is rotated through 2π radians about the x-axis. [7]

(c) (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{x^2 + 4}$. Hence, evaluate $\int_0^2 \frac{12 - 3x^2}{(x^2 + 4)^2}\,\mathrm{d}x$. [6]

(ii) Given that $\int_0^2 4f(x)\,\mathrm{d}x = 12$, find the value of k where $\int_0^2 (kx^3 - 2f(x))\,\mathrm{d}x = 1$. [6]

2 (a) A spherical balloon is being inflated in such a way that its volume is increasing at a constant rate of $300\pi\ \mathrm{cm}^3\ \mathrm{s}^{-1}$. At time t seconds, the radius of the balloon is r cm.

(i) Find the rate at which the radius is increasing when $r = 25$ cm. [3]

(ii) Find the rate of increase of the surface area of the balloon when its radius is 25 cm. [3]

(b) A curve has equation $y = x + \frac{6}{x^2}$.

(i) Show that $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 3$. [5]

(ii) Find the equation of the normal to the curve at the point where $x = 1$. [4]

(c) (i) Given that $y = \sin x^2$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$. Hence or otherwise, evaluate $\int_0^{\frac{\pi}{2}} x\cos x^2\,\mathrm{d}x$. [5]

(ii) Find the general solution of the differential equation $x^3\frac{\mathrm{d}y}{\mathrm{d}x} = x + 1$. [5]

3 (a) Differentiate with respect to x.

(i) $(x^2 + 2)\tan x$ [4]

(ii) $\cos\sqrt{5x^3 - 2x}$ [4]

(b) A function f is defined as follows:

$$f(x) = \begin{cases} x & x < -2 \\ ax + b & -2 \leq x \leq 2 \\ \frac{1}{2}x - 1 & x > 2 \end{cases}$$

Determine the values of a and b such that f is continuous over the interval $-\infty < x < \infty$. [7]

(c) Evaluate the following limits

(i) $\lim\limits_{x \to 4} \dfrac{\sqrt{x + 5} - 3}{x - 4}$ [6]

(ii) $\lim\limits_{x \to 0} \dfrac{\tan 2x}{x}$ [4]

4 (a) Find and classify all maximum points, minimum points and points of inflexion of $y = x^3 - 6x^2 + 9x + 1$. [8]

(b) Given that $\int_0^p (x - 2)^3\,dx = 0$ and $p > 0$, find the value of p. [5]

(c) The equation of a curve C is given by $y = \dfrac{2x + 1}{x - 4}$.

(i) Find the asymptotes of C. [4]

(ii) Show that there are no turning points on C. [4]

(iii) Sketch the graph of C. [4]

Unit 1 Multiple Choice Tests

Multiple Choice Test 1

1 $\dfrac{1}{(\sqrt{3} + \sqrt{2})^2}$ is

A $5 - 2\sqrt{6}$ **B** $5 + 2\sqrt{6}$

C $-5 + 2\sqrt{6}$ **D** $-5 - 2\sqrt{6}$

2 The range of values of x such that $3x^2 - 13x + 14 < 0$ is

A $\{x: x < 2\} \cup \left\{x: x < \frac{7}{3}\right\}$ **B** $\left\{x: 2 < x < \frac{7}{3}\right\}$

C $\{x: x > 2\} \cup \left\{x: x < \frac{7}{3}\right\}$ **D** $\{x: x < -2\} \cup \left\{x : x > \frac{-7}{3}\right\}$

3 Let p be the statement 'Samir learns calculus' and q the statement 'Samir will get an A in calculus'. The statement $p \rightarrow q$ as a statement in words is

A 'Samir will get an A in calculus when he learns calculus'

B 'Samir learns calculus only when he gets an A'

C 'Samir will get an A in calculus if and only if he learns calculus'

D 'Samir does not get an A in calculus if he learns calculus'

4 $\sum_{r=10}^{50} r$ is

A 1230 **B** 2130 **C** 1220 **D** 2120

5 If $x - 2$ is a factor of $4x^3 + ax^2 + 7x + 2$, then a is

A -12 **B** -48 **C** 48 **D** -24

6 $2 \log_e(5p) - 3 \log_e(2f) + 2$ expressed as a single logarithm in its simplest form is

A $\log_e\left(\dfrac{25e^2}{8p}\right)$ **B** $\log_e\left(\dfrac{50p}{8p^3}\right)$ **C** $\log_e\left(\dfrac{25p^2}{8p^3}\right) + 2$ **D** $\log_e\left(\dfrac{25pe^3}{8}\right)$

7 The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of n years, the new population was 240×1.06^n. The year in which the population first reached 2500 was

A 40 **B** 1840 **C** 1839 **D** 1841

8 If $f(x) = 4x - 7$, $x \in \mathbb{R}$ and $fg(x) = x + 1$, then $g(x)$ is

A $\frac{1}{4}x + 2, x \in \mathbb{R}$ **B** $\frac{1}{4}x + \frac{7}{4}, x \in \mathbb{R}$

C $x + \frac{7}{4}, x \in \mathbb{R}$ **D** $x + \frac{1}{2}, x \in \mathbb{R}$

9 The function g is defined by $g(x) = \dfrac{x+1}{x-2}$ and h is defined by $h(x) = \dfrac{ax+3}{x}$, $x \leq 0$. Given that $hg^{-1}(4) = 6$, the value of a is

A 6 **B** 5 **C** 4 **D** -5

10 Given that the roots of $x^3 - 6x^2 + 11x - 6 = 0$ are α, β and γ, the value of $\alpha^2 + \beta^2 + \gamma^2$ is

A 36 **B** -16 **C** 14 **D** -36

11 If the roots of the equation $2x^3 - x^2 + 3x - 4 = 0$ are α, β and γ, then the equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$ is

A $4x^3 - 3x^2 + x - 2 = 0$ **B** $\frac{2}{x^3} - \frac{1}{x^2} + \frac{3}{x} - 4 = 0$

