19. KINEMATICS

Kinematics is the branch of mathematics that deals with the motion of a particle in relation to time. In this topic, we are concerned with quantities such as displacement (and distance), velocity (and speed) and acceleration. In studying the motion of objects, we will examine travel graphs, laws of motion and the application of calculus to problems in kinematics.

Speed, distance and time

When a particle moves at a constant speed, its rate of change of distance remains the same. Constant speed is calculated from the basic formula

$$\text{Speed} = \frac{\text{Distance}}{\text{Time}}$$

If speed changes in an interval, the average speed during the period of travel can be calculated from the formula:

$$\text{Average Speed} = \frac{\text{Total distance covered or travelled}}{\text{Total time taken}}$$

Units

In calculations involving speed, distance and time, units must be consistent. So, if the units in a given situation are not consistent, then we ought to change the units to successfully solve the problem.

Example 1

If a car travels at a speed of $10ms^{-1}$ for 3 minutes, calculate the distance it covers.

Solution

To solve this problem, we must change the 3 minutes to 180 seconds, because speed is given in *metres per second*.

Using, Distance = speed × time.

Distance travelled = 10 × 180 =1800 m=1.8 km

Velocity and acceleration

Velocity is the speed of a particle measured in a specified direction of motion. Therefore, velocity is a vector quantity, whereas speed is a scalar quantity. So, we define velocity as the rate of change of displacement.

$$\text{Velocity}=\frac{\text{Displacement}}{\text{Time}}$$

When the velocity (or speed) of a moving object is increasing the object is ***accelerating***. If the velocity or speed decreases it is said to be ***decelerating***. Acceleration is, therefore, the rate of change of velocity. Constant acceleration is calculated from the formula

$$\text{Acceleration} = \frac{\text{Velocity}}{\text{Time}}$$

When speed is constant, there is no acceleration, that is, acceleration is equal to zero.

Since acceleration is velocity per unit time, units for acceleration will take the form such as ms^{-2}, cms^{-2}, kmh^{-2}. For example:

$$\text{Acceleration} = \frac{\text{Velocity}}{\text{Time}} = \frac{ms^{-1}}{s} = ms^{-1-1} = ms^{-2}$$

Example 2

A car starts from rest and within 10 seconds is moving at a velocity of $20ms^{-1}$. What is its acceleration, given that the acceleration is constant?

Solution

At rest, velocity = 0 ms^{-1}

Velocity after 10 s = 20 ms^{-1}

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time}} = \frac{(20-0)ms^{-1}}{10\ s} = 2ms^{-2}$$

Example 3

A particle moving in a straight line at 18 ms^{-1} decelerates at a constant rate, reaching a speed of 6 ms^{-1} after a period of 24 seconds.

Calculate the deceleration.

Solution

Initial speed = 18 ms^{-1} Final speed = 6 ms^{-1}

$$\text{Acceleration} = \frac{\text{Change in velocity}}{\text{Time taken}}$$

$$= \frac{\text{Final velocity-Initial velocity}}{\text{Time taken}}$$

$$= \frac{(6-18)\ ms^{-1}}{24\ s} = -\frac{1}{2}\ ms^{-2}$$

(The negative sign indicates that the particle is decelerating)

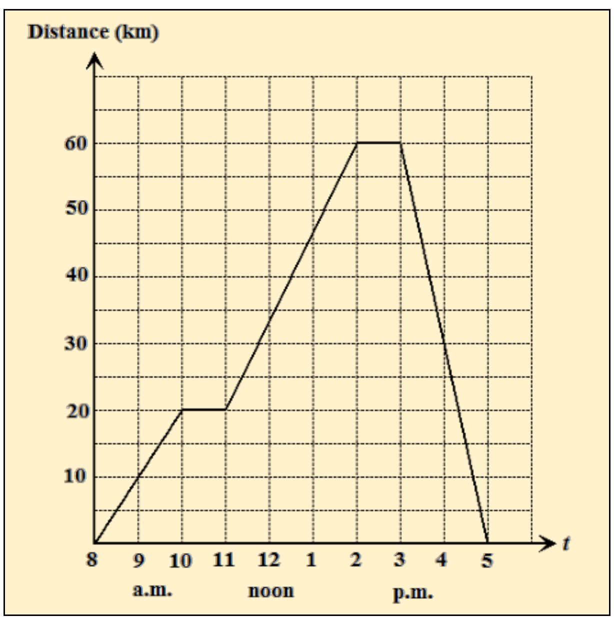
$\therefore$ Deceleration is $\frac{1}{2}$ ms^{-2}

Travel graphs

Travel graphs display the motion of a particle over a period of time. Some of the simplest kinds are graphs of distance (or displacement) versus time and velocity (or speed) versus time. On travel graphs, time always appears on the horizontal axis because it is the independent variable.

