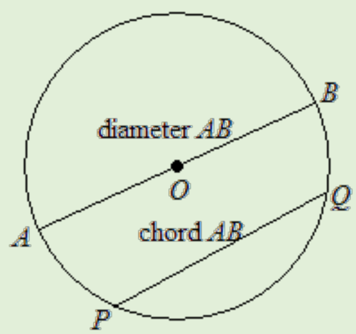
20. GEOMETRY OF THE CIRCLE

PARTS OF THE CIRCLE

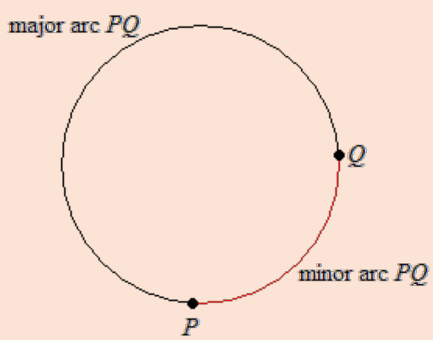
When we speak of a circle we may be referring to the plane figure itself **or** the boundary of the shape, called the circumference. In solving problems involving the circle, we must be familiar with several theorems. In order to understand these theorems, we review the names given to parts of a circle.

Diameter and chord



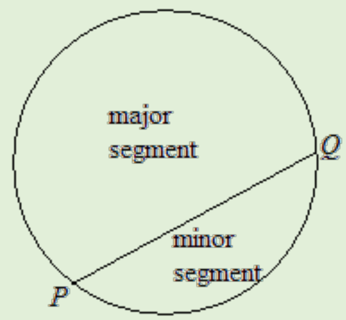
The straight line joining any two points on the circle is called a chord.
A diameter is a chord that passes through the center of the circle. It is, therefore, the longest possible chord of a circle.
In the diagram, O is the center of the circle, AB is a diameter and PQ is also a chord.

Arcs



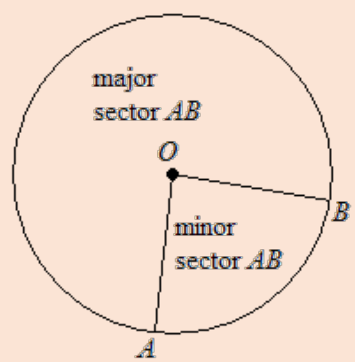
An arc of a circle is the part of the circumference of the circle that is cut off by a chord. The shorter length is called the minor arc and the longer length is called the major arc. If the chord PQ is a diameter, the arcs are equal in length and in this special case, there are no minor or major arcs.

Segments



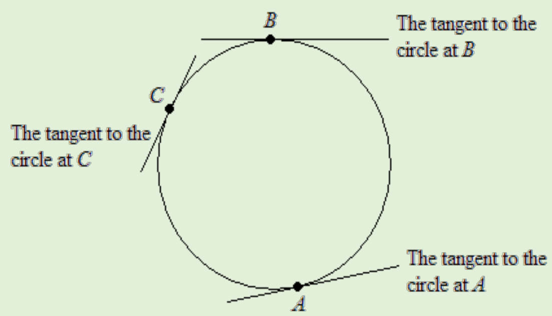
The region that is encompassed between an arc and a chord is called a segment.
The region between the chord and the minor arc is called the **minor** segment.
The region between the chord and the major arc is called the **major** segment.
If the chord is a diameter, then both segments are equal and are called semi-circles.

Sectors



The region that is enclosed by any two radii and an arc is called a sector.
If the region is bounded by the two radii and a minor arc, then it is called the minor sector.
If the region is bounded by two radii and the major arc, it is called the major sector.