C $3x^3 - x^2 + x - 4 = 0$ **D** $x^3 - x^2 + x - 4 = 0$

12 If $|2x + 1| - 3 = 0$, then x is

A $4, -5$ **B** $-5, 4$ **C** $1, -2$ **D** $3, 2$

13 $\dfrac{16^{x+1} + 4^{2x}}{2^{x-3}8^{x+2}}$ is

A $\frac{17}{8}$ **B** $\frac{13}{8}$ **C** $\frac{8}{13}$ **D** $\frac{17}{3}$

14 If $|2x + 3| + 2x = 1$, then x is

A 2 **B** 1 **C** $-\frac{1}{2}$ **D** -2

15 If the roots of the equation $x^3 + px^2 + qx + r = 0$ are $1, -2$ and 3, then p, q and r are

A $p = 2, q = 5, r = -6$ **B** $p = 5, q = 2, r = 6$

C $p = 1, q = 2, r = 4$ **D** $p = -2, q = -5, r = 6$

16 The general solution of $\sin\theta = \frac{1}{2}$ is

A $n\pi - \frac{\pi}{6}, n \in \mathbb{Z}$ **B** $2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$

C $n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$ **D** $n\pi + (-1)^n\left(\frac{\pi}{6}\right), n \in \mathbb{Z}$

17 The expression $\cos 7\theta + \cos 4\theta$ may be written as

A $\cos 11\theta$ **B** $-2\sin\left(\frac{3\theta}{2}\right)\sin\left(\frac{11\theta}{2}\right)$

C $2\cos\left(\frac{11\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$ **D** $2\cos\left(\frac{11\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$

18 If $4\cos\theta + 3\sin\theta = r\cos(\theta - \propto)$ where $r > 0$ and $0 < \propto < \frac{\pi}{2}$, then the minimum value of the expression is

A $\sqrt{5}$ **B** -5 **C** 5 **D** $-\sqrt{5}$

19 The equation $3x^2 + 6y^2 + 6x - 12y = 0$ is the equation of

A a circle **B** an ellipse **C** a hyperbola **D** a parabola

20 The equation $4x^2 + y^2 - 8x + 4y + 6 = 0$

A a circle with centre $(4, -2)$ **B** an ellipse with centre $(1, -2)$

C an ellipse with centre $(-1, 2)$ **D** a circle with centre $(-4, 2)$

21 The cosine of the angle between the vectors $\begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is

A $\frac{4}{9}$ **B** $\frac{8}{9}$ **C** $\frac{4}{81}$ **D** $\frac{8}{81}$

22 Given that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}$ with respect to an origin O, the value of k for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$ is

A 0 **B** $\frac{3}{14}$ **C** $\frac{-14}{3}$ **D** $\frac{14}{3}$

23 The equation of the line passing through the points $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$ and $\overrightarrow{OB} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ is

A $\mathrm{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$

B $\mathrm{r} = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$

C $\mathrm{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}, \lambda \in \mathbb{R}$

D $\mathrm{r} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 9 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

24 If $\dfrac{\cos(A - B)}{\cos(A + B)} = \dfrac{7}{3}$, then

A $5 \cot A = 2 \cot B$

B $5 \tan A = 2 \cot B$

C $5 \tan A = \cot B$

D $5 \tan B = 2 \tan A$

25 Given that $\tan\theta = t$ and that θ is acute, express $\cot(\theta + 45°)$ in terms of t.

A $\dfrac{2t}{1 + t^2}$ **B** $\dfrac{1 + t}{1 - t}$ **C** $\dfrac{1 - t}{1 + t}$ **D** $\dfrac{1 - t^2}{1 + t^2}$

26 $\dfrac{\tan 2\theta}{1 + \sec 2\theta}$ is

A $\cot\theta$ **B** $\tan\theta$ **C** $\sec\theta$ **D** $\operatorname{cosec}\theta$

27 Given that $\cos A = \frac{3}{4}$, $\cos 4A$ is

A $\frac{1}{8}$ **B** $\frac{1}{32}$ **C** $-\frac{31}{32}$ **D** $-\frac{1}{8}$

28 The curve with parametric equation $x = 2\cos t$, $y = \sin t + 1$ has Cartesian equation

A $4(y - 1)^2 + x^2 = 4$

B $(y + 1)^2 + (x - 1)^2 = 2$

C $(y - 1)^2 + x^2 = 2$

D $(y + 2)^2 = x^2$

Questions **29** and **30** refer to $5(x^2 + y^2) - 4x - 22y + 20 = 0$.

29 The centre and radius of the circle is

A $(2, 11), r = 1$

B $\left(\frac{2}{5}, \frac{11}{5}\right), r = 1$

C $\left(-\frac{2}{5}, \frac{11}{5}\right), r = 1$

D $\left(\frac{-2}{5}, \frac{-11}{5}\right), r = 1$

30 The gradient of the tangent to the circle at $\left(\frac{6}{5}, \frac{8}{5}\right)$ is

A $\frac{3}{4}$ **B** $-\frac{3}{4}$ **C** $\frac{4}{3}$ **D** $-\frac{4}{3}$

31 $\lim_{x \to 0} \frac{\sin 3x}{2x}$ is

A $\frac{1}{2}$ **B** $\frac{3}{2}$ **C** $\frac{2}{3}$ **D** $\frac{1}{3}$

32 If $\lim_{x \to 2} [4f(x)] = 7$, then $\lim_{x \to 2} [f(x) + 2x]$ is

A 11 **B** 4 **C** $\frac{23}{4}$ **D** $\frac{9}{4}$

33 The values of x for which $f(x) = \frac{x+2}{x(x+1)}$ is not continuous are

A 0, 1 **B** $0, -1$ **C** $0, -1, -2$ **D** $-2, -1$

34 $\frac{d}{dx}\sin x^3$ is

A $x^2 \sin x^3$ **B** $\cos x^3$ **C** $3x^2 \cos x^3$ **D** $-3x^2 \cos x^3$

35 If $f(x) = x^2 e^x$, then $f'(0)$ is

A 0 **B** 1 **C** 0.42 **D** 0.61

36 The perimeter of a rectangle has a constant value of 40 cm. One side, of length x cm, is increasing at a rate of $0.5\,\text{cm s}^{-1}$. Find the rate at which the area is increasing at the instant when $x = 3$.

