13: TRIGONOMETRY II

Graphs of trigonometric functions

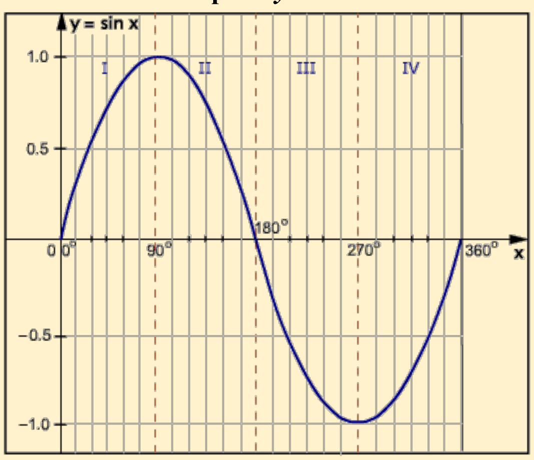
All trigonometric graphs are periodic functions, that is, they repeat themselves after a fixed interval called the period. We will focus on the three basic trigonometric functions in this section.

The graph of $y = \sin x$

The graph is obtained by plotting key points and drawing a curve as shown below.

x(deg)	0	30	90	180	270	360
$\sin x$	0	0.5	1.0	0	-1	0

Graph of $y = \sin x$



For the domain, $0° \leq x \leq 360°$.

Properties of the sine graph

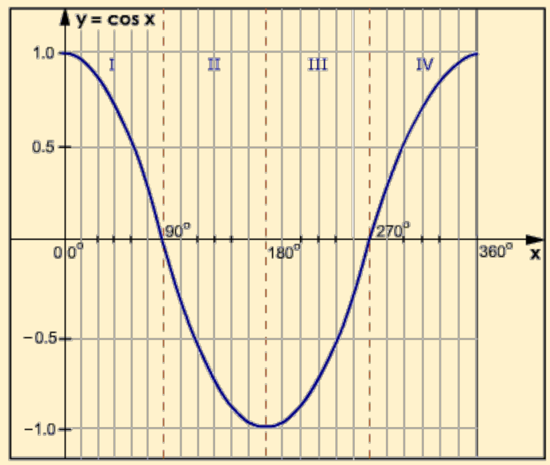
1. The curve $y = \sin x$ has a maximum value of 1 at $x = 90°$ and a minimum value of -1 at $x = 270°$.
2. In quadrants I and II the sine function is positive, and in quadrants III and IV, it is negative.
3. The portion of the curve from 0° to 180° is symmetrical about the line, $x = 90°$ and the portion from 180° to 360° is symmetrical about the line, $x = 270°$.
4. It has a period of 360°, because the entire curve repeats itself in intervals of 360^0.
5. The range is $-1 \leq \sin x \leq 1$.

The graph of $y = \cos x$

The graph is obtained by plotting key points and drawing a curve as shown below.

x(deg)	0	60	90	180	270	360
$\cos x$	1.0	0.5	0	-1	0	1

Graph of $y = \cos x$



For the domain, $0° \leq x \leq 360°$.

Properties of the cosine graph

1. The graph of $y = \cos x$ has a maximum value of 1 at $x = 0$ and 360°, and a minimum value of -1 at $x = 180°$.
2. In quadrants I and IV the cosine function is positive, and in quadrants II and III, it is negative.
3. The region from 0° to 180° is reflected in the line $x = 180°$ to give the region from 180^o to 360^o.
4. It has a period of 360° because the entire curve repeats itself in intervals of 360^0.
5. The range is $-1 \leq \cos x \leq 1$.

The cosine graph is really a shift of 90^0 of the sine graph and therefore the properties of both graphs are quite similar. We should note that

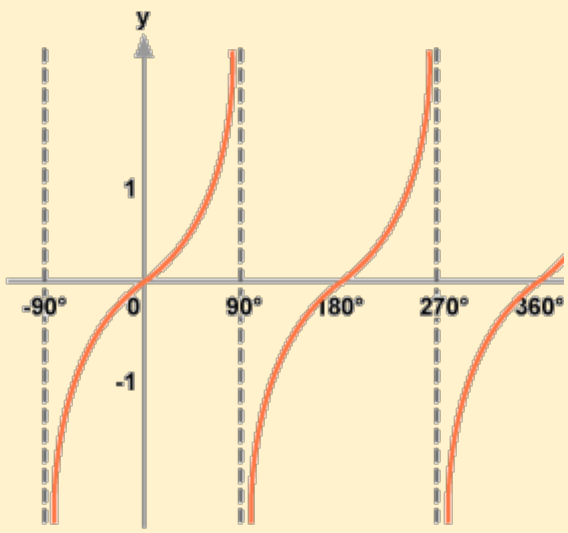
$$\sin \theta = \cos (90 - \theta)$$

The graph of $y = \tan x$

The graph is obtained by plotting key points and drawing a curve as shown below.

x(deg)	-90	0	45	90	180	270	360
$\tan x$	∞	0	1.0	∞	0	∞	0

Graph of $y = \tan x$



For the domain $-90^0 \leq x \leq 360^0$

Properties of the tangent graph

1. The graph of $y = \tan x$ has vertical asymptotes at $x = -90°$ and at intervals of 180^0 before and after these values.
2. In quadrants I and III the tangent function is positive, and in quadrants II and IV, it is negative.
3. The tangent graph has a period of $180°$ because the entire curve repeats itself in intervals of 180^0.
4. The graph of $\tan x$ may take any real value.

Transformed trigonometric functions

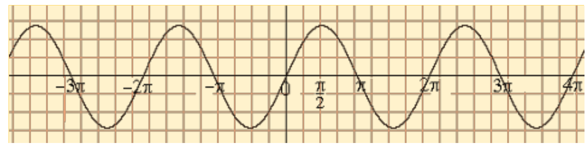
In the following section, we will examine how graphs of the trigonometric functions are transformed when we change certain constants in their equations.

