

12. COORDINATE GEOMETRY

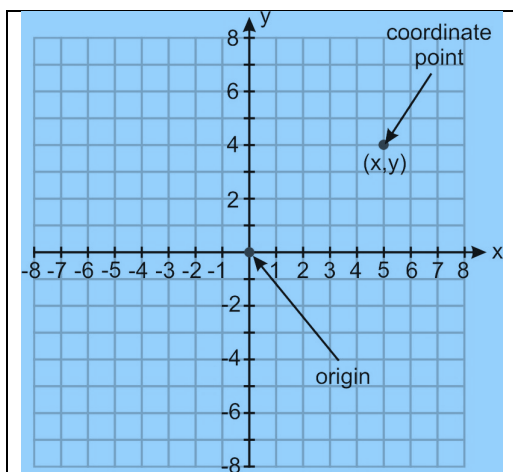
THE CARTESIAN PLANE

We recall from our study of plane geometry that a plane is a two-dimensional region. We now wish to describe the position of a point on a plane. To do so, we use the Cartesian Coordinate system.

The plane consists of a horizontal number-line called the **x-axis**, and a vertical number-line called the **y-axis**. The point of intersection of these two axes is the origin, the reference point from which all positions are measured. We can describe any point in the plane using an ordered pair of numbers with reference to these axes.

The Cartesian plane is really infinite and so we use arrows at the ends of the axes when we represent it. This is illustrated in Figure 1.

A point in the plane is located by stating its coordinates, a pair of numbers enclosed in parentheses: (x, y) . The first number, x , gives its horizontal distance of the point from the origin and the second number, y , gives its vertical distance of the point from the origin. All displacements are measured relative to the origin, O , whose coordinates are $(0, 0)$. In the figure below, the point shown has coordinates $(5, 4)$.

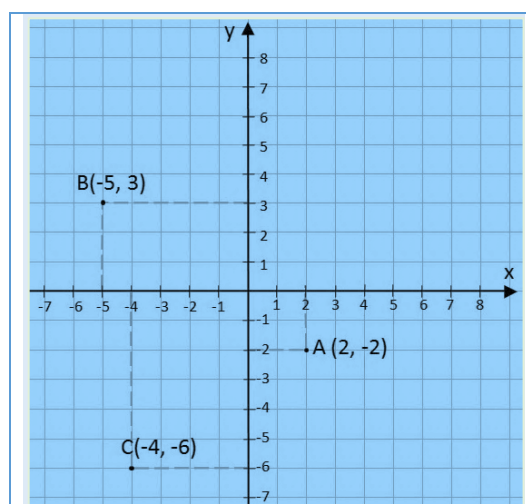


The Cartesian Plane is divided into four quadrants. The point $(5, 4)$ lies in the first quadrant and both coordinates are positive. This is because both the horizontal and vertical displacements are positive.

The point $B(-5, 3)$ lies in the second quadrant. The displacement parallel to the x -axis is -5 units and the displacement parallel to the y -axis is 3 units.

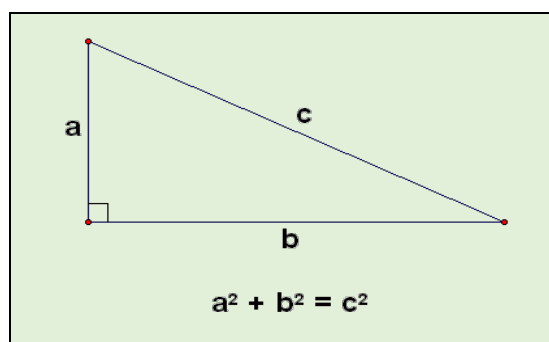
The point, $C(-4, -6)$ lies in the third quadrant. The displacement parallel to the x -axis is -4 units and the displacement parallel to the y -axis is -6 units.

The point $A(2, -2)$ in the diagram below lies in the fourth quadrant. The displacement parallel to the x -axis is positive, but the displacement parallel to the y -axis is negative.



Length of a straight line

In our study of geometry, we noted that in right-angled triangles, there is a relationship between the lengths of the three sides. If we know the length of any two sides, the third side may be calculated by the use of Pythagoras' Theorem. The diagram below illustrates the relationship between all the sides of a right-angle triangle as stated in the theorem.