A $5\,\text{cm}^2\,\text{s}^{-1}$ **B** $\frac{1}{5}\,\text{cm}^2\,\text{s}^{-1}$ **C** $\frac{7}{5}\,\text{cm}^2\,\text{s}^{-1}$ **D** $7\,\text{cm}^2\,\text{s}^{-1}$

Questions **37** and **38** refer to this information.

A cuboid has a total surface area of $150\,\text{cm}^2$ and is such that its base is a square of side x cm.

37 The height, h cm, of the cuboid is

A $h = \frac{150 - x^2}{x}$ **B** $h = \frac{75 - x^2}{2x}$

C $h = 150x - x^3$ **D** $h = 125x - x^2$

38 The maximum volume is

A $105\,\text{cm}^3$ **B** $115\,\text{cm}^3$ **C** $125\,\text{cm}^3$ **D** $150\,\text{cm}^3$

39 The coordinates of the turning point of the curve $y = 8x + \frac{1}{2x^2}$ is

A (2, 6) **B** $\left(-\frac{1}{2}, 4\right)$ **C** $\left(\frac{1}{2}, 6\right)$ **D** $\left(-\frac{1}{2}, -6\right)$

40 $\lim_{x \to 0} \frac{\cos 4x - 1}{x}$ is

A $\frac{1}{4}$ **B** 4 **C** 0 **D** ∞

41 $\int 4\cos 6\theta \cos 2\theta \, d\theta$ is

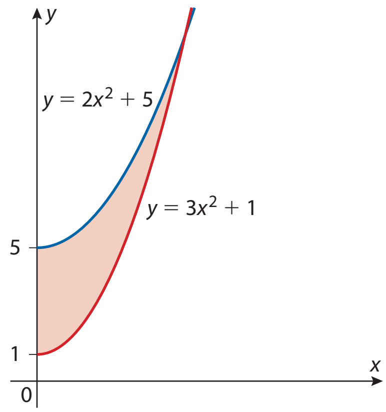
A $\frac{1}{4}\sin 8\theta + \frac{1}{2}\sin 4\theta + c$ **B** $\frac{1}{8}\sin 4\theta + \frac{1}{2}\cos 8\theta + c$

C $\frac{1}{4}\cos 8\theta + \frac{1}{2}\cos 4\theta + c$ **D** $\sin 6\theta \sin 2\theta + c$

42 $\int_0^{\frac{\pi}{2}} \tan^2 2x \, dx$ is

A $\frac{1}{2} - \frac{\pi}{2}$ **B** $-\frac{\pi}{2}$ **C** $\frac{\pi}{2}$ **D** $\frac{\pi}{2} - \frac{1}{2}$

Items **43** and **44** refer to the diagram. The diagram shows parts of the curves $y = 2x^2 + 5$ and $y = 3x^2 + 1$.



43 The area of the shaded region is

A $\frac{14}{3}$ **B** $\frac{16}{3}$

C $\frac{10}{3}$ **D** $\frac{13}{3}$

44 The volume generated when the shaded region is rotate through 360° about the y-axis is

A 25.1 **B** 12.5 **C** 52.1 **D** 125

45 Given that $\int_2^5 f(x)\,dx = 12$, then $\int_2^4 \{f(x) + 4\}\,dx + \int_4^5 f(x)\,dx$ is

A 12 **B** 16 **C** 22 **D** 20

Multiple Choice Test 2

1 The solution set for x in the inequality $|x^2 - 3| < 1$ is

A $\{x: \sqrt{2} < x < 2\}$ **B** $\{x: -2 < x < -\sqrt{2} \text{ or } \sqrt{2} < x < 2\}$

C $\{x: -\sqrt{2} < x < \sqrt{2}\}$ **D** $\{x: -\sqrt{2} < x < \sqrt{2} \text{ or } -2 < x < 2\}$

2 $\log_2 \frac{\sqrt{2}}{8} =$

A $\frac{5}{2}$ **B** $\frac{1}{2}$ **C** $-\frac{5}{2}$ **D** $\frac{3}{2}$

3 If $x - 2$ is a factor of $x^3 - 7x^2 + kx - 12$, then k is

A 16 **B** 2 **C** 32 **D** -32

4 What is the contrapositive of this statement?

'Presentation College Chaguanas win whenever is it raining'

A 'If it is raining, then Presentation College Chaguanas win'

B 'If Presentation College Chaguanas do not win, then it is not raining'

C 'If Presentation College Chaguanas win, then it is raining'

D 'If it is not raining, then Presentation College Chaguanas do not win'

5 The proposition $\sim(p \Rightarrow q)$ is logically equivalent to

Which of the statements below is true?