We will examine four types of transformations. The first two transformations refer to **stretches** along the horizontal and vertical axes. The next two transformations refer to **shifts** along the vertical and horizontal axes.

The graph of $y = \sin ax$

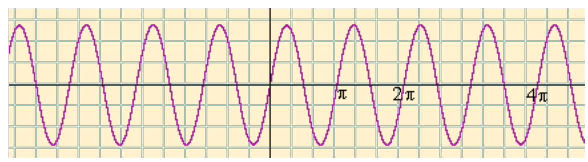
Let us first sketch the graph of $y = \sin x$. Since the sine function has a period of 360^0 or 2π the graph repeats itself in this interval.

The graph of $y = \sin x$ has a period of 2π or 360^0.

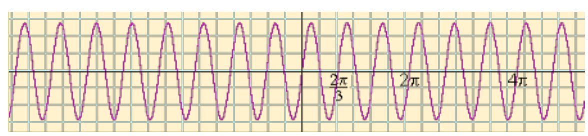


Now, examine the graph of $y = \sin 2x$. The graph repeats itself in a shorter interval than the graph of $y = \sin x$. In this case, there is a horizontal contraction since there are two (2) repetitions in the interval from 0 to 2π.

The graph of $y = \sin 2x$ has a period of π or 180^0.



Now let us examine the graph of $y = \sin 3x$. There is even further contraction and the interval for one (1) repetition is now $\frac{2\pi}{3}$. There are three (3) repetitions of the graph between 0 and 2π.



The graph of $y = \sin 3x$ has a period of $\frac{2\pi}{3}$ or 120^0.

Period of $y = \sin ax$

The graph of $y = \sin x$ has a one (1) repetition in the interval 2π. Hence, its period is $\frac{2\pi}{1} = 2\pi$ or 360^0.

The graph of $y = \sin 2x$ has two (2) repetitions in the interval 2π. Hence, its period is $\frac{2\pi}{2} = \pi$ or 180^0.

The graph of $y = \sin 3x$ has three (3) repetitions in the interval 2π. Hence, its period is $\frac{2\pi}{3}$ or120^0.

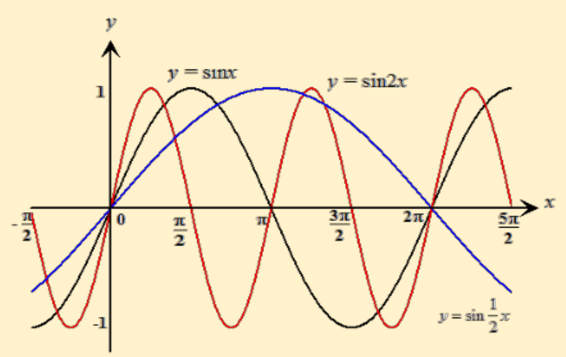
In general, the graph of $y = \sin ax$ has a period of $\frac{2\pi}{a}$.

Horizontal stretch factor of y = sin ax

As the period of a trigonometric graph changes, its ***horizontal stretch factor*** also changes. This stretch factor gives an indication of the amount of stretch or contraction along the horizontal axis. The graph $y = \sin x$ has a stretch factor of one and this is used as a basis for comparison.

Three sine graphs are shown below for the domain $0 \le x \le 2\pi$. Notice the graph of y = sin $2x$ (red) is more 'contracted' than the graph y = sin x, shown in black. On the other hand, the graph of $y = \sin \frac{1}{2}x$, shown in blue, is more 'stretched' than y = sin x.

Sine graphs comparing horizontal stretch factor



Hence, if we consider
y = sin x to have a horizontal stretch factor of 1, then by comparison,

y = sin $2x$ has a stretch factor of $\frac{1}{2}$

$y = \sin \frac{1}{2}x$ has a stretch factor of 2 .

Note that the stretch factor is inversely proportional to the periodicity.

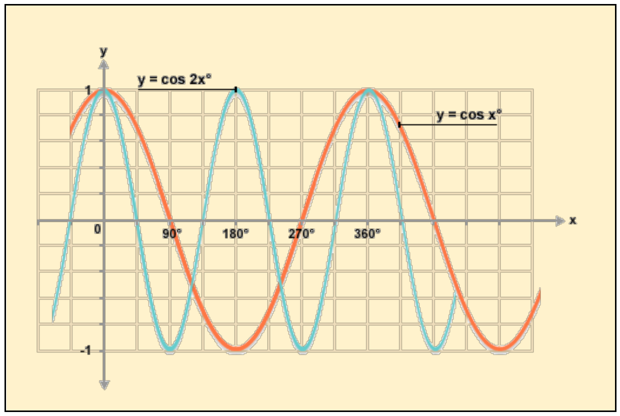
In general, y = sin ax has a horizontal stretch factor of $\frac{1}{a}$.

The graph of y = cos ax

The graph of y = cos ax follows a similar pattern as the sine graph as values of a varies.

Let us examine the graphs of y = cos x and compare it with y = cos $2x$. As shown in the diagram below, the graph of y = cos $2x$ repeats its shape in a shorter interval than the graph of $y = \cos x$. Just like the sine graph, there are two (2) repetitions in the interval from 0 to 2π.

The graph of $y = \cos 2x$ has a period of π or 180^0.
If we were to draw the graph of y = cos $3x$, we would observe that there are three repititions of the curve between 0^0 and 360^0.



Period of y = cos ax

By observation, we note that:

$y = \cos x$ has a period of 2π radians or 360^0.