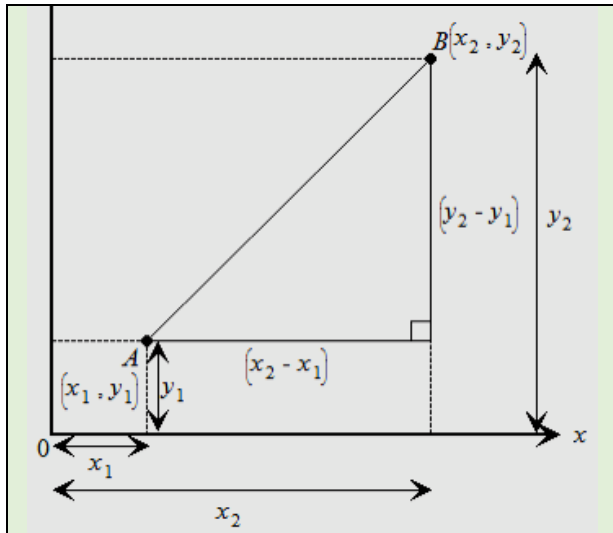


We will use this relationship to calculate the length of a line on the Cartesian plane from coordinates. In the diagram below, let A and B represent two points, $A(x_1, y_1)$ and $B(x_2, y_2)$.

The vertical and the horizontal distances can be calculated as follows:

$$\text{Vertical Side} = y_2 - y_1$$

$$\text{Horizontal Side} = x_2 - x_1$$



By Pythagoras' Theorem, the length of the straight line is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If A and B represent two points, such that $A(x_1, y_1)$ and $B(x_2, y_2)$, then the distance AB is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1

Given $A = (2, 3)$ and $B = (5, 7)$ calculate the length of AB .

Solution

$$\begin{aligned} \text{The length of a line is } & \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\ \text{Hence, the length of } AB &= \sqrt{(5 - 2)^2 + (7 - 3)^2} \\ &= \sqrt{(3)^2 + (4)^2} \\ &= \sqrt{9 + 16} \\ &= \sqrt{25} = 5 \text{ units.} \end{aligned}$$

Example 2

Calculate the length of PQ , where $P = (1, -1)$ and $Q = (3, 5)$.

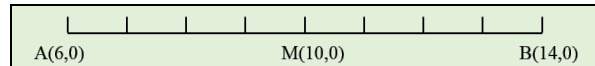
Solution

$$\begin{aligned} \text{Length of } PQ &= \sqrt{(3 - 1)^2 + (5 - (-1))^2} = \sqrt{(2)^2 + (6)^2} \\ &= \sqrt{40} \quad (\text{exact}) \\ &= 6.32 \text{ units (correct to 2 dec places)} \end{aligned}$$

As shown in the example above, the length of a line may not compute to an exact integer value and sometimes we may have to use a calculator to approximate this length to any required degree of accuracy or simply leave the exact answer in a *surd* form.

Midpoint of a straight line

If a point, M is midway between two other points A and B , its distance is the arithmetic average of the coordinates.

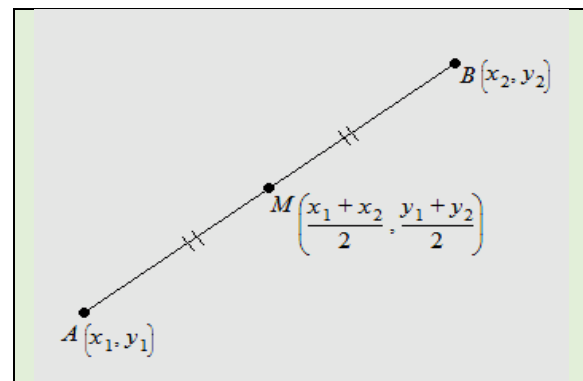


Since the x -coordinates of A and B are 6 and 14 respectively, the x coordinate of M is $\frac{6+14}{2} = 10$.

We can use this principle to determine the mid-point of any line given the coordinates of any two points on the line.

Let A and B represent two points,

$A(x_1, y_1)$ and $B(x_2, y_2)$. Let the mid-point be $M(\bar{x}, \bar{y})$. The x -coordinate of M is the average of x_1 and x_2 . The y -coordinate of M is the average of y_1 and y_2 .



In general, the midpoint of the straight line joining (x_1, y_1) and (x_2, y_2) has coordinates:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right).$$

Example 3

If $A = (3, 4)$ and $B = (7, 6)$, calculate the coordinates of the mid-point of AB .

Solution

Applying the formula for the mid-point $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.
Hence, the midpoint of AB is $\left(\frac{3+7}{2}, \frac{4+6}{2} \right) = (5, 5)$

Example 4

If $P = (1, -3)$ and $Q = \left(\frac{1}{2}, 4 \right)$, calculate the coordinates of the mid-point of PQ .

Solution

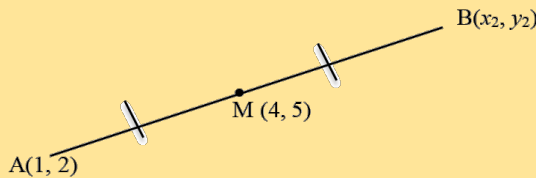
$$\text{Midpoint of } PQ = \left(\frac{1 + \frac{1}{2}}{2}, \frac{-3 + 4}{2} \right) = \left(\frac{3}{4}, \frac{1}{2} \right)$$

Example 5

Given that $A(2, 3)$ and $M(4, 5)$, where M is the midpoint of AB . Find the coordinates of B .