(a) Displacement/Distance-time graphs

In these graphs, the vertical axis displays the distance from a certain point and the horizontal axis displays the time. When a body is moving at constant speed, a straight line distance-time graph is obtained. The journey of a cyclist is shown below.



The different segments of the journey are all displayed using straight lines. Consider his starting point as home. From the graph we can deduce:

i. The cyclist left home at 8 a.m. and returned home at 5 p.m. (the ending point on the graph). Note that at 8 a.m. and 5 p.m., he was zero distance away from home.

ii. At 9 a.m. he was 10 km away from home and at 10 a.m. he was 20 km away from home. Also, between 2 p.m. and 3 p.m. he was 60 km away from home. This was the maximum distance travelled.

iii. At 3 p.m., he began his return journey and between 3 p.m. and 5 p.m., he was travelling towards home.

iv. Between 10 and 11 a.m. and between 2 and 3 p.m. the cyclist was at rest. Horizontal lines on the graph for these periods indicate that there is no motion.

v. The cyclist travelled at a constant speed during the periods 8 a.m. to 10 a.m., 11 a.m. to 2 p.m., and 3 p.m. to 5 p.m.

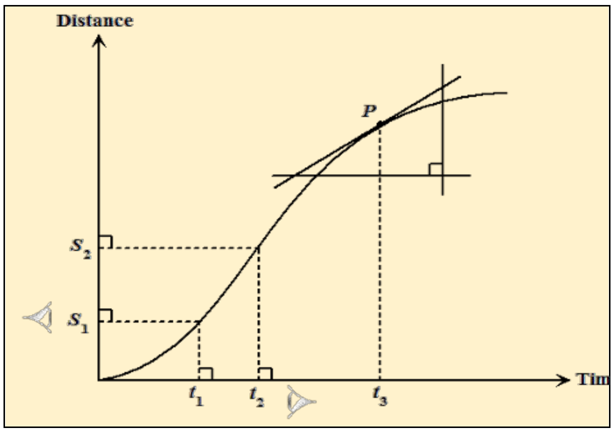
vi. The speed during each of the 3 segments is calculated from the gradient of the lines as shown below.

Speed from 8 a.m.to 10 a.m $= \frac{20\text{ km}}{2\text{ hr}} = 10\text{kmh}^{-1}$
Speed from 11 a.m.to 2 p.m $= \frac{40\text{ km}}{3\text{ hr}} = 13.3\text{ kmh}^{-1}$
Speed from 3 p.m.to 5 p.m $= \frac{60\text{ km}}{2\text{ hr}} = 30\text{ kmh}^{-1}$

If the speed or velocity of an object is not constant, the distance/displacement-time graph will be represented by a curve. From such graphs, we calculate the speed/velocity at an instant by finding the gradient of a curve at a point. This is done by drawing a tangent at that point and finding the gradient. From a graph of distance/displacement against time we can obtain:

i. The distance covered at any time by a read off.
ii. The time taken to cover any distance by a read off, and
iii. the speed/velocity at a given point by calculating the gradient at that point on the graph.

When speed/velocity is not constant, we obtain a curved distance/time graph as shown below.



In the graph shown,

(i) the distance is s_1, at the time, t_1.

(ii) the time taken to cover a distance of s_2 is t_2

(these are obtained by a simple read off)

(iii) the gradient of the curve at *P* is the speed at t_3

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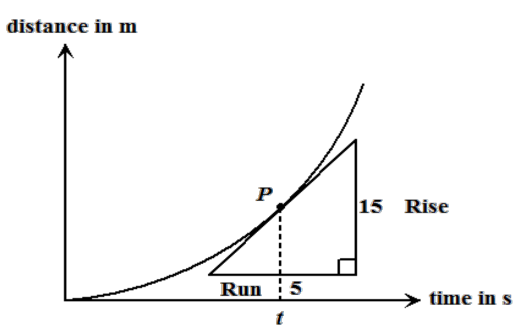
Example 4

The graph shows the distance-time curve for a particle. Find the speed at time, *t*.