A $p \Rightarrow q$ **B** $\sim p \Rightarrow \sim q$ **C** $p \wedge \sim q$ **D** $p \vee \sim q$

6 The function f is defined by $f(x) = \frac{2}{3 + x}$. For what values of x is $ff(x)$ undefined?

A $\{-3, 0\}$ **B** $\left\{-3, \frac{-11}{3}\right\}$ **C** $\left\{-\frac{1}{3}, \frac{-11}{3}\right\}$ **D** $\left\{\frac{1}{3}, \frac{11}{3}\right\}$

7 The solution set of $\frac{|x + 1|}{|2x + 1|} = 3$ is

A $\left\{\frac{-2}{5}, \frac{-4}{7}\right\}$ **B** $\left\{\frac{2}{5}, \frac{4}{7}\right\}$ **C** $\left\{\frac{-2}{5}, \frac{4}{7}\right\}$ **D** $\left\{\frac{2}{5}, \frac{-4}{7}\right\}$

8 $\log_3\left(\frac{1}{27}\right)$ is

A 3 **B** $\frac{1}{3}$ **C** $\frac{-1}{3}$ **D** -3

9 The roots of an equation are 2, -3 and $\frac{3}{4}$. What is the equation?

A $x^3 + x^2 - 27x - 18$ **B** $4x^3 - 7x^2 - 27x + 18$

C $4x^3 + x^2 - 27x + 18$ **D** $4x^3 - x^2 + 27x + 18$

10 What is the solution set of $\frac{x}{x-2} < 0$?

A $\{x: x < 0\}$ **B** $\{x: 0 < x < 2\}$

C $\{x: x > 2\}$ **D** $\{x: x < -2\}$

11 $\frac{1}{\sqrt{x+2} - 2}$ can be simplified to

A $\frac{\sqrt{x+2} + 2}{x+2}$ **B** $\sqrt{x+2} + 2$

C $\frac{\sqrt{x+2} - 2}{x-2}$ **D** $\frac{\sqrt{x+2} + 2}{x-2}$

12 If $(3x)^{\frac{3}{2}} + \left(\frac{8}{3}x\right)^{\frac{2}{3}} = 31$, what is x?

A 3 **B** 6 **C** 10 **D** 7

13 The roots of $x^3 + 6x^2 + 11x + 6 = 0$ are

A 1, 2, 3 **B** 1, 2, -3 **C** $-1, -2, -3$ **D** $-1, -2, 3$

14 If α, β and γ are the roots of the equation $x^3 - 2x^2 + 4x - 7 = 0$, then $\alpha^2 + \beta^2 + \gamma^2$ is

A 5 **B** 9 **C** -4 **D** -9

15 The binary operation $*$ on the set of real numbers is defined as: $a * b = 2ab + a - 3$ for any $a, b \in \mathbb{R}$. The identity element for $a \in \mathbb{R}$ is

A $\frac{3+a}{2a+1}, a \neq \frac{-1}{2}$ **B** $\frac{3}{2a}, a \neq 0$

C $3a - 3$ **D** $a - 3$

16 $\sin 9\theta - \sin 3\theta$ is equal to

A $-2\sin 6\theta \cos 3\theta$ **B** $2\cos 6\theta \cos 3\theta$

C $2\sin 6\theta \cos 3\theta$ **D** $2\cos 6\theta \sin 3\theta$

17 If $\theta = \tan^{-1}(2)$, when θ is acute the exact value of $\sin\theta$ is

A $\frac{2}{\sqrt{5}}$ **B** $\frac{-2}{\sqrt{5}}$ **C** $\frac{\sqrt{5}}{2}$ **D** $\frac{-\sqrt{5}}{2}$

Questions **18** to **20** refer to $f(x) = 1 - 6\sin x\cos x + 4\cos^2 x$.

18 $f(x)$ expressed as $K - R\sin(2x - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$ is

A $3 - \sqrt{13}\sin(2x + 33.7°)$ **B** $3 - \sqrt{13}\sin(2x - 33.7°)$

C $3 + \sqrt{13}\sin(2x - 33.7°)$ **D** $-3 - \sqrt{13}\sin(2x + 33.7°)$

19 The maximum value of $f(x)$ is

A $3 + \sqrt{13}$ **B** $3 - \sqrt{13}$ **C** 2 **D** $-3 + \sqrt{13}$

20 When $f(x)$ is a maximum and $0° < x < 180°$, the value of x is

A 16.9° **B** 61.9° **C** 151.9° **D** −16.9°

21 The position vectors of points A, B and C are $-2\mathbf{i} + 2\mathbf{j} - k$, $-3\mathbf{i} + (m + 2)\mathbf{j} - k$, $-2\mathbf{i} + 4\mathbf{j} - 5k$.

The value of m for which $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BC}$ is

A 0 **B** −1 **C** 1 **D** 2

Questions **22** and **23** refer to this information.

A, B and C are points where A is (1, 2, −1), B is (3, 4, 0), and C is (1, 5, −2). O is the origin.

22 $\overrightarrow{AB} \cdot \overrightarrow{AC}$ is

A 5 **B** −5 **C** 6 **D** 7

23 Angle CAB is

A 121.8° **B** 58.2° **C** 85.2° **D** 112.8°

24 With respect to an origin O, the vector $\overrightarrow{OA} = \begin{pmatrix}1\\2\\4\end{pmatrix}$ is on the line l and the vector $\begin{pmatrix}2\\3\\1\end{pmatrix}$ is parallel to l. The Cartesian equation of l is

A $\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\1\end{pmatrix}, \lambda \in \mathbb{R}$ **B** $\dfrac{x-1}{2} = \dfrac{y-2}{3} = z - 4$

C $\dfrac{x-2}{1} = \dfrac{y-3}{2} = \dfrac{z-1}{4}$ **D** $x + 2 = \dfrac{y+2}{3} = \dfrac{z+4}{1}$

25 The line with equation $\mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\2\\0\end{pmatrix}, \lambda \in \mathbb{R}$ is perpendicular to the plane $\mathbf{r} \cdot \mathbf{n} = D$. The plane passes through the point (2, 1, 3). The equation of the plane is