Solution

Let $A(x_1, y_1)$ and $B(x_2, y_2)$



We recall the midpoint formula: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$

$$4 = \frac{1 + x_2}{2} \qquad 5 = \frac{2 + y_2}{2}$$

$$1 + x_2 = 8 \qquad 2 + y_2 = 10$$

$$x_2 = 8 - 1 \qquad y_2 = 10 - 2$$

$$x_2 = 7 \qquad y_2 = 8$$

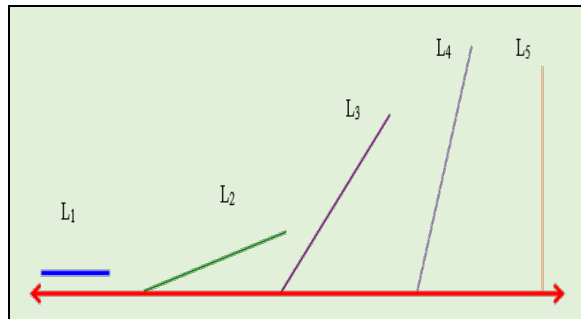
Therefore, B is the point with coordinates, $(7, 8)$.

Gradient of a straight line

When we speak of the gradient or the slope of a straight line, we refer to the measure of the steepness of the line. Lines can have varying degrees of steepness depending on their orientation.

Consider the five lines, shown below, L_1, L_2, L_3, L_4 and L_5 . We may observe that all five lines have different degrees of steepness.

Examine the lines L_2, L_3 and L_4 . If we were to order these three lines in ascending order of steepness, then we can deduce, from observation, that L_2 is the least steep of all three, then L_3 , and after that, L_4 is the steepest of all three.



L_1 and L_5 are special lines. L_1 is a horizontal line and is regarded as having no steepness. L_5 is a vertical line and has the maximum possible steepness.

In mathematics, we need to be very precise when comparing the steepness of lines. We can only do so if we measure the steepness and assign numbers to this measure. We refer to this measure of **steepness**, as the **slope** or the **gradient** of the straight line.

We can think of gradient as a rate of change of the vertical displacement (or **rise**) with respect to its horizontal displacement (or **run**, sometimes referred to as **step**). When both the horizontal and vertical distances are expressed in the same units, we can express the gradient as a ratio of the rise to the run.

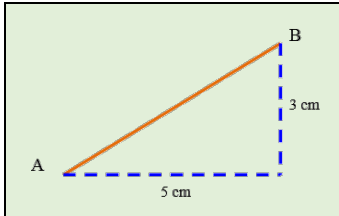
We can think of gradient, which is often denoted by the letter m , in mathematics, as any one of the following ratios.

$$\text{Gradient}(m) = \frac{\text{Vertical Displacement}}{\text{Horizontal Displacement}}$$

$$\text{Gradient}(m) = \frac{\text{Change in } y}{\text{Change in } x}$$

Calculating gradient using measurement

If the line is not drawn on a grid, then we would have to take measurements (likely by a ruler) to determine the gradient.



By measurement,
 Vertical displacement = 3cm
 Horizontal displacement = 5cm
 Gradient = $\frac{\text{Vertical Displacement}}{\text{Horizontal Displacement}} = \frac{3}{5}$

Gradient from coordinates

We can use the coordinates of any two points on a line to compute the gradient. This method does not require the use of a graph and it is widely used in coordinate geometry.

Given A (x_1, y_1) and B (x_2, y_2) are any two points on the line. The vertical displacement represents a change in y values and the horizontal displacement represents a change in x values. Hence the gradient of AB is calculated as follows:

$$\text{Gradient} = \frac{\text{Change in } y}{\text{Change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

In general, the gradient of the straight line joining A (x_1, y_1) and B (x_2, y_2) is $\frac{y_2 - y_1}{x_2 - x_1}$

Example

Calculate the gradient of a line joining the two points (1, 4) and (6, 7).