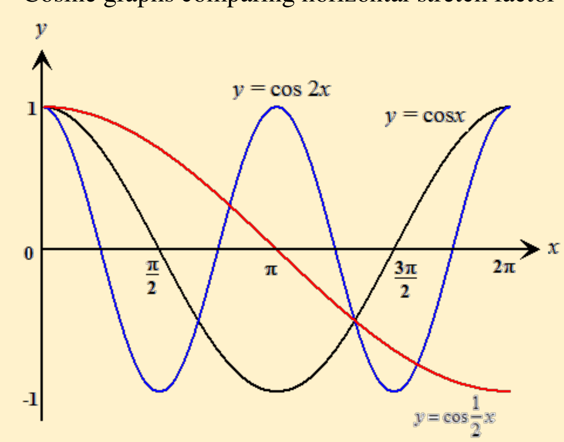
$y = \cos 2x$ has a period of π radians or 180^0.

In general, the graph of y = cos ax has a period of $\frac{2\pi}{a}$

Horizontal stretch factor of y = cos ax

The graphs of cos x, cos $2x$ and $\cos \frac{1}{2}x$ are shown below for the domain $0 \le x \le 2\pi$.

Cosine graphs comparing horizontal stretch factor



Hence, if we consider y = cos x to have a horizontal stretch factor of 1, then by comparison,

y = cos $2x$ has a stretch factor of $\frac{1}{2}$

y = cos ½ x has a stretch factor of 2

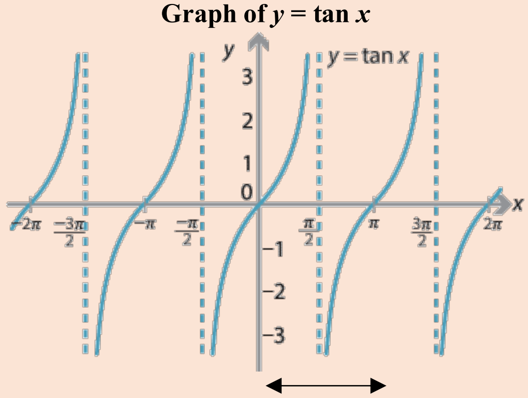
In general, y = cos ax has a horizontal stretch factor of $\frac{1}{a}$.

The graph of $y = \tan ax$

The tangent graph is not a continuous graph and there are no maximum and minimum points. The vertical lines are asymptotes or tangents to the graph at infinity.

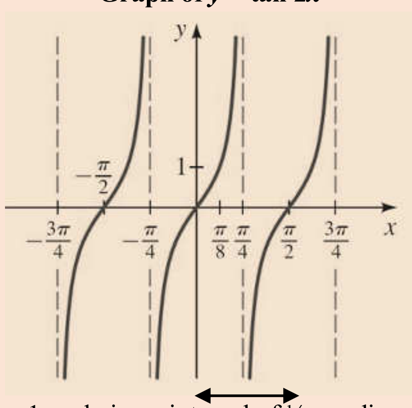
The graphs of $y = tan\, x$ and $y = tan\, 2x$ shown below.

Graph of $y = \tan x$



1 cycle in an interval of π radians or180^0

Graph of $y = \tan 2x$



1 cycle in an interval of ½ π radians or 90^0

The graph of $y = \tan x$ has a period of 180^0 or π radians. Note that the pattern is repeated at intervals of 180^0. The graph of $y = \tan 2x$ repeats itself after 90^0 or $\frac{\pi}{2}$ radians.

Period of $y = \tan x$

$y = \tan x$ has a period of π radians or 180^0.

$y = \tan 2x$ has a period of $\frac{\pi}{2}$ radians or 90^0.

In general, the graph of $y = \tan ax$ has a period of $\frac{\pi}{a}$.

Horizontal stretch factor of $y = \tan x$

It can be deduced that the horizontal stretch factors for the tangent graphs are as follows:

$y = \tan x$ has a stretch factor of 1

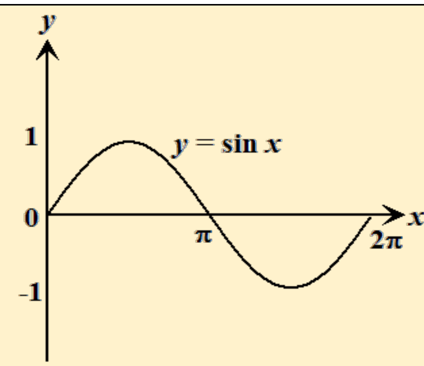
$y = \tan 2x$ has a stretch factor of $\frac{1}{2}$

$y = \tan \frac{1}{2} x$ has a stretch factor of 2

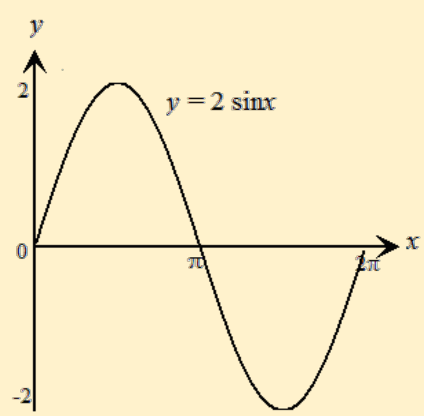
In general, $y = \tan ax$ has a stretch factor of $\frac{1}{a}$.

Graphs of $y = a\, sinx$ and $y = a\, cosx$

Changing the constant in front of the trigonometric function results in a **vertical stretch** or a change of **amplitude.** This will also result in a change in the range of the graph.