Solution



The speed at time *t* is the gradient of the curve at *t*

$$\frac{\text{Rise}}{\text{Run}} = \frac{15\text{ m}}{5\text{ s}}$$

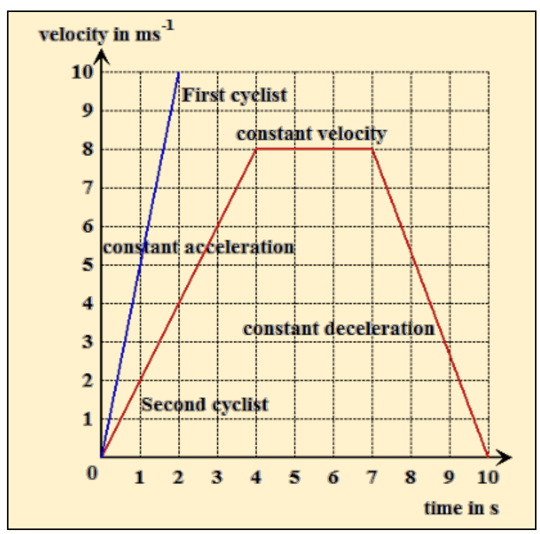
$$= 3\text{ ms}^{-1}$$

(b) Velocity/Speed -time graphs

A speed/velocity-time graph displays the velocity or speed of the particle on the vertical axis and time on the horizontal axis. For a body moving with uniform acceleration, a straight-line velocity-time graph is obtained. From such a graph, we can deduce that:

1. A horizontal line indicates that the object is at rest, that is, the velocity is zero.
2. The gradient gives the constant acceleration or deceleration. A positive slope indicates acceleration while a negative slope indicates deceleration
3. The area under the graph is the distance covered.

The graph below shows a straight-line velocity-time graph displaying the journey of two cyclists. The first cyclist (blue line) travelled at constant acceleration throughout the journey. The second cyclist travelled at constant acceleration then stopped accelerating and travelled at a constant velocity. Afterwards, the cyclist travelled at constant deceleration and finally came to a stop.



We can make the following deductions:

i. The acceleration of the first cyclist was

$$\frac{10ms^{-1}}{2s} = 5ms^{-2}$$

ii. The acceleration of the second cyclist was
$$\frac{8ms^{-1}}{4s} = 2ms^{-2}$$

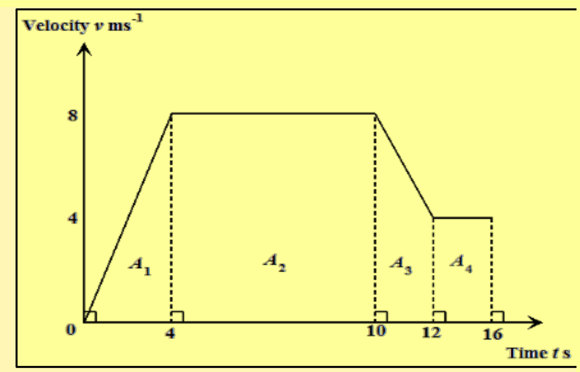
iii. Between 4-7 seconds the second cyclist travelled at a constant velocity.

iv. The deceleration of the second cyclist was $\frac{8\,\text{ms}^{-1}}{3\,\text{s}} = 2.67\,\text{ms}^{-2}$(to 2 d.p)

v. The total distance travelled by the second cyclist is found by the area of the trapezium
$$= \frac{1}{2}\times 8(10+3)\text{ metres} = 52\text{ metres}$$

Example 5

A velocity/time graph is shown below.



Calculate:

a. the acceleration from $t = 10$ to $t = 12$

b. the distance covered in the first 16 seconds

Solution

a. The acceleration during the period $t = 10$ to $t = 12$ is the gradient of the line joining (10, 8) to (12, 4) $= \frac{8-4}{10-12} = \frac{4}{-2} = -2$

Acceleration = –2 ms^{-2} (the negative sign indicates a deceleration)

b. The area under the graph will give the distance covered. The region is divided into 4 regions, as shown in the diagram above.