Solution

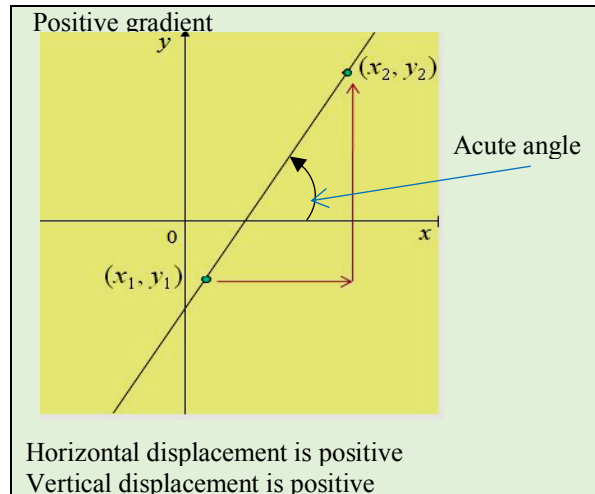
We let (x_1, y_1) and (x_2, y_2) represent the coordinates (1, 4) and (6, 7) respectively. We now substitute the coordinates in the formula to obtain:

$$\text{Gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 4}{6 - 1} = \frac{3}{5}$$

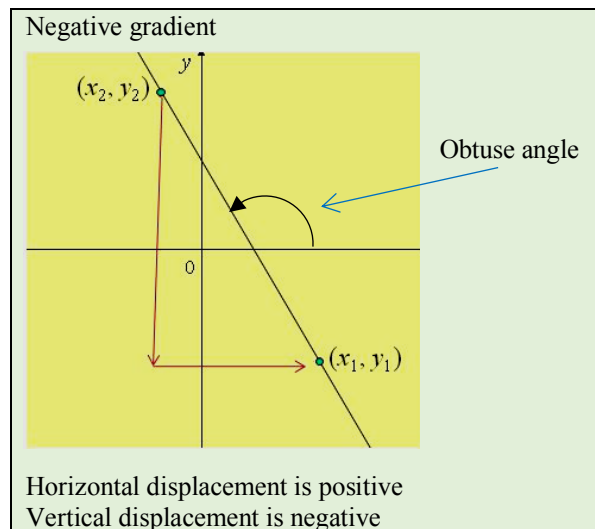
Positive and negative gradient

When we measure gradient, our resulting ratio can be positive or negative. The numerical value is the magnitude of the gradient and the sign (positive or negative) is the direction of the gradient.

When both vertical and horizontal displacements have the same direction the gradient of the line is positive.



When one of the displacements is negative, the gradient of the line is negative.



Positive and negative gradients can also be identified by considering the angle formed with the positive x -axis. When this angle is acute the gradient is positive and when it is obtuse the gradient is negative. Notice that when the slope is positive, the angle is acute and when the slope is negative the angle is obtuse. Note also that the direction of turn is anticlockwise.

Gradient of horizontal and vertical lines

The following should be noted:

- A horizontal line has no steepness and its gradient is defined as zero. This is because it has no 'rise' regardless of the 'run' and $(0 \div \text{any number}) = 0$
- A vertical line has maximum steepness and its gradient is infinite. This is because it has 'rise' but no 'run' and $(\text{any number}) \div 0 \rightarrow \infty$

Horizontal lines have run, but absolutely no rise.

$$\text{And so the gradient} = \frac{\text{Zero}}{\text{Run}} = 0$$

Vertical lines have rise, but absolutely no run.

$$\text{And so the gradient} = \frac{\text{Rise}}{\text{Zero}} = \infty$$

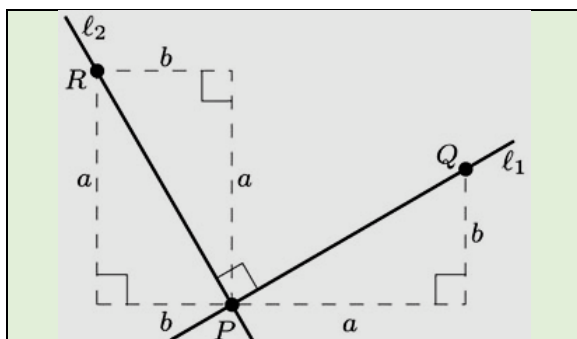
Gradient of parallel lines

Parallel lines have the same steepness and therefore have the same or equal gradients. This is easy to see since they slant in the same direction and the ratio of vertical rise to the horizontal run is the same.

Conversely, lines that have the same gradient are parallel to each other.