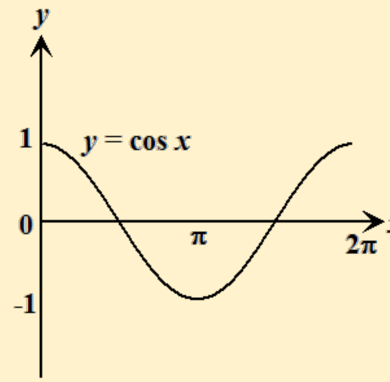


The amplitude of $y = \sin x$ is 1, and the range is 2(1) =2

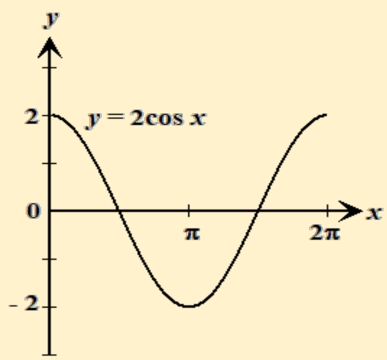


The amplitude of $y = 2 \sin x$ is 2, and the range is 2(2) = 4

The cosine graph is affected in a similar manner.



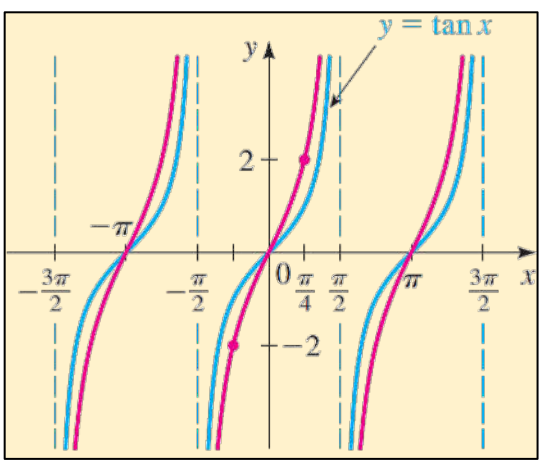
The amplitude of y = cos x is 1, and the range is 2(1) =2



The amplitude of y = 2 cos x is 2, and the range is 2(2) = 4

The graph of $y = a$ tan x

The tangent graph does not have a finite range and although there is a vertical stretch when y = tan x is transformed to y = 2 tan x, this change is barely noticeable. As shown below, the graph of y = 2 tan x (red) has a steeper curve than y = tan x.

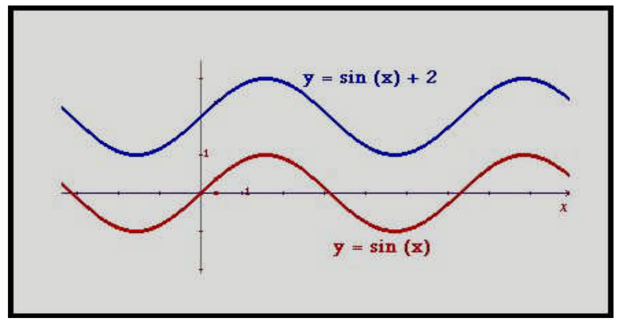


Vertical Shifts of trigonometric functions

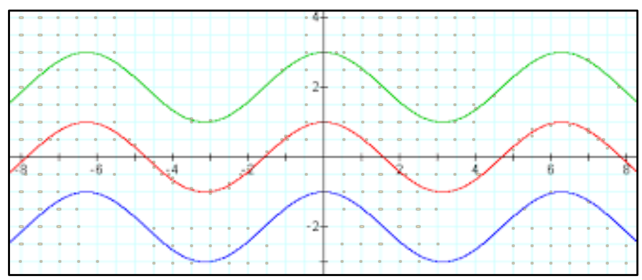
When we add a constant to a trigonometric function in the manner shown below, we merely shift the graph along a **vertical** axis. The transformation is identical to what happens to any function when we add a constant. In general, the graph of $f(x)$ is mapped onto $f(x)+a$ by a vertical shift of a units vertically upwards (a > 0).

$$f(x) \xrightarrow{T=\begin{pmatrix}0\\a\end{pmatrix}} f(x)+a$$

The graphs of
$y = \sin x$ and $y = \sin x + 2$ are shown below



The graphs of
$y = \cos x, y = \cos x + 2$ and $y = \cos x - 2$ are shown below



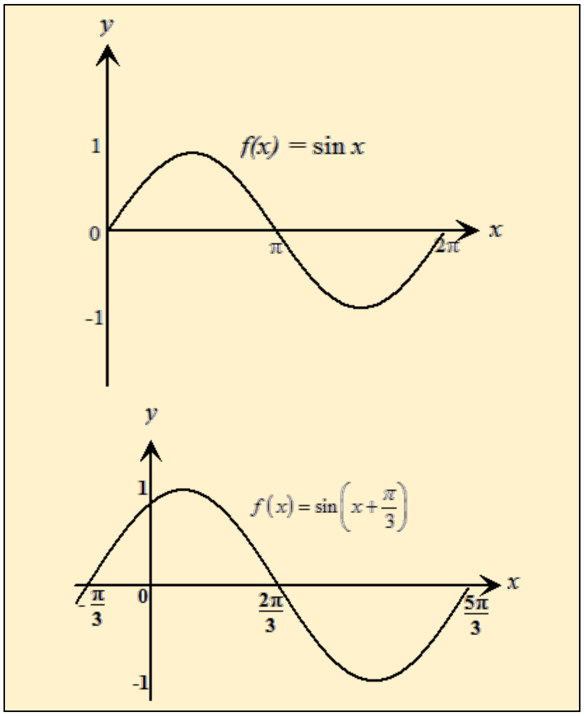
Horizontal Shift of trigonometric functions

The graph of $f(x)$ is mapped onto $f(x+a)$ by a horizontal shift of a units to the left. This is because everything happens a units earlier.