Area of $A_1 = \frac{4\times 8}{2} = 16$

Area of $A_2 = 8\times(10-4) = 48$

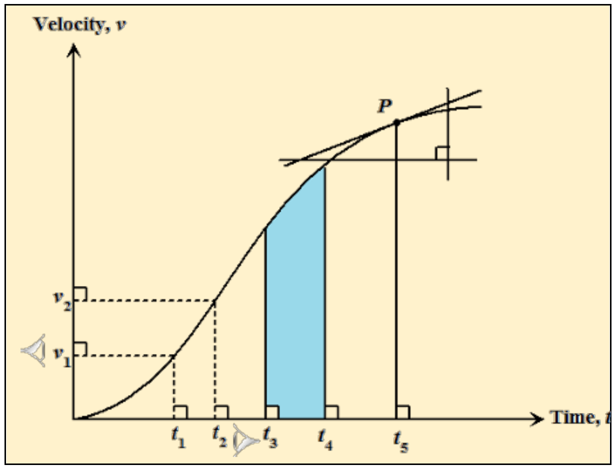
Area of $A_3 = \frac{1}{2}(8+4)\times(12-10) = 12$

Area of $A_4 = 4(16-12) = 16$

$\therefore$ Total area $= 16+48+12+16 = 92$

The distance covered is 92 m.

For a body moving with non-uniform or variable acceleration, a curved velocity-time graph is obtained as shown below.



From this graph, we can deduce the following:

1. The velocity at any time **or** the time taken to attain any velocity can be obtained by a read-off. For example, it took a time of t_2 to attain a velocity of v_2.
2. The area under the curve gives the total distance covered. For example, the area of the shaded region gives the total distance covered between t_3 and t_4.
3. The gradient of the tangent at a point gives the acceleration at that point. For example, the gradient of the tangent at P gives the acceleration at t_5.

Example 6

The velocity, v ms^{-1}, of a ball at time t is given by $v = -0.5t^2 + 4t + 6$.

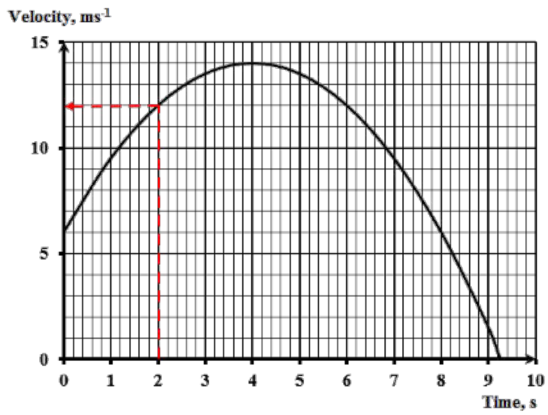
i. On graph paper draw the velocity-time graph for the first 10 seconds.

ii. Determine the velocity of the ball after 2 seconds.

iii. Estimate the acceleration of the ball after 6 seconds.

Solution

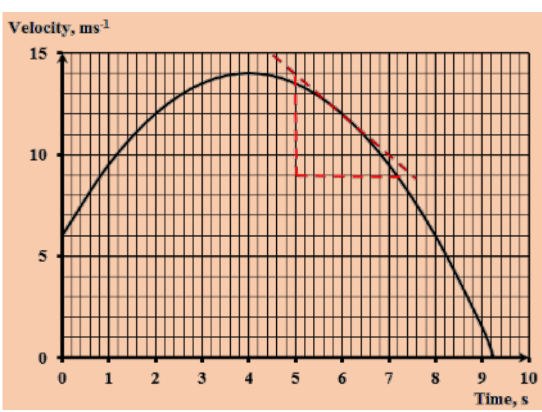
(i). The velocity-time graph for the first 10 seconds is shown in the diagram shown below.



(ii). The velocity after 2 seconds can be obtained by a read-off from the graph. As shown on the graph, the velocity at $t = 2$ is 12 ms^{-1}.

(iii). To obtain acceleration at $t = 6$, we draw the tangent at this point and find the gradient of the tangent. We calculate, $\frac{\text{Rise}}{\text{Run}} = \frac{5}{-2.5} = -2\ \text{ms}^{-2}$.

Hence, the acceleration at t = 6, is estimated as $-2\ \text{ms}^{-2}$.



The motion of a particle in a straight line under constant acceleration

For linear motion under constant acceleration, we use equations of motion to calculate unknown quantities.