The tangent of a circle



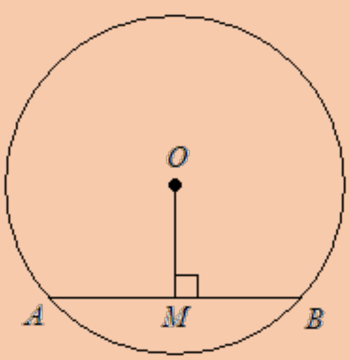
Any straight line that 'just touches' a circle at only one point, is called the tangent to the circle at that point. There can be only one tangent drawn to a circle at a point.

CIRCLE THEOREMS

A theorem is a statement of geometrical truth that has been proven from facts already proven or assumed. In our study of theorems at this level, we will not present the proofs. For convenience, the theorems presented below are numbered from 1-9. When referring to a theorem, we must be careful to quote it fully which is called its general enunciation.

Theorem 1

The straight line drawn from the center of a circle and perpendicular to a chord must bisect the chord.

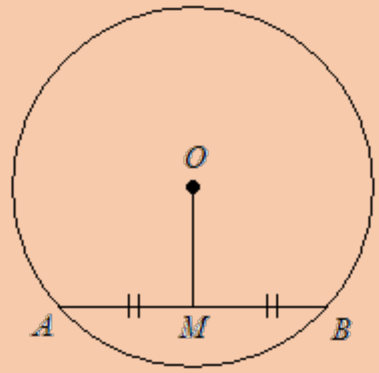


AB is a chord of a circle, center *O*.
If *OM* is perpendicular to *AB*, then *OM* bisects *AB*.
and *AM* = *BM*.

Theorem 2

The straight line is drawn from the center of a circle to the midpoint of a chord

is perpendicular to the chord.



AB is a chord of a circle, center *O*.
Since *OM* bisects the chord *AB*, *M* is the midpoint of the chord *AB*. Hence, *OM* is perpendicular to *AB*, that is

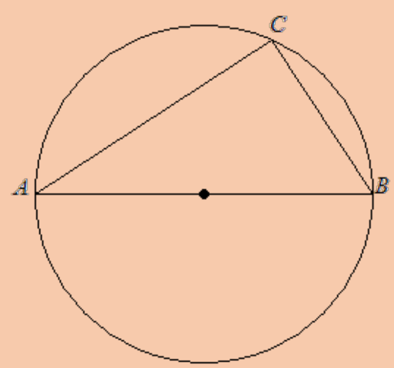
$$O\hat{M}A = O\hat{M}B = 90^\circ.$$

Theorem 3

A diameter subtends a right angle at the circumference of a circle.

OR

The angle is a semi-circle is a right angle.

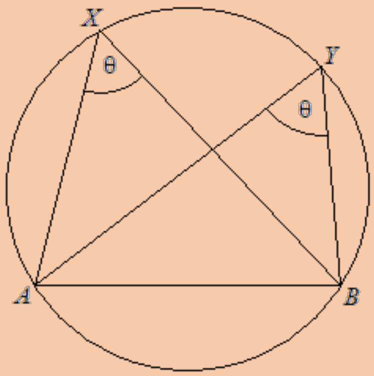


AB is a diameter of the circle and *C* is a point on the circumference. Hence,

$$\angle ACB = 90^0$$

Theorem 4

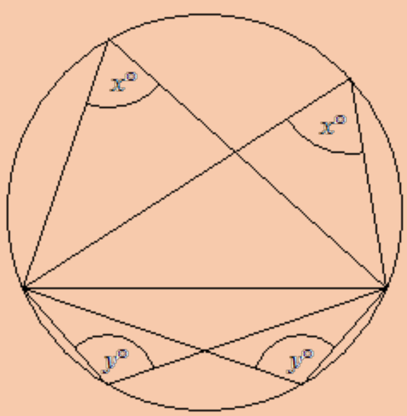
The angles subtended by a chord at the circumference of a circle and standing on the same arc are equal.



AB is a chord. *X* and *Y* are two points on the circumference, in the same segment. Hence

$$A\hat{X}B = A\hat{Y}B$$

It is important to note that the angles subtended by the chord, in the other or alternate segment, are also equal to each other.