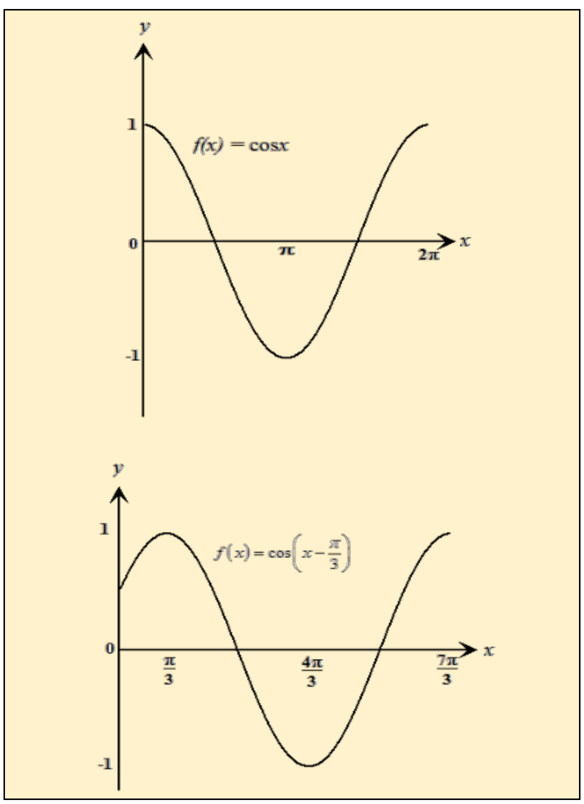
$$f(x) = \xrightarrow{T=\begin{pmatrix}-a\\0\end{pmatrix}} f(x+a)$$

$f(x) = \sin x$ and $f(x) = \sin(x + \frac{\pi}{3})$ are shown below

The graph shifts $\frac{\pi}{3}$ units to the left.



Graphs showing $\cos x \longrightarrow \cos\left(x - \frac{\pi}{3}\right)$

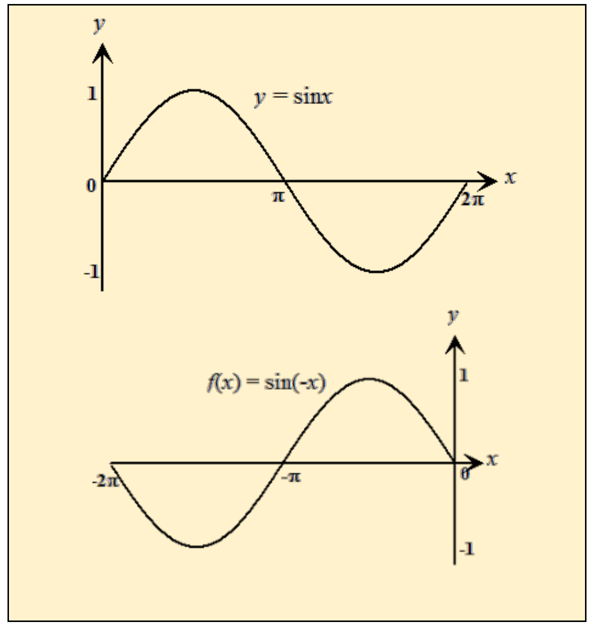


Reflection about the *x* and *y*-axes

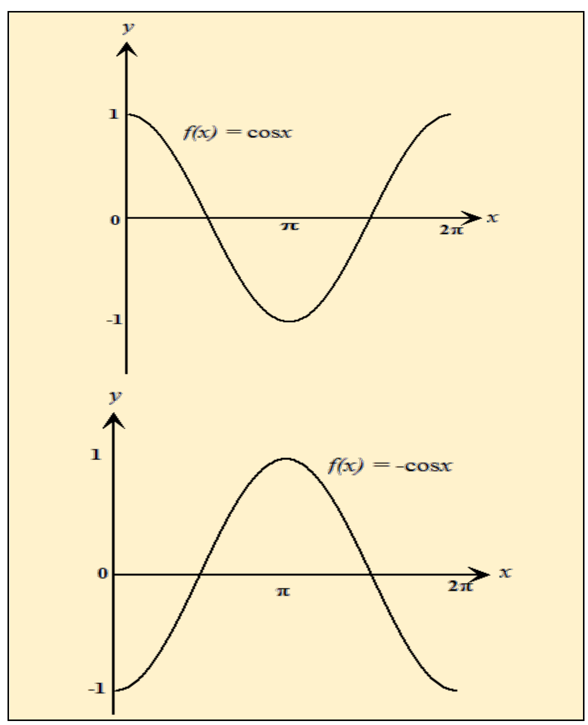
The graph of $f(x)$ is mapped onto $f(-x)$ by a reflection in the y-axis.

The graph of $f(x)$ is a mapped onto $-f(x)$ by a reflection in the x-axis.

$$\sin x \xrightarrow{\text{Reflection in the } y\text{-axis}} \sin(-x)$$



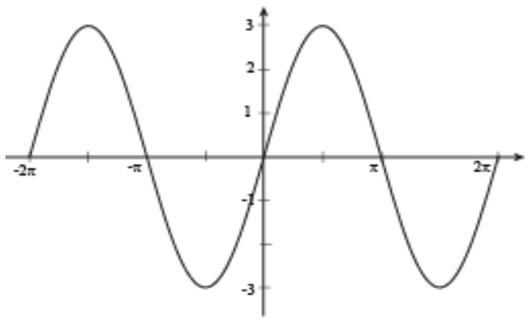
$$\cos x \xrightarrow{\text{Reflection in the } x\text{-axis}} -\cos x$$



Example 1

Draw the graph of $y = 3sinx$ between -2π and 2π

Solution

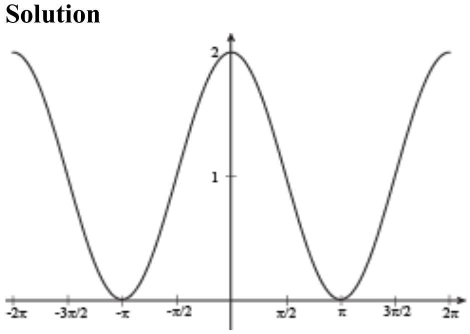


The amplitude is 3 and so the graph has a maximum at $y = 3$ and a minimum at $y = -3$.