Notation

The following symbols are used to represent the variables we are interested in.

u = initial velocity	v = final velocity
a = acceleration	t = time
s = displacement from a fixed point at time, $t = 0$.	

First Law

By definition,

$$\text{Constant Acceleration} = \frac{\text{Change in velocity}}{\text{Time}}$$

$$\therefore a = \frac{v-u}{t} \quad \text{or} \quad v - u = at \qquad (1)$$

Second law

By definition, for constant acceleration

Displacement = average velocity $\times$ time

$$s = \left(\frac{u+v}{2}\right) \times t$$

Replace v by $u + at$

$$s = \left(\frac{u+u+at}{2}\right) \times t = \left(\frac{2u+at}{2}\right) \times t = (u + \tfrac{1}{2}at) \times t$$

$$s = ut + \frac{1}{2}at^2 \qquad (2)$$

Third law

If we square the first equation:

$$(v = u + at)^2$$

$$v^2 = u^2 + 2uat + a^2t^2 = u^2 + 2a\left(ut + \frac{1}{2}at^2\right)$$

Recall: $s = ut + \frac{1}{2}at^2$

$$\therefore v^2 = u^2 + 2as \qquad (3)$$

The equations (1), (2) and (3) are called the three equations of linear motion and may be used in calculations where there is **linear motion under a constant acceleration**.

The equations of linear motion are:

1. $v = u + at$
2. $s = ut + \frac{1}{2}at^2$
3. $v^2 = u^2 + 2as$

Example 7

> A particle starts from a point O and accelerates uniformly for 8 seconds at a constant rate of 2 ms^{-2}, until it reaches a point P. Calculate:
> i. the velocity at P.
> ii. the distance covered.

Solution

i Let u = initial velocity = 0 ms^{-1} (at O), t = 8 s

v = final velocity or velocity at P.

Acceleration, a = 2 ms^{-2}

$$v = u + at$$

$$v = 0 + 2(8) = 16 \text{ ms}^{-1}$$

The velocity at P is 16 ms^{-1}

ii Let s = distance covered, then

$$s = ut + \frac{1}{2}at^2$$

$$= 0(8) + \frac{1}{2}(2)(8)^2$$

$$= 64$$

Distance covered = 64 m.

Example 8

> A particle passes a fixed point, A, and decelerates uniformly at 3 ms^{-2} for 12 s until it reaches a point B. The velocity at B is 24 ms^{-1}.
> i. Calculate the velocity at A.
> ii. Calculate the distance from A to B.
> iii. From B, how long will the particle take to come to rest?

Solution

(i) Let velocity at A be u ms-1 and acceleration, the deceleration, $a = -3ms^{-2}$, time, t = 12 s and the final velocity (or velocity at B), $v = 24\,\text{ms}^{-1}$

$$v = u + at$$

$$\therefore 24 = u + (-3)(12)$$

$$u = 24 + 36$$

$$= 60 \text{ ms}^{-1}$$

(ii) Let the distance from A to B be s

$$s = ut + \frac{1}{2}at^2$$

$$s = 60(12) + \frac{1}{2}(-3)(12)^2$$

$$s = 720 * (-216)$$

$$s = 504\, m$$

(iii) From B to the rest position.

u = initial velocity = 24 ms^{-1} v = final velocity = 0 ms^{-1} and a = – 3 ms^{-2}

$$v = u + at$$

$$0 = 24 + (-3)t$$

$$3t = 24$$

$$t = 8 \text{ s}$$

From B, the particle takes 8 s to come to rest.

Using calculus to solve Kinematics Problems

We can use differentiation and integration to solve problems in kinematics. We use the ideas of rate of change to obtain the relationships.

Notation

The following symbols are used to represent the variables in kinematics.

> v = velocity t = time a = acceleration
>
> x = displacement from a fixed point at time, $t = 0$.

Recall that velocity is the rate of change of distance ($v = \frac{dx}{dt}$), and acceleration is the rate of change of velocity ($a = \frac{dv}{dt}$). From these two relationships, we can use calculus to derive other relationships as shown below.