Gradient of perpendicular lines

To demonstrate the relationship between the gradients of perpendicular lines, we need some knowledge of rotation. If we rotate a line through 90° about a fixed point, we observe that the gradient will change from positive to negative. Also, the ratio of rise to run is inverted as the line turns through 90° . The diagram below illustrates this principle.



$$\text{Gradient of } PQ = \frac{b}{a} \quad \text{Gradient of } PR = -\frac{a}{b}$$

$$\text{Product of the gradients} = \frac{b}{a} \times -\frac{a}{b} = -1$$

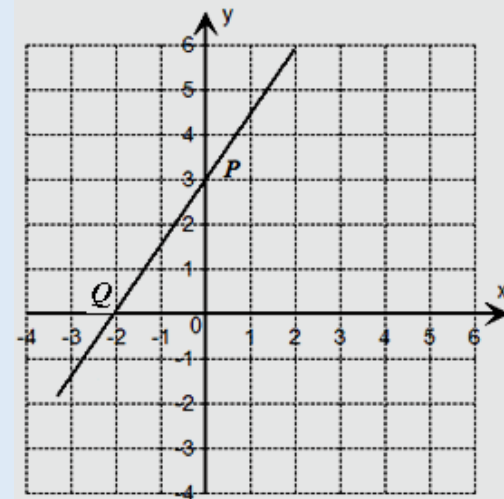
Conversely, lines whose product of their gradients is negative one are perpendicular to each other.

If two lines have gradients m_1 and m_2 , then

- the lines are parallel when $m_1 = m_2$
- the lines are perpendicular when $m_1 \times m_2 = -1$

Example 6

The line PQ cuts the x -axis at P and the y -axis at Q .



- State the coordinates of P and of Q
- Calculate the gradient of PQ
- Calculate the gradient of a line PS that is perpendicular to PQ

Solution

- Line cuts the y -axis at 3, $P(0, 3)$
Line cuts the x -axis at -2 , $Q(-2, 0)$
- Gradient of $PQ = \frac{3}{2}$ (line a rise of $+3$ units and a run of $+2$ units).
- The product of the gradients of perpendicular lines is -1 . Let gradient of PS be m_2

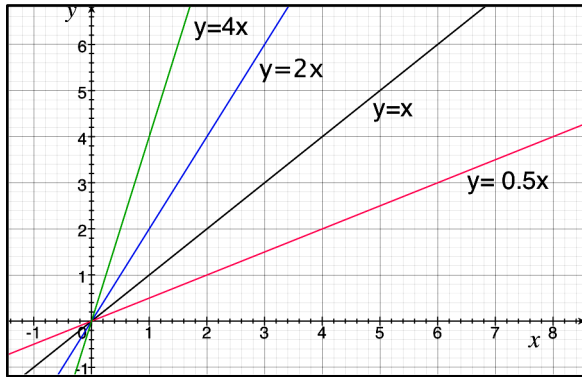
$$\frac{3}{2} \times m_2 = -1$$

$$m_2 = -\frac{2}{3}$$

Our knowledge of the gradient or steepness of a straight line is important in helping us to understand other properties of a straight line. We will now revisit the Cartesian Plane and observe other features of a straight line besides its gradient.

THE EQUATION OF A STRAIGHT LINE

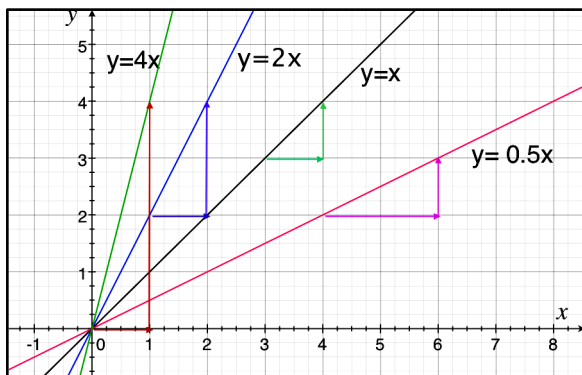
The equation of a straight line is an expression of the relationship between the x and y values in a linear equation. Let us examine a family of straight lines drawn through the origin but having different gradients.



The relationship between the x and y coordinates for each line is stated below.

Equation of line	Points on the line
$y = \frac{1}{2}x$	(2, 1) (4, 2) (6, 3) the y - coordinate is one half of the x - coordinate.
$y = x$	(1, 1) (2, 2) (3, 3) the y - coordinate is equal to the x - coordinate.
$y = 2x$	(1, 2) (2, 4) (2.5, 5) the y - coordinate is always twice the x - coordinate.
$y = 4x$	(0.25, 1) (0.5, 2) (1, 4) the y - coordinate is always four times the x - coordinate.

We now determine the gradient of each line by drawing a right-angled triangle and recording the rise over the run. This is shown below.



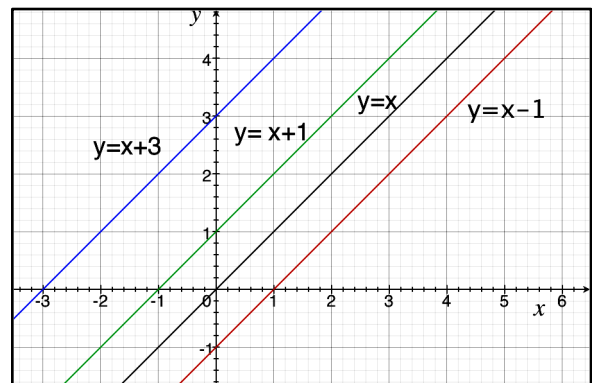
All the displacements are positive and so the ratio of vertical rise to horizontal run is positive. The magnitude of the gradient is recorded below.