Example 2

Draw the graph of $y = cosx + 1$ between -2π and 2π

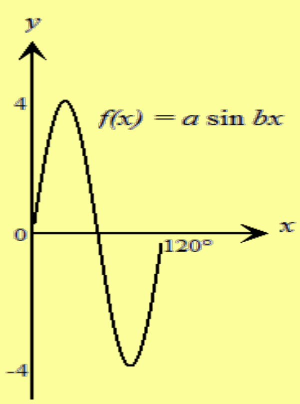
Solution



The graph is a vertical shift of $y = cosx$ one unit upwards. The axis is the line $y = 1$.

Example 3

The diagram shows the graph of $f(x) = a\sin bx$. State (i) the amplitude of $f(x)$, (ii) the value of a and of b.



Solution

By observation, the amplitude, $a = 4$ since the height above and below the x-axis is 4 units.

Since the period is 120^0 the graph repeats itself 3 times in the interval from $0^0 - 360^0$, so $b = 3$

$f(x) = a\sin bx$ can now be written as

$f(x) = 4\sin 3x$, where $a = 4$ and $b = 3$.

Solving trigonometric equations

We can now apply our knowledge of the graphs in the previous section to solving trigonometric equations. In solving these equations, we are interested in finding the values of x that satisfy equations that would take the form:

$2\,sin\,x\ -1 = 0$, $cos2x\ + 2cos\,x\ - 1 = 0$ and $cos\,x\ + \ 2\,sin\,x\ = \ 0$ and so on.

Since the trigonometric graphs are periodic functions, there will be an infinite number of solutions to such equations. We are usually given the domain of the function so that we can limit our solutions to values within this domain.

Example 4

Solve the equation $2\sin x - 1 = 0$, where x lies between 0° to 360°.

Solution

$2\sin x - 1 = 0$

$2\sin x = 1$

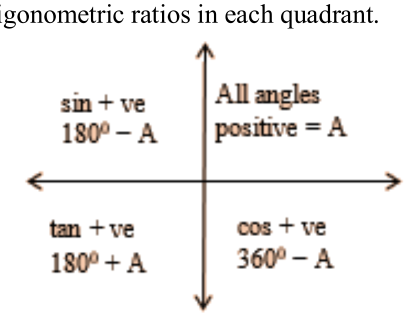
$\sin x = \frac{1}{2}$

$x = \sin^{-1}\left(\frac{1}{2}\right)$

$\sin^{-1}(0.5) = 30^0$

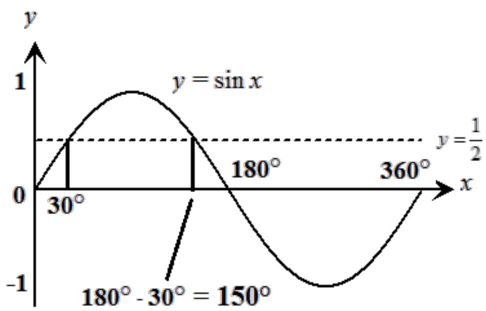
$x = 30^0$

This is the basic acute angle. To obtain all the solutions between 0° to 360°, consider the signs of the trigonometric ratios in each quadrant.



Note that $\sin x$ is positive in quadrants 1 and 2.
The solutions are:
$x = 30°$ and $x = 180^0 - 30^0$
$x = 30°, 150°$

The solution is illustrated on a graph of $\sin x$.



This diagram illustrates the solution.
The line $y = 0.5$ cuts the graph at two points. By a read-off, we can obtain the solutions.

Example 5

Solve for x from 0° to 360° in $2\sin x + 1 = 0$.

Solution

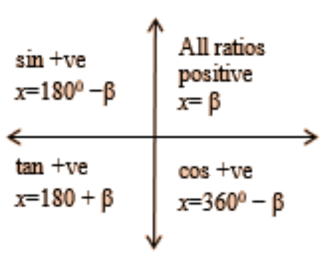
$2\sin x + 1 = 0$

$2\sin x = -1$

$\sin x = -\dfrac{1}{2}$

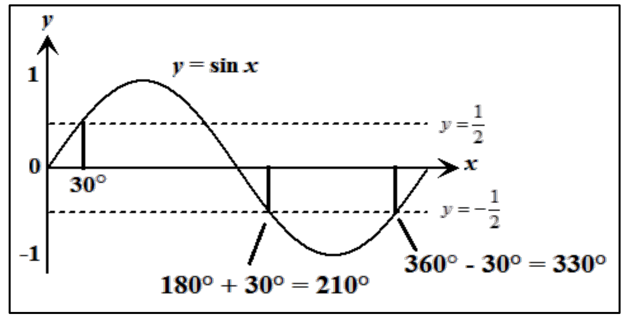
Ignore the negative sign and obtain the basic acute angle, β where β = $\sin^{-1}(0.5) = 30^0$

To obtain all the solutions between 0° to 360°, consider the signs of the trigonometric ratios in each quadrant.



The sine ratio is negative in quadrants 3 and 4.
In quadrant 3
$x = 180^0 + 30^0 = 150^0$
In quadrant 4,
$x = 360^0 - 30^0 = 330^0$
$x = 150^0, 330^0$

The graphical solution is shown below.