$v = \frac{dx}{dt}$ $dx = vdt$ $\int dx = \int vdt$ $x = \int vdt$	$a = \frac{dv}{dt}$ $dv = adt$ $\int dv = \int adt$ $v = \int adt$

$$v = \frac{dx}{dt}$$

$$\frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{dx}{dt}\right)$$

$$\frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2}$$

Summary

It is important to note that in using calculus to solve problems in kinematics we must be given an expression for x or v or a, in terms of t. We then choose to integrate or differentiate, with respect to t, to solve for the required unknown quantity.

$v = \frac{dx}{dt}$ $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	$\int a\,dt = v$ $\int v\,dt = x$

Example 9

The velocity, v ms^{-1}, of a particle moving from O, in a straight line, is given by $v = 6t^2 - 5t + 3$, where t is the time in seconds after leaving O. Calculate

a. the initial acceleration of the particle

b. the distance from O when $t = 4$.

Solution

a. The initial acceleration is the acceleration when t = 0.

$v = 6t^2 - 5t + 3$

Let the acceleration at time t be a.

$$a = \frac{dv}{dt} = 6(2t) - 5 = 12t - 5$$

Initial acceleration is at t = 0,

$a = 12(0) - 7 = -5$ ms^{-2}.

b. Let the distance from O at time t be s.

$$s = \int v\,dt = \int \left(6t^2 - 5t + 3\right) dt = \frac{6t^3}{3} - \frac{5t^2}{2} + 3t + C,$$

where C is a constant

$$\therefore s = 2t^3 - \frac{5}{2}t^2 + 3t + C$$

$s = 0$ at $t = 0$ (data)

$$\therefore s = 2(0)^3 - \frac{5}{2}(0)^2 + 3(0) + C$$

$$\therefore C = 0$$

$$s = 2t^3 - \frac{5}{2}t^2 + 3t$$

$$s = 2(4)^3 - \frac{5}{2}(4)^2 + 3(4) = 100$$

Distance from O when $t = 4$ is 100 m.

Example 10

A particle starts to move at O and travels in a straight line with velocity v cms^{-1}, where $v = 2t - t^2$ and t is the time in seconds after leaving O. The particle comes to rest at A. Calculate

a. the value of t at A.

b. the acceleration, a cms^{-2} at A.

c. the distance, s cm, from O to A.

d. the total distance covered from $t = 0$ to $t = 4$.

Solution

a. $v = 2t - t^2$, At A, particle is at rest, so $v = 0$

$2t - t^2 = 0$

$t(2 - t) = 0$

$t = 0$ or $t = 2$

$t = 0$ at O and

$t = 2$ at A.

b. The acceleration at A

$v = 2t - t^2$

$a = \frac{dv}{dt}$, $\therefore a = 2 - 2t$

At A, $a = 2 - 2(2) = -2$ cms^{-2}.

c. The distance from O to A

$$s = \int v\,dt$$

$$s = \int \left(2t - t^2\right) dt = t^2 - \frac{t^3}{3} + C$$

$s = 0$ at $t = 0$

$$\therefore s = (0)^2 - \frac{(0)^3}{3} + C, \quad C = 0$$

$$\therefore s = t^2 - \frac{t^3}{3}, \quad s = (2)^2 - \frac{(2)^3}{3} = 1\frac{1}{3}$$

d. The total distance covered from $t = 0$ to $t = 4$.

From part (a), we note that the particle was at rest at $t = 2$ sec.

Total distance covered from 2 to 4 seconds is

$$\int_2^4 (2t - t^2)\ dt = \left[t^2 - \frac{t^3}{3}\right]_2^4$$

$$= \left(16 - \tfrac{64}{3}\right) - \left(4 - \tfrac{8}{3}\right) = -5\tfrac{1}{3} - 1\tfrac{1}{3} = -6\tfrac{2}{3}\text{cm}$$

Total distance covered

$= 1\frac{1}{3} + 6\frac{2}{3} = 8\ cm$

Example 11

A particle passes through a fixed point O and at a time t seconds after leaving O, its velocity is v cms^{-1} is given by $v = pt^3 + qt + 6$ where p and q are constants. It is also given that the distance, s cm, from O at time $t = 2$, is 16 cm and the acceleration, a cms^{-2}, is 32 cms^{-2}.

Calculate

a. the value of p and of q.

b. the velocity at the time when $a = 0$.