Equation of line	Gradient of the line
$y = \frac{1}{2}x$	$\frac{1}{2}$
$y = x$	1
$y = 2x$	2
$y = 4x$	4

By observation, the gradient of each line is the same as the coefficient of x in the equation. We are now in a position to state a rule that relates to the equation of straight lines passing through the origin.

Lines passing through the origin have equations of the form $y = mx$, where m is the gradient of the line.

We will now examine another family of lines. These lines do not pass through the origin but they are parallel. It is easy to see that the gradient of each line is the same and equal to one.

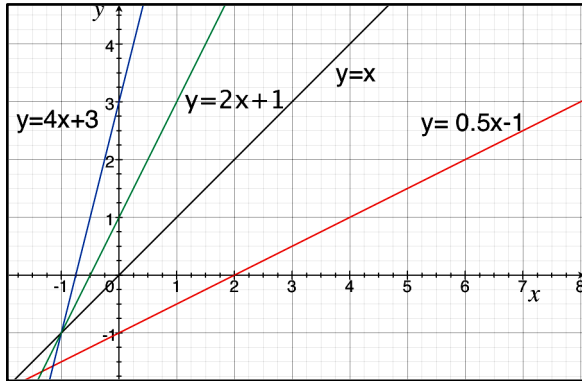


The above lines differ in that each one passes through a different point on the y -axis. This value is called the y -intercept for each line. These values are recorded below. We may think of the line $y = x$ as the base and the line $y = x + 2$ as an upward shift of 2 units. Similar an upward shift of 3 units will produce the line $y = x + 3$ while a downward shift of 1 unit will produce the line $y = x - 1$.

Equation of line	y -intercept	
$y = x$	0	Line passes through the origin
$y = x + 1$	1	Upward shift of 1 units.
$y = x + 3$	3	Upward shift of 3 units
$y = x - 1$	-1	Downward shift of 1 unit.

From the above table, we can conclude that the constant term in an equation of the form $y = mx + c$ is the y -intercept when the line is drawn.

We will now examine a new set of lines that are neither parallel nor equal in gradient.



The gradient and y -intercept for each of the four lines are shown below.

Equation	Gradient	y -intercept
$y = x$	1	0
$y = 2x + 1$	2	1
$y = 4x + 3$	4	3
$y = 0.5x - 1$	0.5	-1

From the table, we observe the following:

If the equation of a straight line is written in the form $y = mx + c$, then the gradient of line is the coefficient of x in the equation and the constant is the y -intercept.

In summary, equations of the form $y = mx$ pass through the origin $(0, 0)$, and have a gradient, m . Equations of the form $y = mx + c$ pass through the point $(0, c)$, where c is the intercept on the y -axis and m is the gradient.

Example 7

State the gradient and y -intercept for the line $y = 3x + 2$.

Solution

$y = 3x + 2$, is of the form, $y = mx + c$, where $m = 3$ and $c = 2$. The line, therefore, has a gradient of 3 and cuts the y -axis at 2 or at the point $(0, 2)$.

Example 8

State the gradient and y -intercept for the line $2y + 8x - 1 = 0$.

Solution

The given equation is not in the form, $y = mx + c$. We need to use some of our acquired algebraic skills to rearrange the terms so that y is the subject.

$$2y = -8x + 1$$

$y = -4x + \frac{1}{2}$, is of the form $y = mx + c$, where $m = -4$ and $c = \frac{1}{2}$.

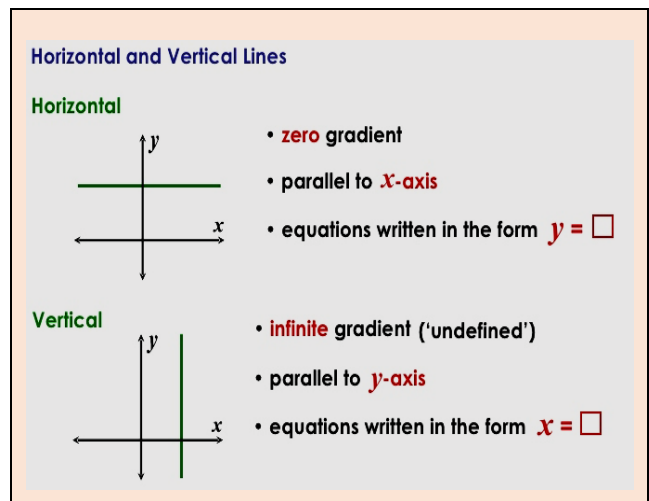
Hence the line has a gradient of -4 and cuts the y -axis at $(0, \frac{1}{2})$.