A $2x + y = 5$ **B** $2x + y + 3z = 10$

C $x + 2y - z = 8$ **D** $6x + 3y + z = 12$

26 What is the length of the radius of the circle given by the equation $x^2 + 4x - 2y + y^2 + 2 = 0$?

A $\sqrt{3}$ **B** 3^3 **C** 6^6 **D** 2

27 The Cartesian equation of the curve given by $x = 2\sec t$ and $y = \tan t + 1$ is

A $y = \dfrac{\sqrt{x^2 - 4} - 2}{2}$ **B** $y = \dfrac{\sqrt{x^2 - 4} + 2}{2}$

C $y = \dfrac{\sqrt{x^2 + 4} - 2}{2}$ **D** $y = 2\sqrt{x^2 - 3} + 2$

28 The parametric equations of a curve is $x = 2 + \cos\theta$, $y = 3 + \sin\theta$. The curve is

A a circle with centre (2, 3) **B** an ellipse with centre (2, 3)

C a circle with centre (3, 2) **D** an ellipse with centre (3, 2)

29 The parametric equations of the line passing through the points with position vectors $\begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}$ is

A $\left.\begin{array}{l} x = 2 + 6\lambda \\ y = 1 + 2\lambda \\ z = -4 + \lambda \end{array}\right\} \lambda \in \mathbb{R}$

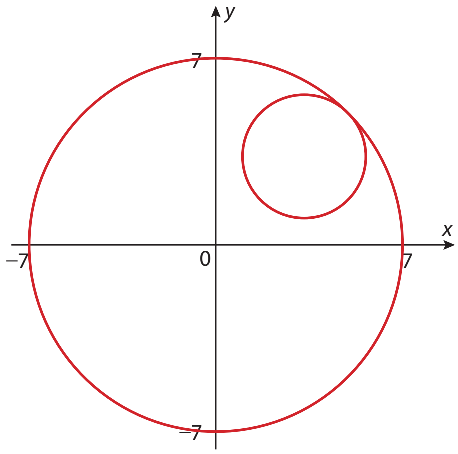
B $\left.\begin{array}{l} x = 2 + 4\lambda \\ y = 1 + \lambda \\ z = -4 + 5\lambda \end{array}\right\} \lambda \in \mathbb{R}$

C $\left.\begin{array}{l} x = 6 + 2\lambda \\ y = 2 + \lambda \\ z = 1 - 4\lambda \end{array}\right\} \lambda \in \mathbb{R}$

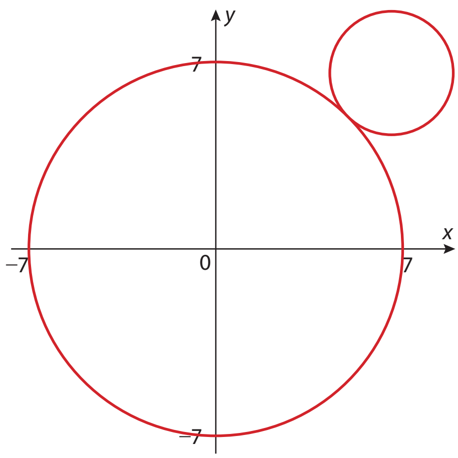
D $\dfrac{x-2}{6} = \dfrac{y-1}{2} = z + 4$

30 Which one of the graphs could represent the system of equations below?
$x^2 + y^2 = 49, x^2 + y^2 - 6x - 8y + 21 = 0$

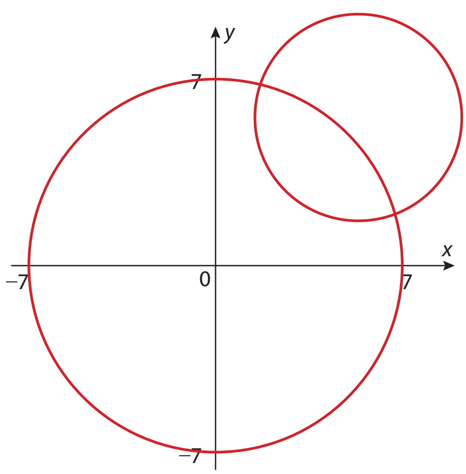
A



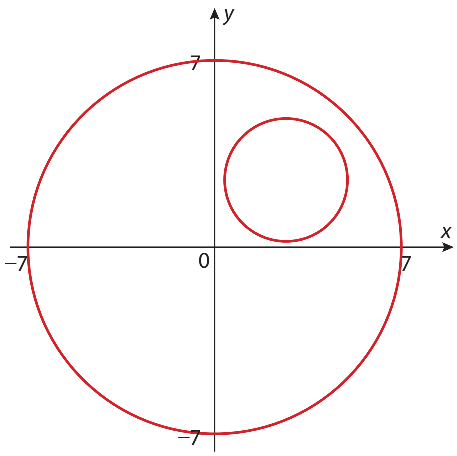
B



C



D



31 $\lim\limits_{x \to 0} \dfrac{\sin 4x}{x}$ is

A $\frac{1}{4}$ **B** 4 **C** 1 **D** 0

32 What is $\lim\limits_{x \to -2} \dfrac{4x^2 + 10x + 4}{3x + 6}$?

A -2 **B** 0 **C** 1 **D** 4

33 In the function $f(x) = \dfrac{2x^2 + 3x + 5}{x^2 - 5x + 5}$, x cannot be

A 3.15 and 6.24 **B** 10.2 and 2.3

C 3.62 and 1.38 **D** 0.25 and 1.37

34 If, $f(x) = \dfrac{x - 1}{x^3 - 3x^2 + 2x}$ for what value(s) of x is $f(x)$ undefined?

A 1 and 2 **B** 0 and 2 **C** 0, 1 and 2 **D** 0, -1 and -2

35 $\lim\limits_{x \to 0} \left(\dfrac{x^2 - 3x}{2}\right) =$

A -3 **B** 0 **C** 2 **D** ∞

36 Given that $f(x) = x\cos 3x$, $f'(x)$ is

A $\cos 3x - 3x\sin 3x$ **B** $\cos 3x + 3x\sin 3x$

C $3x\sin 3x$ **D** $3\sin 3x$

37 The equation of the tangent line to $y = ax^2 + bx - 3$ at $(-4, -31)$ is $y = 9x + 5$. The values of a and b are

A $a = 2, b = 5$ **B** $a = -\frac{1}{2}, b = 5$

C $a = \frac{1}{2}, b = 5$ **D** $a = 2, b = -5$

38 The coordinates of the stationary points on the curve $y = 2x^3 - 9x^2 + 12x$ are

A (1, 5) and (2, 14) **B** (1, 5) and (2, 4)