Example 6

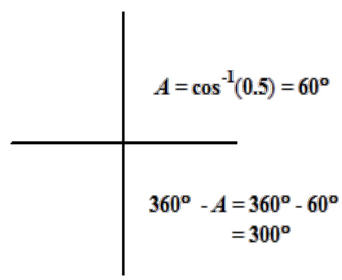
Solve for x from 0° to 360° in the equation, $\cos 3x = 0.5$

Solution

$\cos 3x = 0.5$

$3x = \cos^{-1}(0.5)$

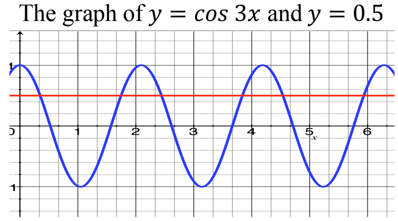
Solutions lie in quadrants 1 and 4 since cosine is positive in these quadrants. Let $A = 3x$



$A = 60^0$ or 300^0

These solutions satisfy the equation $y = \cos A$ for $0^0 \leq A \leq 360^0$ but we are solving an equation involving $y = \cos 3x$, which has 3 repetitions in the interval $0° \leq x \leq 360°$.
The line $y = 0.5$ cuts the graph of $y = \cos 3x$ at six points from 0 to 360^0 (or 0 to 2π radians), so there are 6 solutions.

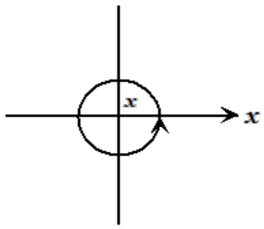
The graph of $y = cos\ 3x$ and $y = 0.5$



To obtain all six (6) solutions we need to complete three (3) rotations as shown below. For each rotation, we are interested only in the angles in the first and fourth quadrant.

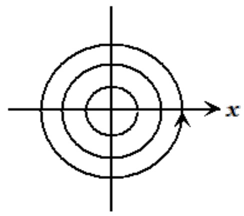
If $0° \le x \le 360°$ then $3(0°) \le 3x \le 3(360°)$.

$0° \le x \le 360°$



One rotation

$0° \le 3x \le 1080°$



Three rotations

Recall that angles in the fourth quadrant are calculated by subtracting the basic acute angle (in this case 60^0 from 360^0).

First rotation, solutions are 60^0 (first quadrant) and $360^0 - 60^0 = 300^0$ (fourth quadrant). For the other solutions, we keep adding 360^0 to these angles.

Second rotation, solutions are

$360^0 + 60^0, 360^0 + 300^0$

Third rotation, solutions are

$360^0 + (360^0 + 60^0), 360^0 + (360^0 + 300^0)$

Equating $3x$ to all 6 solutions, we have

$3x = 60^0, 300^0, 420^0, 660^0, 780^0, 1020^0$

$x = 20^0, 100^0, 140^0, 220^0, 260^0, 340^0$

Since all solutions fall within the required range, we keep all.

Example 7

Solve for x, in $2\cos\frac{1}{2}x - 1 = 0, \ 0^0 \le x \le 360^0$,

Solution

$$2\cos\frac{1}{2}x - 1 = 0$$

$$2\cos\frac{1}{2}x = 1$$

$$\cos\frac{1}{2}x = \frac{1}{2}$$

$$\frac{1}{2}x = \cos^{-1}\left(\frac{1}{2}\right) = 60^0$$

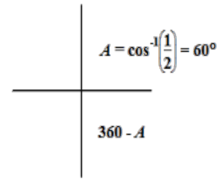
Cosine is positive in quadrants 1 and 4.

$$\frac{1}{2}x = 60^0, 360^0 - 60^0$$

$$\frac{1}{2}x = 60^0, 300^0$$

$x = 120^0$ only

[quadrant 4 is out of range]



Example 8

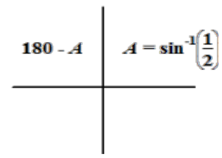
Solve for x where $0° \le x \le 360°$ in $2\sin^2 x - \sin x = 0$.

Solution

$$2\sin^2 x - \sin x = 0$$

$$\sin x(2\sin x - 1) = 0$$

$$\sin x = 0 \text{ or } \sin = \frac{1}{2}$$



When $\sin x = 0$

$$x = 0°, 180°, 360°$$

When $\sin x = \frac{1}{2}$

$x = 30°, 150°$

$x = 0°, 30°, 150°, 180°, 360°$

Example 9

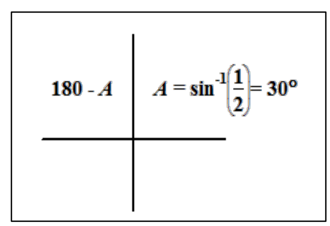
Solve for x from $0°$ to $360°$ in $2\sin\frac{1}{4}x - 1 = 0$.

Solution

$$2\sin\frac{1}{4}x - 1 = 0$$

$$\sin\frac{1}{4}x = \frac{1}{2}$$

$$\frac{1}{4}x = \sin^{-1}\frac{1}{2} = 30^0$$



Sine is positive in Quadrants 1 and 2. The solutions lie in quadrants 1 and 2.

$$\therefore \frac{1}{4}x = 30°, 180° - 30°$$

$$= 30°, 150°$$

$x = 120^0, 600^0$

$x = 120^0$ only

Expressing solutions in the range −180° to 180°

Sometimes we are asked to express the answer to a trigonometric equation in the range −180°to 180°. The following example illustrates how this can be done.

Example 10

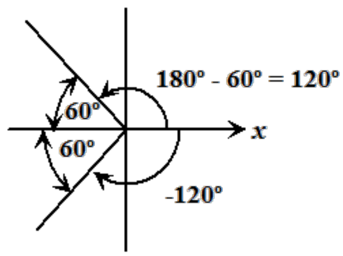
Solve for x, $\cos x = -0.5$, where $-180^0 < x < 180^0$.