Horizontal and vertical lines

Horizontal lines have a gradient of 0. A horizontal line that cuts the y -axis at a , has equation $y = a$ (where a is a constant). This is because all points on the line will have a y coordinate of a .

Vertical lines have a gradient that approaches infinity (∞). A vertical line has equation, $x = b$ (where b is a constant) and cuts the x -axis at b . This because all points on the line will have an x coordinate of b .

The diagram below summarises the properties of horizontal and vertical lines.



Linear and non-linear equations

Equations of the form, $y = ax + b$, where x and y are variables are called linear equations. They can be recognised by examining the highest power of the variable. The power of the variables x and y in a linear expression or equation is one. The equation $y = x^2 + 3$ is an example of a non-linear equation because the power of x is 2. Such an equation is said to be of degree 2.

Graphing linear equation

To draw the graph of a linear equation we need to obtain the coordinates of at least two points on the line. Since the linear equation gives the relationship between the variables x and y we can use it to create a table of values which is really the coordinates of a set of points on the graph.

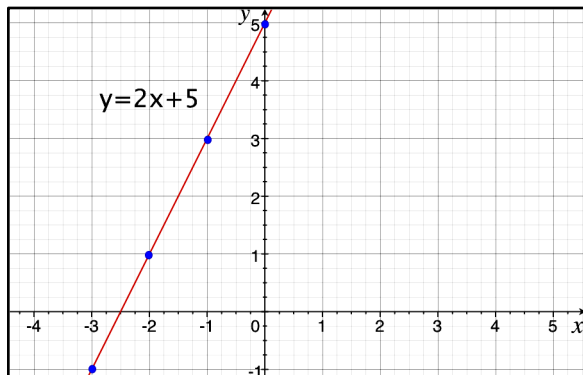
To draw the graph of $y = 2x + 5$, $x \in R$, for values of x in the interval $-3 \leq x$ we create a table of values by listing the integer values of x in the interval (called the domain) then calculate the corresponding members of y by substituting values of x in the equation.

$y = 2x + 5$
When $x = -3$, $y = 2(-3) + 5 = -1$
When $x = -2$, $y = 2(-2) + 5 = 1$
When $x = -1$, $y = 2(-1) + 5 = 3$
When $x = 0$, $y = 2(0) + 5 = 5$

The table of values for the equation $y = 2x + 5$.

x	-3	-2	-1	0
$y = 2x + 5$	-1	1	3	5

Next, we plot these points on graph paper using a suitable scale. The plotted points are joined to form a straight line. The line can be extended to any length.



Realistically, we need only two points to draw a straight line. After joining the two points, the line may be extended to any desired length.

To determine the equation of a straight line

To determine the equation of a straight line we must have specific information about the line. The general equation for a straight line is $y = mx + c$, and so if we need to find the equation of a special line some information must be given about the line. Our general equation gives us the relationship between x and y , so x and y are variables. To find a specific equation, we need to know the values of the unknowns, m and c .

(a) Given the values of m and c

This is the simplest case as the following example illustrates.

Example 8

State the equation of a line whose gradient is 3 and whose y -intercept is 2.

Solution

So $m = 3$ and $c = 2$
Substituting these values in $y = mx + c$, we have
 $y = 3x + 2$.

Example 9

A line cuts the y -axis at $(0, 5)$ and has a gradient of 4. State its equation.

Solution

Since the line cuts the y -axis at $(0, 5)$, its y -intercept is 5. So $m = 4$ and $c = 5$.
Substituting these values in $y = mx + c$, gives
 $y = 4x + 5$.

(b) Given the gradient and a point on the line

If we are given the gradient and one point, we can use the following method to obtain the equation. We can use the following method.

Example 10

Determine the equation of a line passing through the point $(3, 5)$ with a gradient of 2.

Solution

We calculate the value of c by substituting the value of m and the coordinates in the equation $y = mx + c$

Substituting $m = 2, x = 3$ and $y = 5$:

$$5 = 2(3) + c$$

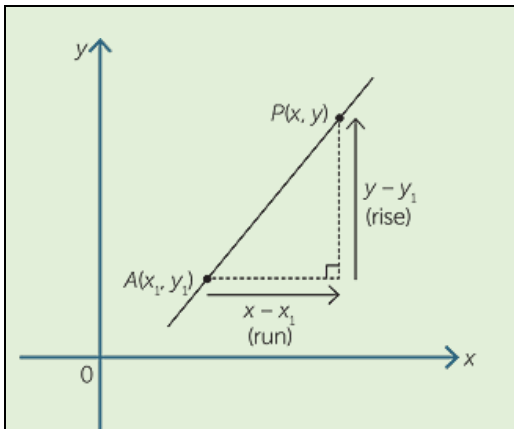
$$c = -1$$

The equation of the line is $y = 2x - 1$

Alternative method

This method relies on a formula, which will now be developed.