Solution

a. $v = pt^3 + qt + 6$

$s = \int v\,dt = \int \left(pt^3 + qt + 6\right)dt$

$s = \frac{pt^4}{4} + \frac{qt^2}{2} + 6t + C$, C is a constant

$s = 0$ when $t = 0$

$\therefore 0 = \frac{p(0)^4}{4} + \frac{q(0)^2}{2} + 6(0) + C$

$\therefore C = 0$

$\therefore s = \frac{p}{4}t^4 + \frac{q}{2}t^2 + 6t$

$s = 16$ when $t = 2$

$\therefore 16 = \frac{p(2)^4}{4} + \frac{q(2)^2}{2} + 6(2)$

$4p + 2q = 4$

$2p + q = 2 \ldots(1)$

$a = 32$ when $t = 2$

$a = \frac{dv}{dt} = p \times 3t^2 + q$

and $32 = 3p(2)^2 + q$

$12p + q = 32 \ldots(2)$

Equation (2) – Equation (1)

$12p + q = 32$

$2p + q = 2$

$10p = 30$ and $p = 3$

From (1) $q = 2 - 2(3) = -4$

$\therefore p = 3$ and $q = -4$

b. The velocity when $a = 0$

Substituting the values of p and q in v:

$v = 3t^3 - 4t + 6$

$a = \frac{dv}{dt} = 9t^2 - 4$

When $a = 0$, $9t^2 - 4 = 0$

$t = \pm\frac{2}{3}$ $t > 0$ $\therefore t = \frac{2}{3}$ only

The velocity when $a = 0$,

$v = 3\left(\frac{2}{3}\right)^3 - 4\left(\frac{2}{3}\right) + 6$

$v = \frac{8}{9} - \frac{8}{3} + 6$

$v = 4\frac{2}{9}\ cms^{-1}$

Example 12

A particle, P, moves in a straight line so that its displacement, s cm, from O, at time t seconds after leaving O, is given by $s = 28 + 4t - 5t^2 - t^3$ cm.

Another particle, Q, moves in the same straight line as P and starts from O at the same time as P.

The initial velocity of Q is 2 cms^{-1} and its acceleration is a cms^{-2} given by $a = 2 - 6t$.

Find t when P and Q collide and determine whether P and Q are travelling in the same or opposite direction at the instant of collision.

Solution

For particle Q

$a = 2 - 6t$ $\therefore v = \int (2 - 6t)dt$

$v = 2t - 3t^2 + C$

Since the initial velocity of Q is 2 cms^{-1},when

$v = 2$, $t = 0$

$\therefore 2 = 2(0) - 3(0)^2 + C$

$\therefore C = 2$

$v = 2t - 3t^2 + 2$

Let s be the distance travelled by Q after t seconds,

$s = \int v\,dt = \int \left(2t - 3t^2 + 2\right)dt$

$s = \frac{2t^2}{2} - \frac{3t^3}{3} + 2t + K$

$s = t^2 - t^3 + 2t + K$

When $s = 0$, $t = 0$

$\therefore 0 = (0)^2 - (0)^3 + 2(0) + K$,

Hence, $K = 0$

$\therefore s = t^2 - t^3 + 2t$

At the point of collision, both particles would have travelled the same distance from *O*.

$\therefore 28 + 4t - 5t^2 - t^3 = t^2 - t^3 + 2t$

$-6t^2 + 2t + 28 = 0$

$6t^2 - 2t - 28 = 0$

$3t^2 - t - 14 = 0$

$(3t - 7)(t + 2) = 0$

$t = 2\frac{1}{3}$ or $t = -2$ Since $t > 0$, $t = 2\frac{1}{3}$ only

For *P*

$s = 28 + 4t - 5t^2 - t^3$

$\therefore v = \frac{ds}{dt} = 4 - 10t - 3t^2$

When the velocity of *P* $v = 4 - 10\left(\frac{7}{3}\right) - 3\left(\frac{7}{3}\right)^2$

$= 4 - 23\frac{1}{3} - 16\frac{1}{3} =$ negative

For Q

When $t = 2\frac{1}{3}$ $v = 2\left(\frac{7}{3}\right) - 3\left(\frac{7}{3}\right)^2 + 2$

$= 4\frac{2}{3} - 16\frac{1}{3} + 2$

$= 4\frac{2}{3} - 16\frac{1}{3} + 2 =$ negative

$\therefore$ At the point of collision, both *P* and *Q* are travelling in the same direction.