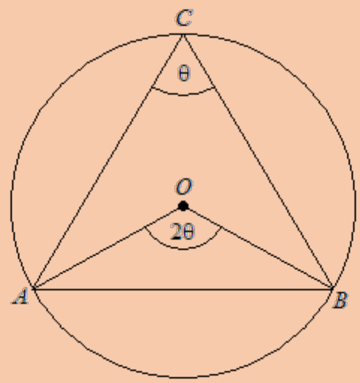
Note that angles in the same segment are equal once they stand on the same chord.

The angles labelled as *x* are in the major segment and the angles labelled *y* are in the minor segment of this circle.

Note also that *x* is not equal to *y*.

Theorem 5

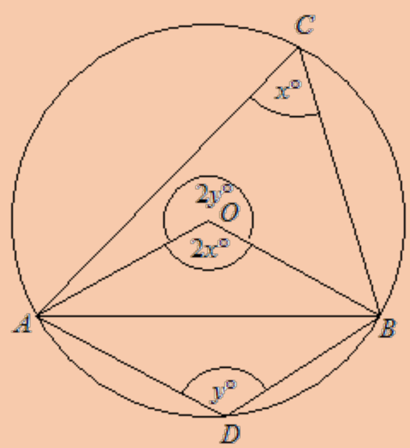
The angle subtended by a chord at the center of a circle is twice the angle that the chord subtends at the circumference, standing on the same arc.



AB is a chord of the circle, center *O*. *C* lies on the circumference. The angle subtended at the center is $A\hat{O}B$. The angle subtended at the circumference is $A\hat{C}B$. Hence,

$$A\hat{O}B = 2A\hat{C}B.$$

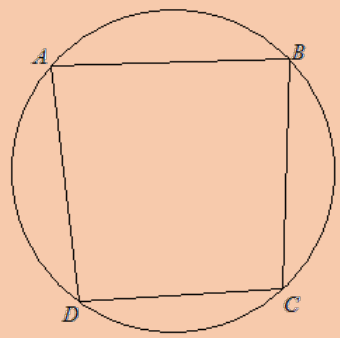
This theorem is also applicable to the reflex angle *AOB*, but in this case, it will be twice the angle subtended by *AB* in the alternate segment.



Note that the reflex angle $A\hat{O}B$ is twice the angle in the alternate segment. That is
Reflex $\angle AOB = 2 \times ADB$

Theorem 6

The opposite angles of a cyclic quadrilateral are supplementary.



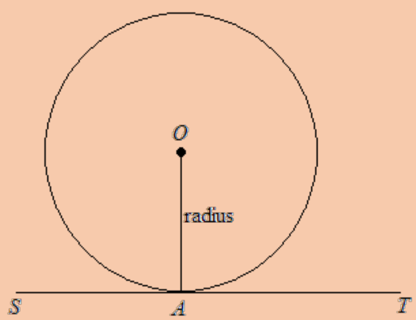
A cyclic quadrilateral has all of its four vertices on the circumference of a circle.
Supplementary angles add up to 180^0. Since A, B, C and D all lie on the circumference of the circle,

$$\hat{A}+\hat{C}=\hat{B}+\hat{D}=180°$$

The converse is also true. That is, if the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

Theorem 7

The angle formed by the tangent to a circle and a radius, at the point of contact, is a right angle.

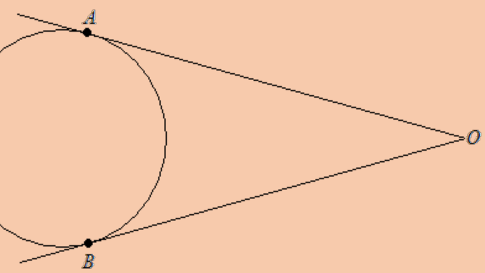


O is the center of the circle.
SAT is the tangent to the circle at A.
Therefore,

$$O\hat{A}T=O\hat{A}S=90°$$

Theorem 8

The two tangents that can be drawn to a circle from a point outside the circle are equal in length.