Let $P(x, y)$ be any point on the line whose gradient is m . Let the given point be $A(x_1, y_1)$.



The gradient of the line is:

Rise

Run

$$\frac{y - y_1}{x - x_1} = m$$

This can be re-arranged to $y - y_1 = m(x - x_1)$

In general, if (x, y) is any point on the line whose gradient is m and the point (x_1, y_1) lies on the line, the equation of the line is

$$y - y_1 = m(x - x_1)$$

Example 11

Determine the equation of the line passing through the point $(2, 4)$ and whose gradient is 3.

Solution

The equation is $m = \frac{y - y_1}{x - x_1}$, substituting for m, x_1

and y_1 , we have

$$\frac{y - 4}{x - 2} = 3$$

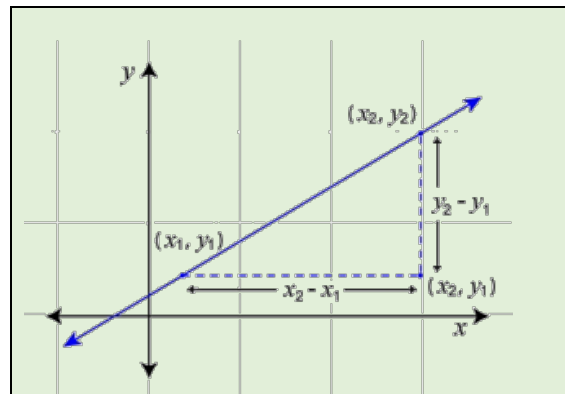
$$y - 4 = 3(x - 2)$$

$$y - 4 = 3x - 6$$

$$y = 3x - 2$$

(c) Given two points on the line

If we are given two points, then we can use the coordinates of the points to calculate the gradient of the line and proceed as we did above. To calculate the gradient we can use the following derived below.



Gradient of line joining

(x_1, y_1) to (x_2, y_2)

$$m = \frac{\text{Rise}}{\text{Run}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

Example 12

Determine the equation of the line that passes through the points $(2, 1)$ and $(4, 7)$.

Solution

The gradient of the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7 - 1}{4 - 2} = \frac{6}{2} = 3$$

Choosing either of the points, say (2, 1),

$$m = \frac{y - y_1}{x - x_1}$$

$$3 = \frac{y - 1}{x - 2}$$

$$y - 1 = 3x - 6$$

$$y = 3x - 5$$

If we had instead used the point (4, 7) and the same gradient, 3, we would have obtained the same equation.

Example 13

Determine the equation of the straight line that passes through O and which is perpendicular to the line with the equation, $y = 2x$.

Solution

The line $y = 2x$ has gradient 2.

∴ The required line has a gradient $= -\frac{1}{2}$

(Using the fact that the product of the gradients of perpendicular lines $= -1$)

The equation of the required line is

$$\frac{y - 0}{x - 0} = -\frac{1}{2} \quad 2y = -x.$$

Example 14

Find the equation of the line passing through the point (2, -1) and which is parallel to the line with the equation, $2y + x - 1 = 0$.

Solution

$$2y + x - 1 = 0$$

$$y = -\frac{1}{2}x + \frac{1}{2}$$

∴ The gradient $= -\frac{1}{2}$

∴ The required line has a gradient $= -\frac{1}{2}$ (parallel lines have the same gradients)

The equation of the required line is

$$\frac{y + 1}{x - 2} = -\frac{1}{2} \quad 2y = -x.$$

Intercepts on the x and y axes

To determine the y coordinate on a line which cuts the x-axis, we let $y = 0$. This is because, at any point where a line cuts the x-axis, the y coordinate is zero.

Similarly, to determine the x-coordinate on a line which cuts the y-axis, we let $x = 0$. This is because, at any point where a line cuts the y-axis, the x coordinate is zero.

The point where a line cuts the y-axis is called the y-intercept. The point where a line cuts the x-axis is called the x-intercept.

Example 15

Determine the coordinates of the points at which the line $y = 4x - 8$ cuts each axis.

Solution

$$y = 4x - 8$$

$$\text{Let } x = 0$$

$$y = 4(0) - 8$$

$$y = 0 - 8$$

$$y = -8$$

$$\text{Let } y = 0$$

$$0 = 4x - 8$$

$$4x = 8$$

$$x = 2$$

∴ Line cuts the y-axis at (0, -8) and the x-axis at (2, 0).

If $y = 4x - 8$ is sketched, the figure may look like:

