7: THE COORDINATE GEOMETRY OF THE CIRCLE

The equation of a circle

The equation of a circle can be expressed in different forms. To define a circle on the coordinate plane, we must know the coordinates of the centre and the length of the radius.

Equation of a circle, centre origin (0, 0)

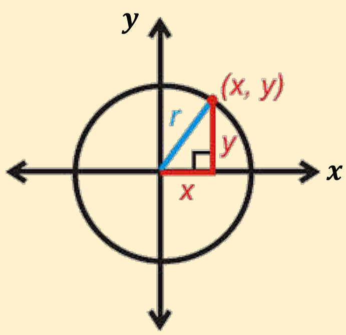
The equation of a circle is a rule satisfied by the coordinates (x, y) of any point that lies on the circumference. Points that do not lie on the circle will not satisfy the equation.

The equation of a circle will vary depending on its size (radius) and its position on the Cartesian Plane.

If a circle of radius r, passes through the origin, we can obtain its equation by considering any point (x, y) on the circumference and deriving a relationship between x, y and r using Pythagoras' Theorem.

Equation of a circle, centre origin

$$x^2 + y^2 = r^2$$



The right-angle triangle shown has sides of length x and y and the hypotenuse r, is the length of the radius. Applying Pythagoras' Theorem,

$$x^2 + y^2 = r^2$$

where (x, y) is the coordinates of any point on the circle and r is the radius.

Hence, a circle of radius 5 units, will have equation

$$x^2 + y^2 = 5^2$$

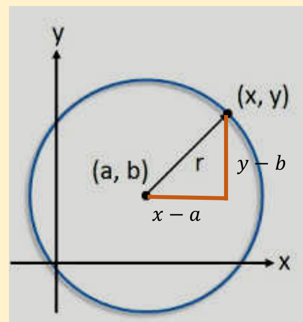
Points such as $(3, 4)$ and $(4,3)$ will lie on the circle but points such as $(1,2)$ and $(6, -2)$ will not lie on the circle. Note also that $(1,2)$ lies inside the circle and $(6, -1)$ lies outside the circle.

Equation of a circle, centre (a, b)

When the centre of the circle is (a, b), we still apply Phythagoras' Theorem to determine the equation.

Equation of a circle, centre (a, b)

$$(x-a)^2 + (y-b)^2 = r^2$$



The right-angle triangle has sides $(x - a)$ and $(y - b)$ and the hypotenuse, r, is the length of the radius. Applying Pythagoras' Theorem,

$r^2 = (x-a)^2 + (y-b)^2$

where (x, y) is the coordinates of any point on the circle and r is the radius.

For example, the equation of the circle, centre (3, 5) and radius 2 is $(x-3)^2 + (y-5)^2 = 2^2$.

Equation of a circle, centre $(-g, -f)$

The equation of a circle is sometimes seen in another form and can be derived from the one above.

$$(x-a)^2 + (y-b)^2 = r^2$$
$$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$$
$$x^2 + y^2 - 2ax - 2by = r^2 - (a^2 + b^2)$$

We know that the centre of the circle is (a, b).

Let $C(-g, -f)$, hence, $a = -g$ and $b = -f$.

Replacing a and b by $-g$ and $-f$, we can rewrite the equation in the form

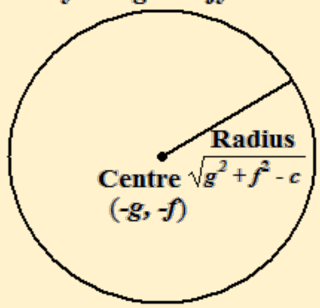
$$x^2 + y^2 + 2gx + 2fy = r^2 - (g^2 + f^2)$$
$$x^2 + 2gx + 2fy + (g^2 + f^2) - r^2 = 0$$

Letting $c = (g^2 + f^2) - r^2$, the equation becomes:

$$x^2 + +y^2 + 2gx + 2fy + c = 0$$

Equation of a circle, centre $(-g, -f)$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$



R= $\sqrt{g^2+f^2-c}$

In solving problems on the circle, we can choose any of the general forms, depending on the information given.

Example 1

Given that $(x-6)^2+(y+2)^2=45$ is the equation of a circle, state the coordinates of the centre and the radius.

Solution

The equation is of the form $(x-a)^2+(y-b)^2=r^2$ where (a, b) is the centre and r is the radius.

Hence, $a = 6$, $b = -2$, and r = $\sqrt{45}$

Example 2

A circle has an equation

$x^2+y^2+4x-2y-4=0$

State the coordinates of the centre and the radius of the circle.

Solution

The general form of the equation is:

$x^2+y^2+2gx+2fy+c=0$

The equation can be expressed as

$x^2+y^2+2(2)x+2(-1)y+(-4)=0$.

Hence, $g = 2$ and $f = -1$.

The centre of this circle is ($-g$, $-f$), so

$(-(2), -(-1)) = (-2, 1)$

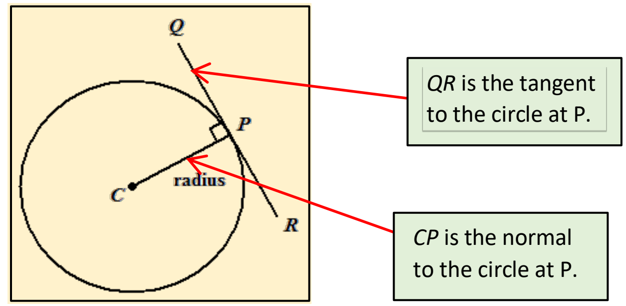
and the radius , $r = \sqrt{(2)^2+(-1)^2-(-4)} = 3$ units.

Tangent and normal to a circle

We define the tangent and normal to a circle as follows:

The tangent to a circle at a point is the straight line that 'just touches' the circle at that point.

The normal to a circle at a point is the straight line that is perpendicular to the tangent at that point.



The gradient of the tangent and normal to a circle

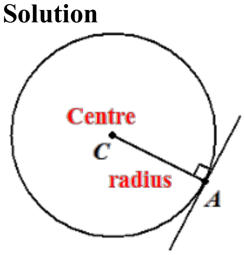
We recall that the angle made by the tangent to a circle and a radius, at the point of contact, is a right angle. Also, the product of the gradients of perpendicular lines is -1. Hence, if we obtain the gradient of either one of these lines, then we can deduce the gradient of the other.

To find the gradient of the tangent of a circle, centre, C at a given point, P, we calculate the gradient of the normal CP and then obtain the negative reciprocal of the gradient of CP.

Example 3

Find the gradient of the tangent of a circle at the point A (4, 7) whose centre is the origin.

Solution



The gradient of the normal, AC $= \frac{7-0}{4-0} = \frac{7}{4}$ where O is (0,0) and A(4, 7),

Hence, the gradient of the tangent at

$A = \dfrac{-1}{\text{Gradient of } AC}$

$= -\frac{4}{7}$.

The equation of the tangent and normal at a point on the circle

The equation of the tangent/normal will take form $y = mx + c$, and so to we need a point on the line and the gradient to find the specific equation.

Example 5

Find the equation of the (i) normal and (ii) the tangent to the circle $x^2 + y^2 + 2x - y - 5 = 0$ at the point (1, 2).

Solution

Rewriting the equation in the form

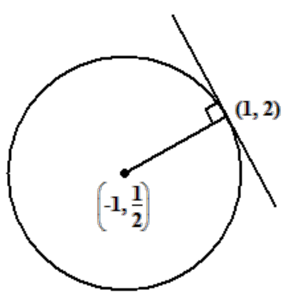
$x^2 + y^2 + 2gx + 2fy + c = 0$

we have:

$x^2 + y^2 + 2(1)x + 2\left(-\frac{1}{2}\right)y + (-5) = 0$

Hence, g=1 and f = ½

The centre of the circle is $\left(-1, \frac{1}{2}\right)$.



The gradient of the normal to the circle is the gradient of the radius drawn from the centre to the point (1, 2)

$$= \frac{2 - \frac{1}{2}}{1 - (-1)} = \frac{1\frac{1}{2}}{2} = \frac{3}{4}$$

The gradient of the normal is $\frac{3}{4}$, so the equation of the normal at (1, 2) is

$$\frac{y-2}{x-1} = \frac{3}{4}$$
$$4(y-2) = 3(x-1)$$
$$4y - 8 = 3x - 3$$
$$4y = 3x + 5$$

The gradient of the tangent is $-\frac{4}{3}$

The equation of the tangent at (1, 2) is

$$\frac{y-2}{x-1} = -\frac{4}{3}$$
$$3(y-2) = -4(x-1)$$
$$3y - 6 = -4x + 4$$
$$3y = -4x + 10$$

Alternative method to find the gradient of a tangent to a circle

The gradient of the tangent to a circle at any point can also be found by direct substitution using a formula. This formula is derived from implicit differentiation, a topic that is done at a higher level.

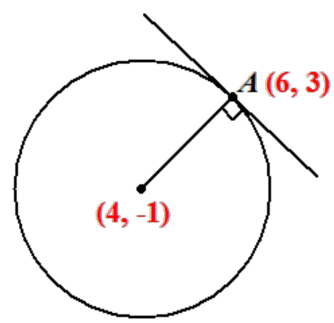
The gradient of the tangent to a circle with equation $x^2 + y^2 + 2gx + 2fy + c = 0$ at the point (x, y) is $\frac{-(x+g)}{y+f}$.

This is a more direct method and does not rely on computing the gradient of the normal.

Example 4

A circle has centre (4, −1). Find the gradient of the tangent to the circle at the point A (6, 3).

Solution



Centre $(4, -1) \Rightarrow g = -4,\ f = 1$

$A = (6, 3) \Rightarrow x = 6,\ y = 3$

The gradient of the tangent at A is obtained by substituting in the formula:

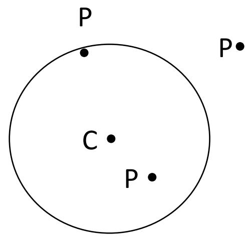
$$\frac{-(x+g)}{y+f} = \frac{-(6+(-4))}{3+1} = \frac{-2}{4} = -\frac{1}{2}$$

The position of a point relative to a circle

To determine whether a point, P lies within the circle or on the circle or outside the circle, we need information on the position of its centre, C and the length of the radius, r.

The following guidelines are suggested.

1. Determine the coordinates of C, the centre of the circle and calculate, r, the radius.
2. Calculate, the length of CP, which is the distance from the centre to the point, P.



3. If

   $CP < r \Rightarrow$ the point lies within the circle

   $CP = r \Rightarrow$ the point lies on the circumference of the circle.

   $CP > r \Rightarrow$ the point lies outside the circle

Example 6

Determine whether the point (2,4) lies on the circle or within the circle or outside the circle, whose equation is $x^2 + y^2 = 9$.

Solution

The equation of the circle is of the form $x^2 + y^2 = r^2$, hence it's centre is the origin and the radius is √9 =3 units.

Let P be (2,4)

The distance from the centre C to P

$$CP = \sqrt{(4-0)^2 + (2-0)^2} = \sqrt{20} = 4.47$$

The radius of the circle is 3 units.

Since $CP > 3$, then P lies outside the circle

Intersection of a circle and a straight line

In determining the point(s) of intersection of the circle whose equation is of the form $x^2 + y^2 + 2gx + 2fy + c = 0$ and the straight line $y = mx + c$, we note that three possible cases can arise. The line can:

1. Cut the circle at two distinct points,
2. Touch the circle at one point or the line or
3. The circle can have no intersection.

These three cases are illustrated in the figures below.

Intersection of a line and a circle

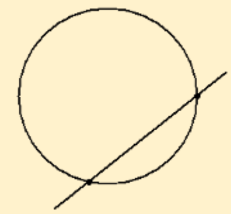


Figure 1

The line cuts the circle at two points.

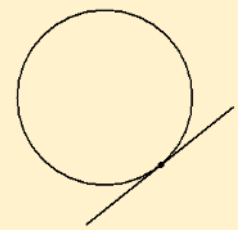


Figure 2

The line touches the circle at one point

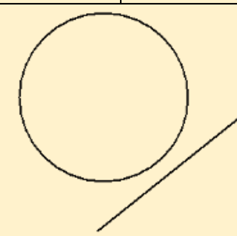


Figure 3

The line and the circle do not intersect

The procedure for solving problems involving the intersection of a line and a circle is outlined below.

To obtain the point (s) of intersection of a circle and a straight line, if they exist.

1. Solve the two equations simultaneously to obtain a quadratic equation.
2. If there are two distinct roots, then there are two points of intersection, as shown in Figure 1.
3. If there is one root, i.e, the roots of the quadratic are equal, then the line 'just touches' the circle and is, therefore, a tangent at that point, as shown in Figure 2
4. If there are no real roots or what is referred to as imaginary roots, then the line and the circle do not meet or intersect, as shown in Figure 3

Example 7

Find the points of intersection of the line $y = 3x - 1$ and the circle with the equation, $x^2 + y^2 + 2x - 4y + 1 = 0$.

Solution

Solving simultaneously
$y = 3x - 1 \ldots(1)$ and
$x^2 + y^2 + 2x - 4y + 1 = 0 \ \ldots(2)$

Substitute eq. (1) into eq.(2)

$x^2 + (3x-1)^2 + 2x - 4(3x-1) + 1 = 0$

$x^2 + 9x^2 - 6x + 1 + 2x - 12x + 4 + 1 = 0$

$10x^2 - 16x + 6 = 0$

$5x^2 - 8x + 3 = 0$

$(5x-3)(x-1) = 0$

$x = 1$, or $x = \frac{3}{5}$

We substitute the values of x in either equation to obtain the corresponding y coordinates.

When $x = 1$, $y = 3(1) - 1 = 2$

And when $x = \frac{3}{5}$ $y = 3\left(\frac{3}{5}\right) - 1 = \frac{4}{5}$

$\therefore$ The line cuts the circle at (1, 2) and $\left(\frac{3}{5}, \frac{4}{5}\right)$.

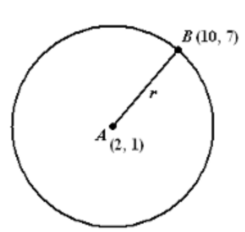
Example 8

A circle with centre $A(2,1)$, passes through the point $B(10,7)$.
(a) Determine the equation of the circle in the form
$$x^2 + y^2 + hx + gy + k = 0$$
where h, g and $k \in Z$.

(b) The line, l, is a tangent to the circle at B. Determine the equation of l.

Solution

The length of the radius $= \sqrt{(10-2)^2 + (7-1)^2}$
=10 units



The centre of the circle is (2, 1)

$\therefore$ Equation of the circle is

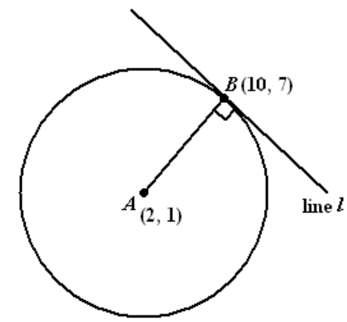
$$(x-2)^2 + (y-1)^2 = (10)^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 100$$

$$x^2 + y^2 - 4x - 2y - 95 = 0$$

which is of the form
$x^2 + y^2 + hx + gy + k = 0$, where
$h = -4 \in Z$, $g = -2 \in Z$, $k = -95 \in Z$

(b)



The gradient of AB, $= \frac{7-1}{10-2} = \frac{6}{8} = \frac{3}{4}$

The equation of l is

$$\frac{y-7}{x-10} = \frac{-4}{3}$$

$$3(y-7) = -4(x-10)$$

$$3y - 21 = -4x + 40$$

$$3y + 4x - 61 = 0$$

$$3y = -4x + 61$$

$$y = -\frac{4}{3}x + 20\frac{1}{3}$$