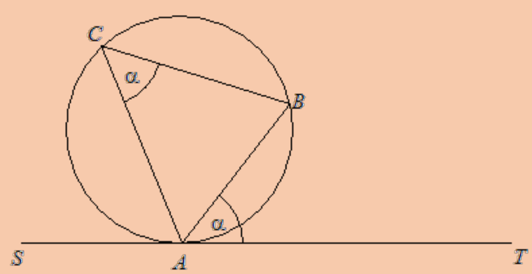


In the above diagram, OA and OB are the two tangents drawn from an external point, O.

Therefore $OA = OB$.

Theorem 9

The angle formed by the tangent to a circle and a chord, at the point of contact, is equal to the angle in the alternate segment.

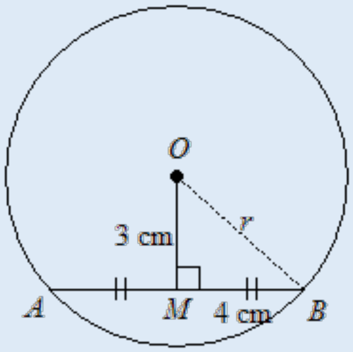


In the above diagram, BAT is the angle between the tangent SAT and chord, AB at A, the point of contact. Angle ACB is the angle in the alternate segment.

Therefore $\angle BAT = \angle ACB$

Example 1

In the figure below, AB is a chord of a circle, center O and M is the midpoint of AB. If AB is 8 cm and OM is 3 cm, find the length of the radius of the circle.



Solution

If M is the midpoint of AB, then MB = 8cm ÷ 2 = 4 cm.
Let radius of the circle, OB, be r, then

$O\hat{M}B = 90°$

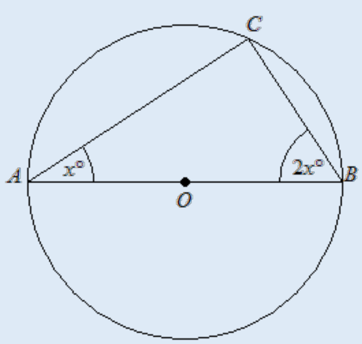
(The straight line drawn from the center of a circle to the midpoint of a chord is perpendicular to the chord).
By Pythagoras' theorem:

$$r^2 = (3)^2 + (4)^2$$
$$r^2 = 9 + 16 = 25$$
$$r = \sqrt{25} = 5$$

The radius of the circle is 5 cm

Example 2

In the figure below, AOB is a diameter of the circle and C lies on the circumference. If $C\hat{A}B = x°$ and $C\hat{B}A = 2x°$, find the value of x.

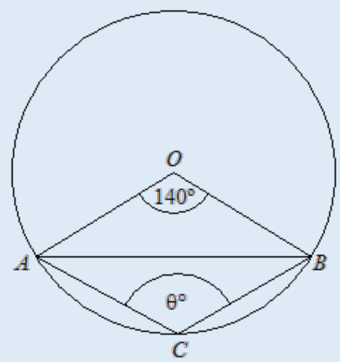


Solution

$A\hat{C}B = 90°$
(The angle in a semi-circle is equal to 90º)
Hence, $x° + 2x° + 90° = 180°$
(Sum of angles in a triangle is equal to 180º)
$3x° = 90°$
$x = 30$

Example 3

AB is a chord of a circle, center O and $A\hat{O}B = 140°$. Calculate the value of θ.



Solution

$A\hat{O}B = 140°$

$A\hat{O}B(\text{reflex}) = 360° - 140°$
$= 220°$

$\therefore A\hat{C}B = \frac{1}{2}(220°)$
$= 110°$

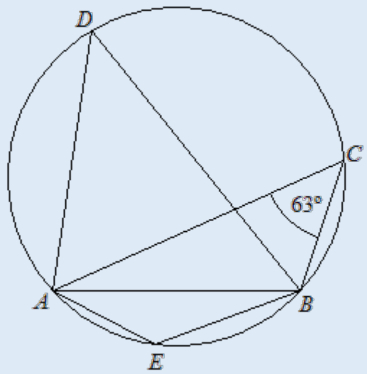
(The angle subtended by a chord at the center of a circle is twice the angle that the chord subtends at the circumference, standing on the same arc).

Example 4

In the circle *AEBCD*, the angle $A\hat{C}B = 63°$.
Calculate the size of the angle

i. $A\hat{D}B$

ii. $A\hat{E}B$



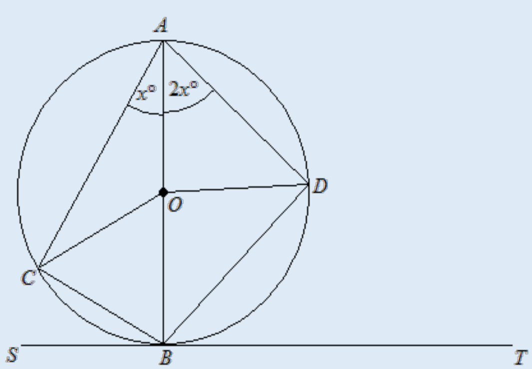
Solution

$\angle ADB = 63^0$ **(**Angles subtended by the same chord in the same segment are equal).

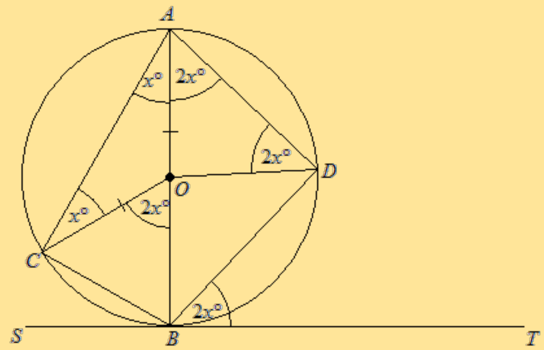
$\angle AEB = 180^0 - 63^0$
$= 117^0$ (Opposite angles of a cyclic quadrilateral are supplementary)

Example 5

AOB is a diameter of a circle, center *O* and *SBT* is the tangent at *B*. *C* and *D* are points on the circumference such that:
$\angle CAB = x^0$ and $\angle OAD = 2x^0$.
Name, and state reasons, three angles, other than the given angle that are equal to $2x^0$..



Solution

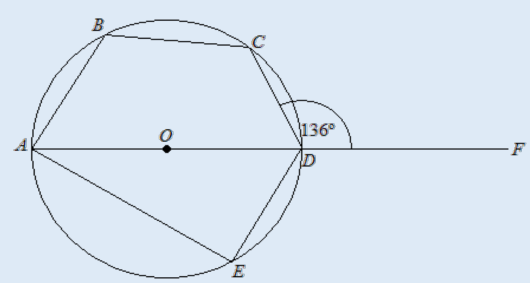


$\angle ODA = 2x$	Base angles of an isosceles triangle are equal (*OA* = *OD* radii)
$\angle DBT = 2x$	Angle formed by the tangent (*BT*)to a circle and a chord (*BD*) at the point of contact is equal to the angle in the alternate segment (Angle *BAD*)
OA = *OC*, Therefore, $\angle ACO = x$	Radii of a circle are equal in length Base angles of the isosceles triangle *ACO* are equal.
$\angle COB = 2x$	The angle at the center of a circle (angle *COB*) is twice that at the circumference, standing (angle *CAB*) on the same arc. **OR** Exterior angle of a triangle (angle *COB*) = sum of the interior opposite angles (angle *OCA* + *OAC*)

Example 6

In the figure, *AOD* is a diameter of a circle, center *O*. *ABCDE* is a pentagon inscribed in the circle. *AD* is produced to *F* and $C\hat{D}F = 136°$ Find the size of