Solution

$\cos x = -0.5$

$x = \cos^{-1}(-0.5)$

Cosine is negative in quadrants 2 and 3. If we were considering positive angles only, the answers are $x = 120^0, 240^0$.



The first answer $x = 120^0$, lies within our range as it is less than 180⁰.

A positive angle of 120^0 has a basic acute angle of 60^0 in the second quadrant.

A positive angle of 240^0 has a basic acute angle of 60^0 in the third quadrant.

Since the range for this question was −180°to 180°, the solution would be

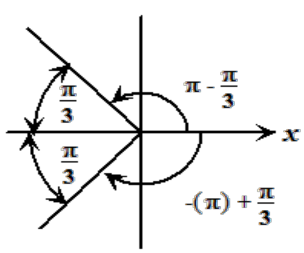
$x = 180^0 - 60^0 = 120^0$ and

$x = -180^0 + 60^0 = -120^0$

In radians, our result would be

$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and

$x = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$



Example 11

Solve for x in $2\sin x - 3\cos x = 0$, $0^0 \le x \le 360^0$

Solution

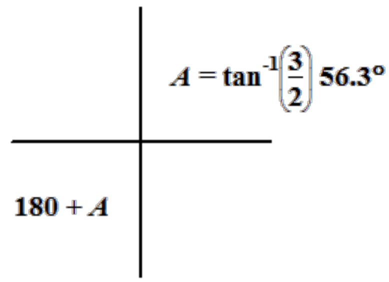
$2\sin x - 3\cos x = 0$

$2\sin x = 3\cos x$

$\frac{\sin x}{\cos x} = \frac{3}{2}$

$\tan x = \frac{3}{2}$

Tangent is positive in quadrants 1 and 3.



$x = 56.3°$ and $180° + 56.3°$

$= 56.3°$ and $236.3°$

Example 12

Solve for x in $\cos^2 x + \cos x = \sin^2 x$, where $0^0 \le x \le 360^0$

Solution

$\cos^2 x + \cos x = \sin^2 x$

$\cos^2 x + \cos x = 1 - \cos^2 x$

$2\cos^2 x + \cos x - 1 = 0$

$(2\cos x - 1)(\cos x + 1) = 0$

$\cos x = \frac{1}{2}$ or -1

When $\cos x = -1$ $x = 180°$

When $\cos x = \frac{1}{2}$,

$x = 60^0$ or 300^0,

Hence, $x = 180°$, 60^0 or 300^0

Example 13

Solve for θ : $8\sin^2\theta = 5 - 10\cos\theta,\ 0° \le \theta \le 360°$

Solution

$8\sin^2\theta = 5 - 10\cos\theta$

Recall: $\sin^2\theta + \cos^2\theta = 1$

$\therefore \sin^2\theta = 1 - \cos^2\theta$

Substituting this expression in the original equation, we obtain,

$$8\left(1 - \cos^2\theta\right) = 5 - 10\cos\theta$$

$8 - 8\cos^2\theta - 5 + 10\cos\theta = 0$

$-8\cos^2\theta + 10\cos\theta + 3 = 0$

$\times -1$

$8\cos^2\theta - 10\cos\theta - 3 = 0$

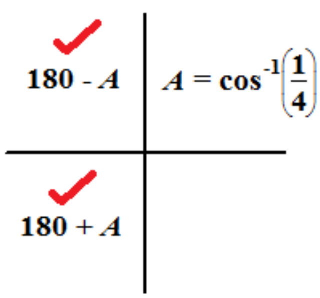
$(4\cos\theta + 1)(2\cos\theta - 3) = 0$

$$\therefore \cos\theta = -\frac{1}{4} \text{ or } \frac{3}{2}$$

$-1 \le \cos\theta \le 1 \ \ \forall\theta$.

Hence, $\cos\theta = \dfrac{3}{2}$ has no real solutions.

However, $\cos\theta = -\dfrac{1}{4}$, has solutions in quadrants 2 and 3, as shown below.



$A = 75.52°$

$\therefore \theta = 180° - 75.52°,\ 180° + 75.52°$

$= 104.48°,\ 255.52°$

$= 104.5°$ and $255.5°$ (correct to 1 decimal place)

Example 14

Solve the equation

$\sin^2\theta + 3\cos 2\theta = 2,\quad 0 \le \theta \le \pi$.

Give your answer(s) to 1 decimal place.

Solution

$\sin^2\theta + 3\cos 2\theta = 2$

Recall $\cos 2\theta = 1 - 2\sin^2\theta$

Substituting for $\cos 2\theta$

$$\sin^2\theta + 3\left(1 - 2\sin^2\theta\right) = 2$$

$$\sin^2\theta + 3 - 6\sin^2\theta = 2$$

$$-5\sin^2\theta = 2 - 3$$

$$-5\sin^2\theta = -1$$

$$5\sin^2\theta = 1$$

$$\sin^2\theta = \frac{1}{5}$$

$$\sin\theta = \pm\sqrt{\frac{1}{5}}$$

When

$\sin\theta = \sqrt{\dfrac{1}{5}}$

$\theta = 0.46$

Sine is positive in quadrants 1 and 2.

$\theta = 0.46$

and

$\theta = \pi - 0.46$

$= 2.67$

When

$\sin\theta = -\sqrt{\dfrac{1}{5}}$

$\theta = -0.46$

Sin is negative in quadrants 3 and 4.

So there are no solutions for the required range

$\theta = 0.46,\ 2.67$ for $0 \le \theta \le \pi$

$\theta = 0.5$ radians, 2.7 radians correct to 1 decimal place for $0 \le \theta \le \pi$