C $(-1, 5)$ and (2, 4) **D** (1, 5) and $(2, -14)$

39 Given that $f(x) = \dfrac{4}{(2x+1)^3}$, then $f'(x)$ equals

A $\dfrac{-12}{(2x+1)^2}$ **B** $\dfrac{24}{(2x+1)^4}$ **C** $\dfrac{-24}{(2x+1)^4}$ **D** $\dfrac{24}{(2x+1)^{-4}}$

40 The real values of x for which the function $f(x) = 2x^2 + 4x - 3$ is decreasing

A $x < -1$ **B** $x > -1$ **C** $-1 < x < 0$ **D** $x > 4$

41 The area enclosed by the curve $y = x^2$ and $y = 2x - x^2$ is

A $\frac{1}{6}$ **B** $\frac{1}{3}$ **C** $\frac{1}{4}$ **D** $\frac{1}{12}$

42 The equation of the curve for which $\frac{dy}{dx} = 2x + 1$, when $x = 0$, $y = 3$ is

A $y = \frac{1}{4}(2x + 1)^2 + 2\frac{3}{4}$ **B** $y = \frac{1}{4}(2x + 3)^2 + 12$

C $y = x^2 + x$ **D** $y = x^2 + 2x + 3$

43 $\int_0^1 x^2(5 - \sqrt{x})\,dx$ is

A $\frac{29}{21}$ **B** $\frac{29}{7}$ **C** $\frac{29}{3}$ **D** $\frac{41}{21}$

44 Find the volume of the solid generated by revolving the region bounded by the graphs $y^2 = 5x + 1$, $x = 0$ and $x = 2$ through a radius of 2π about the x-axis

A 60π **B** 6π **C** 12π **D** 120π

45 The rate of growth of a population (P) of insects is directly proportional to the population, P, of the insects at time t. A model for this growth is given by

A $\frac{dP}{dt} = kP, k > 0$ **B** $\frac{dP}{dt} = kP, k < 0$

C $\frac{dP}{dt} = \frac{k}{P}, k > 0$ **D** $\frac{dP}{dt} = \frac{k}{P}, k < 0$

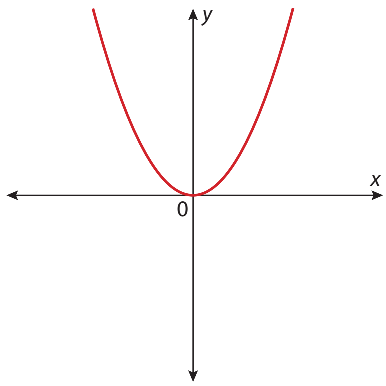
Multiple Choice Test 3

1 If $f(x) = 4x - 2$, then $f^2(2)$ is equal to

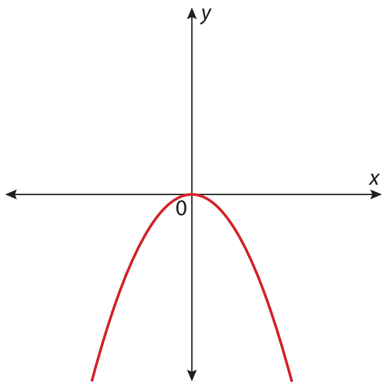
A 6 **B** 22 **C** 2 **D** 42

2 Which of the graphs below represents the function $y = \frac{3}{x^2}$?

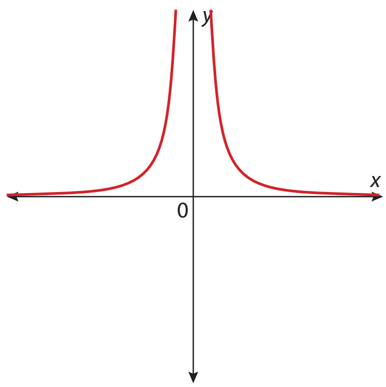
A



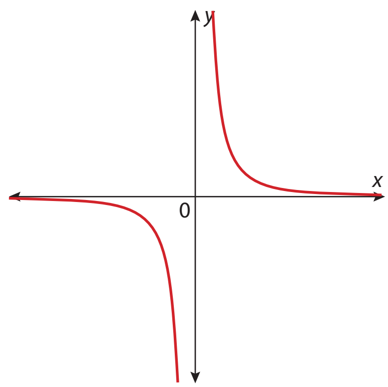
B



C



D



3 $p \Leftrightarrow q$ is logically equivalent to

A $\sim p \Leftrightarrow q$ **B** $p \Leftrightarrow \sim q$ **C** $\sim p \Leftrightarrow \sim q$ **D** $\sim p \Rightarrow q$

4 The set of values of x for which $\frac{x+3}{2x-1} > 0$ is

A $\{x: x < -3\} \cup \left\{x: x > \frac{1}{2}\right\}$ **B** $\{x: x < -3\}$

C $\left\{x: -3 < x < \frac{1}{2}\right\}$ **D** $\left\{x: -\frac{1}{2} < x < 3\right\}$

5 The binary operation $*$ over the set of real numbers is defined by:
$a * b = a + b - 2ab$ for any $a, b \in \mathbb{R}$. The identity element is

A 1 **B** 0 **C** $\frac{1}{1-2a}$ **D** $\frac{a}{2a-1}$

6 $\sum_{r=1}^{20}(3r + 2)$ is

A 670 **B** 632 **C** 631 **D** 1300

7 If $\log\sqrt{x^2 - 9} + \frac{1}{2}\log\left(\frac{x+3}{x-3}\right)$ is expressed as a single logarithm in its simplest form, the result is