(i) $A\hat{E}D$ (ii) $C\hat{D}A$(iii) $A\hat{B}C$



Solution

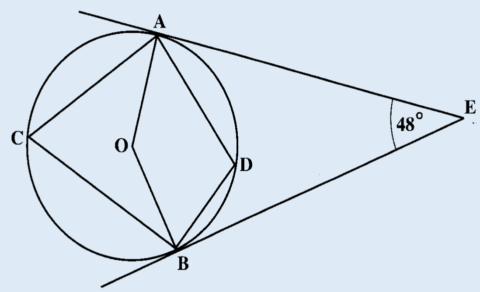
(i) $A\hat{E}D = 90°$
(The angle in a semi-circle is equal to 90º)

(ii) $C\hat{D}A = 180° - 136° = 44^0$
(The angles in a straight line total 180º)

(iii) $A\hat{B}C = 180° - 44° = 136^0$
(Opposite angles of a cyclic quadrilateral are supplementary)

Example 7

The diagram below, not drawn to scale, shows a circle with center, O.
EA and EB are tangents to the circle and angle AEB = 48^0.



Calculate, giving reasons for your answer, the size of EACH of the following angles:
(i) ∠OAE (ii) ∠AOB (iii) ∠ACB
(iv) ∠ADB

Solution

(i) ∠OAE = 90°
(The angle formed by a tangent to a circle and a radius, at the point of contact is equal to 90°, so too, angle OBE= 90°).

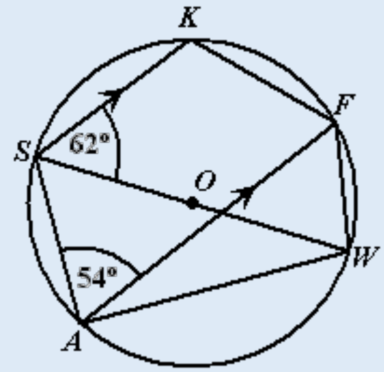
(ii) ∠AOB = 360^0 – (90°+90°+48°) = 132°

(iii) ∠ACB = ½ (132^0) = 66^0
(The angle at the center of a circle is twice that at the circumference, standing on the same arc).

(iv) $\angle ADB = 180^0 - 66^0 = 114^0$
(The opposite angles of cyclic quadrilateral are supplementary).

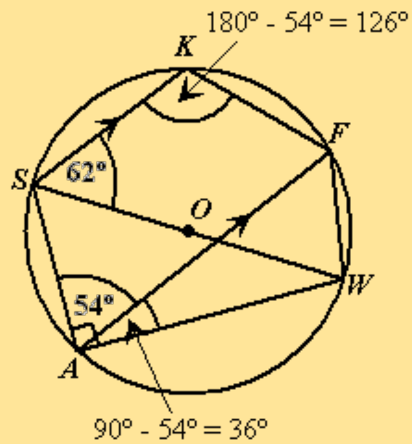
Example 8

In the diagram below, not drawn to scale, *O* is the center of a circle. *SK* and *AF* are parallel, $\angle KSW = 62^0$ and $\angle SAF = 54^0$



Calculate, giving reasons for your answer, the measure of:
(i) $\angle FAW$ (ii) $\angle SKF$ (iii) $\angle ASW$

Solution



(i) $\angle SAW = 90^0$ (the angle in a semi-circle is 90°)
Therefore, $\angle FAW = 90^0 - 54^0 = 36^0$

(ii) Consider the cyclic quadrilateral *SKFA*
$\angle SKF = 180^0 - 54^0 = 126^0$
(The opposite angles of a cyclic quadrilateral are supplementary)

(iii) The angles *KSA* and *SAF* are co-interior opposite angles and are therefore supplementary.

Therefore $54^0 + 62^0$ + angle *ASW* = 180^0
And $\angle ASW = 180^0 - (54^0 + 62^0)$ =64