A $\log(x - 3)$ **B** $\frac{1}{2}\log(x + 3)$

C $\frac{1}{2}\log(x + 3)(x - 3)$ **D** $\log(x + 3)$

8 What is the range of values of x for which $2x^2 - 5x - 3 \geq 0$?

A $\left\{x: -\frac{1}{2} \leq x \leq 3\right\}$ **B** $\left\{x: -3 \leq x \leq \frac{1}{2}\right\}$

C $\left\{x: x \leq -\frac{1}{2}\right\} \cup \{x: x \geq 3\}$ **D** $\left\{x: x \geq -\frac{1}{2}\right\} \cup \{x: x \leq 3\}$

9 The graph of $y = (x - 3)^2 + 2$ is a translation of $y = x^2$ by vector

A $\begin{pmatrix}3\\2\end{pmatrix}$ **B** $\begin{pmatrix}-3\\2\end{pmatrix}$ **C** $\begin{pmatrix}-3\\-2\end{pmatrix}$ **D** $\begin{pmatrix}3\\-2\end{pmatrix}$

10 If α, β and γ are the roots of the equation $x^3 - 4x^2 + 6x - 8 = 0$, the equation whose roots are $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$

A $x^3 - x^2 + \frac{3}{2}x - 2 = 0$ **B** $8x^3 - 6x^2 + 4x - 1 = 0$

C $\frac{1}{8}x^3 - \frac{1}{4}x^2 + \frac{1}{6}x - 1 = 0$ **D** $x^3 + 4x^2 - 6x + 8 = 0$

11 The sum of the squares of the roots of the equation $6x^3 - 3x^2 - 3x + 2 = 0$ is

A $-\frac{5}{4}$ **B** $-\frac{1}{4}$ **C** $\frac{1}{4}$ **D** $\frac{5}{4}$

12 $|2x - 1| > 5$ can be expressed as

A $-2 < x < 3$ **B** $x < -2, x > 3$

C $x < -2, x > -3$ **D** $|x| < 3, |x| > 2$

13 If the range of the function $f(x) = \frac{4}{x} - 3$ is $\left\{1, -1, -\frac{11}{5}, -\frac{5}{2}\right\}$, the domain is

A $\{1, 2, 5, 8\}$ **B** $\left\{1, -7, -\frac{53}{11}, -11\right\}$

C $\left\{1, \frac{1}{2}, \frac{1}{5}, \frac{1}{8}\right\}$ **D** $\left\{1, -1, -\frac{5}{11}, -2\right\}$

14 When $\sqrt{2} + \sqrt{36} + \sqrt{72}$ is expressed in the form $a + b\sqrt{c}$, where a, b and c are rational numbers, the result is

A $6 + 5\sqrt{2}$ **B** $6 + 12\sqrt{2}$ **C** $6 + 7\sqrt{2}$ **D** $6 + 8\sqrt{2}$

15 If $P(x) = x^3 - ax^2 + 2x + 5$ and $x - 2$ is a factor of $P(x)$, then the value of a is

A $-\frac{17}{4}$ **B** 4 **C** $\frac{17}{4}$ **D** $\frac{4}{17}$

16 $\frac{1 - \cos 2\theta}{\sin 2\theta} \equiv$

A $\frac{\sin 2\theta}{1 + \cos 2\theta}$ **B** $\cot\theta$ **C** $\tan 2\theta$ **D** $\tan\theta$

17 Given that A and B are in the first quadrant and $\cos A = \frac{4}{5}$, $\cos B = \frac{5}{13}$, the value of $\cos(A + B)$ is

A $-\frac{16}{65}$ **B** $\frac{56}{65}$ **C** $\frac{33}{65}$ **D** $\frac{63}{65}$

18 When $3\cos\theta + 4\sin\theta$ is converted to $r\cos(\theta - \alpha)$, the result is

A $5\cos\left(\theta + \arctan\frac{4}{3}\right)$ **B** $5\cos\left(\theta - \arctan\frac{3}{4}\right)$

C $5\cos\left(\theta + \arctan\frac{3}{4}\right)$ **D** $5\cos\left(\theta - \arctan\frac{4}{3}\right)$

19 The maximum and minimum values of $6 + 3\sin\theta$ are respectively

A 6 and 3 **B** 9 and 3 **C** -3 and 3 **D** 3 and -3

20 The Cartesian equation of the curve represented parametrically by $x = 2 + 3\cos\theta$, $y = 3 + 2\sin\theta$ is

A $(x - 2)^2 + (y - 3)^2 = 36$

B $4x^2 + 9y^2 - 16x - 54y + 61 = 0$

C $9x^2 + 4y^2 - 16x - 54y + 61 = 0$

D $\frac{(x - 3)^2}{4} + \frac{(y - 2)^2}{9} = 1$

21 Which of the following equations represents circles?

I $x^2 + y^2 - 2x - 4y - 8 = 0$

II $x^2 + y^2 + 2xy - 4y - 9 = 0$

III $3x^2 + 3y^2 - 12x + 15 = 0$

A I only **B** I and II only

C I and III only **D** I, II and III

22 The equation $x^2 + y^2 - 2x - 4y - 4 = 0$ is a circle with

A centre $(1, 2)$, radius $= 3$ **B** centre $(-1, 2)$, radius $= 9$

C centre $(1, 2)$, radius $= 9$ **D** centre $(-1, -2)$, radius $= 3$

23 $\cos^4 x - \sin^4 x \equiv$

A $\sin 2x$ **B** $-\sin 2x$ **C** $\cos 2x$ **D** $2\cos^2 x + 1$

24 Expressed as a product of two trigonometric functions, $\cos 6\theta + \cos 4\theta$ is

A $2\sin 5\theta \cos\theta$ **B** $2\cos 5\theta \sin\theta$

C $2\cos 10\theta \cos 2\theta$ **D** $2\cos 5\theta \cos\theta$

25 Given that $\mathbf{a} = \begin{pmatrix}1\\4\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-2\\1\\p+3\end{pmatrix}$, the value of p for which **a** is perpendicular to **b** is

A $\frac{5}{17}$ **B** $\frac{17}{5}$ **C** $\frac{-5}{17}$ **D** $\frac{-17}{5}$

26 With respect to an origin O, the points P and Q have position vectors $\overrightarrow{OP} = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ and $\overrightarrow{OQ} = 6\mathbf{i} + 12\mathbf{j} - 5\mathbf{k}$ respectively. The midpoint of $\overrightarrow{PQ}$ is

A $4\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ **B** $2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$

C $4\mathbf{i} + 6\mathbf{j} - 4\mathbf{k}$ **D** $8\mathbf{i} + 18\mathbf{j} - 6\mathbf{k}$

27 The distance from the origin to the plane with equation $x + 2y + 2z = 12$ is

A 3 **B** 4 **C** 12 **D** 6

28 The circle $x^2 + y^2 - 4x - 2y + 1 = 0$ has the point (4, 1) at the end of a diameter. The coordinates of the other end of the diameter are

A (6, 2) **B** (3, 1) **C** (0, 1) **D** (1, 0)

29 The general solution of $\cos 2\theta = \frac{1}{2}$ is

A $\theta = 180°n \pm 30°, n \in \mathbb{Z}$ **B** $\theta = 360°n \pm 60°, n \in \mathbb{Z}$

C $\theta = 360°n \pm 30°, n \in \mathbb{Z}$ **D** $\theta = 180°n \pm 60°, n \in \mathbb{Z}$

30 $\cos 75° \cos 15° - \sin 75° \sin 15°$ is equal to

A $\frac{\sqrt{3}}{2}$ **B** $\frac{1}{2}$ **C** $-\frac{1}{2}$ **D** 0

31 $\lim_{x \to 0} \frac{\sin 6x}{x} =$

A $\frac{1}{6}$ **B** 6 **C** 1 **D** 0

32 $\lim_{x \to 3} \frac{x^2 - x - 6}{2x^2 - 5x - 3} =$

A 0 **B** $\frac{7}{5}$ **C** $\frac{5}{7}$ **D** ∞

33 $\lim_{x \to 3} \frac{\sqrt{x+1} - 2}{x - 3} =$

A 0 **B** $\frac{1}{4}$ **C** 4 **D** ∞

34 $\int \frac{x^7 - 2}{x^4}\,dx$ is equal to

A $\frac{x^4}{4} - \frac{2}{x^3} + c$ **B** $x^3 - \frac{2}{x^4} + c$ **C** $\frac{x^3}{3} - \frac{1}{x^4} + c$ **D** $\frac{x^4}{4} + \frac{2}{3x^3} + c$

35 The range of values of x for which the function $f(x) = x^3 - 3x^2 + 2$ is decreasing is

A $0 < x < 2$ **B** $x > 0, x > 2$ **C** $-2 < x < 0$ **D** $x < -2$

36 Given that $f(x) = \cos^3 x$, $f'(x)$ is

A $3\cos^2 x$ **B** $3\sin^2 x$ **C** $3\cos^2 x \sin x$ **D** $-3\cos^2 x \sin x$

37 $\int 2\sin 8\theta \cos 4\theta \, d\theta$

A $\sin 12\theta + \sin 4\theta + c$ **B** $-\frac{1}{12}\cos 12\theta - \frac{1}{4}\cos 4\theta + c$

C $\frac{1}{12}\sin 12\theta + \frac{1}{4}\sin 4\theta + c$ **D** $\frac{1}{12}\cos 12\theta + \frac{1}{4}\cos 4\theta + c$

38 The function $f(x) = \dfrac{x+1}{|x|^2 - 4}$ is discontinuous when x takes the values

A $2, -2$ **B** $4, -4$ **C** $-1, 4$ **D** $1, -4$

39 The volume of the solid generated when the region enclosed by the curve $y = x^2 + 1$, the x-axis and the line $x = 2$ is rotated 360° about the x-axis is

A $\frac{112}{15}\pi$ **B** $\frac{8}{15}\pi$ **C** $\frac{142}{15}\pi$ **D** $\frac{206}{15}\pi$

40 $\lim_{x \to 0} \dfrac{20\sin 10x}{\cos 6x \sin 6x} =$

A 20 **B** $\frac{120}{3}$ **C** $\frac{100}{3}$ **D** $\frac{20}{6}$

41 Given that the phaser description of an alternating current is $\varphi(x) = \sec^4\left(x + \frac{\pi}{4}\right)$. What is $\varphi''(x)$ at the stage where $x = \frac{\pi}{12}$?

A 16 **B** $128\sqrt{3}$ **C** $8\sqrt{3}$ **D** 1024

42 Given that $\int_0^{\frac{\pi}{2}} \cos x \, dx = \int_0^{\frac{\pi}{2}} \cos y \, dy$, what is y?

A $\frac{\pi}{3} - x$ **B** $\pi - x$ **C** $x - \pi$ **D** $2x$

43 A curve C is defined by $y = \dfrac{1}{x^{\frac{3}{2}}}$ for $x \in (1, \infty)$. The area between C and the x-axis is

A 0 **B** 1 **C** 2 **D** ∞

44 Given that $\int_0^c 4(2x+1)^3 \, dx = \frac{15}{2}$, the value of c is

A 16 **B** 3 **C** $\frac{1}{2}$ **D** 2

45 What is the equation of a curve which passes through (2, 0) and whose tangent at a point (x, y) has a gradient of $\frac{x^2}{y^2}$?

A $y^3 = x^3$ **B** $y^2 = x^2 - 8$ **C** $y^3 = 3x^3 - 8$ **D** $y^3 = x^3 - 8$

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