17. MATRICES AND MATRIX TRANSFORMATIONS

MATRICES

A matrix is a rectangular array of numbers (or symbols) enclosed in brackets either curved or square. The constituents of a matrix are called entries or elements. A matrix is usually named by a letter for convenience. Some examples are shown below.

$$X = \begin{pmatrix} 3 & 1 & 4 \\ 2 & -7 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 3 & -1 & 4 & 2 & 6 \\ 0 & 0 & 1 & 0 & -5 \end{pmatrix}$$

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$Y = \begin{bmatrix} 4 & 1 & 2 \\ 3 & -1 & 5 \\ 6 & -2 & 10 \end{bmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rows and Columns

The elements of a matrix are arranged in rows and columns. Elements that are written from left to right (horizontally) are called rows. Elements that are written from top to bottom (vertically) are called columns. The first row is called 'row 1', the second 'row 2', and so on. The first column is called 'column 1, the second 'column 2', and so on.

$$M = \begin{pmatrix} 3 & 1 & 4 & -1 \\ 0 & 11 & -5 & 8 \\ 2 & 1 & 6 & -4 \end{pmatrix} \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccc} \text{Col 1} & \text{Col 2} & \text{Col 3} & \text{Col 4} \end{array} \\ N = & \begin{pmatrix} 3 & 1 & 4 & -1 \\ 0 & 11 & -5 & 8 \\ 2 & 1 & 6 & -4 \end{pmatrix} \end{array}$$

Order of a matrix

The order of a matrix is written as $m \times n$, where m represents the number of rows and n represents the number of columns. A matrix of order 4×3, consists of 4 rows and 3 columns while a matrix of order 3×4 consists of 3 rows and 4 columns.

$$M = \begin{pmatrix} 2 & 1 & 3 & 0 \\ 5 & 7 & -6 & 8 \\ 9 & -2 & 6 & -3 \end{pmatrix} 3 \times 4$$

$$N = \begin{pmatrix} 4 & 5 & 7 \\ -2 & 3 & 6 \\ -7 & 1 & 0 \\ 9 & -5 & 8 \end{pmatrix} 4 \times 3$$

Note that the orders 3×4 and 4×3 are NOT the same.

Row matrices

If a matrix is composed only of one row, then it is called a row matrix (regardless of its number of elements). The matrices J, K and L are row matrices,

$$J = \begin{pmatrix} 4 & 1 & 3 \end{pmatrix} \qquad L = \begin{pmatrix} 3 & 0 & -4 & 2 \end{pmatrix}$$

$$K = \begin{pmatrix} 2 & 1 \end{pmatrix}$$

Column matrices

If a matrix is composed of only one column, then it is called a column matrix (regardless of the number of elements). The matrices P, Q and R are column matrices.

$$P = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad Q = \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} \qquad R = \begin{pmatrix} 6 \\ 7 \\ 8 \\ 9 \end{pmatrix}$$

Square matrices

If a matrix has the same number of rows as the number of columns, then it is called square. For example, a matrix that has 6 rows and 6 columns is a square matrix. We may describe such a matrix as being square of order 6 or simply a 6×6 matrix. An $m \times m$ matrix is a square matrix of order m. Q and S are square matrices.

$$Q = \begin{pmatrix} 1 & -1 \\ 4 & 2 \end{pmatrix} \quad 2 \times 2$$

$$S = \begin{pmatrix} 1 & 2 & 4 \\ -1 & -1 & 2 \\ 0 & 0 & 11 \end{pmatrix} \quad 3 \times 3$$

The position of elements in a matrix

Every single entry in a matrix has a specific position that can be uniquely described. In describing the position, we use the notation a_{ij} where the subscript, i, refers to the row number and the subscript j refers to the column number of the element.
Since each position is unique, no two entries can have the same row and column number. The symbol a represents an element.

a_{ij} belongs to the i^{th} row and the j^{th} column.

a_{42} belongs to the 4th row and the 2nd column.

a_{24} belongs to the 2nd row and the 4th column.

For the matrix, A, defined below, we can assign each element its unique position shown below. Notice $a_{ij} \neq a_{ji}$. For example, $a_{13} \neq a_{31}$.

$$A = \begin{pmatrix} 4 & 1 & 3 & 2 \\ 0 & -11 & 6 & 5 \\ 5 & -10 & -2 & 1 \end{pmatrix}$$

$$\begin{array}{cc} & \begin{array}{cccc} \mathit{Col}\ 1 & \mathit{Col}\ 2 & \mathit{Col}\ 3 & \mathit{Col}\ 4 \end{array} \\ A = & \begin{pmatrix} 4_{11} & 1_{12} & 3_{13} & 2_{14} \\ 0_{21} & -11_{22} & 6_{23} & 5_{24} \\ 5_{31} & -10_{32} & -2_{33} & 1_{34} \end{pmatrix} \begin{array}{l} \mathit{Row}\ 1 \\ \mathit{Row}\ 2 \\ \mathit{Row}\ 3 \end{array} \end{array}$$

Diagonal elements

Diagonal elements are the elements positioned along the diagonal of a square matrix. A and B are square matrices and have diagonal elements of the type a_{ii}.

Square matrix with three diagonal elements

$$A = \begin{pmatrix} 3_{11} & 1 & 4 \\ 2 & 0_{22} & 6 \\ -11 & 5 & -8_{33} \end{pmatrix}$$

Square matrix with two diagonal elements

$$B = \begin{pmatrix} 1_{11} & 4 \\ 0 & 6_{22} \end{pmatrix}$$

Diagonal matrices

A diagonal matrix is a square matrix whose non-diagonal elements are zero. T and V are diagonal matrices.

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \qquad V = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Zero matrix

If all the elements of any matrix are zero(s), then the matrix is called a zero matrix. Some examples are shown below. A zero matrix can be of any order.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Operations on Matrices

In performing operations on matrices, there are some restrictions. Unlike numbers, one cannot always add, subtract or multiply any two matrices. In fact, a division of two matrices is not even possible.

Addition

Matrices can only be added if they are of the same order. This is done by adding or subtracting corresponding entries. The resulting matrix will also be of the same order.

$$A = \underset{2\times 3}{\begin{bmatrix} 1 & 2 & -1 \\ 4 & 3 & -5 \end{bmatrix}} \qquad B = \underset{2\times 3}{\begin{bmatrix} 0 & -4 & 2 \\ -1 & 6 & 3 \end{bmatrix}}$$

$$A + B = \underset{2\times 3}{\begin{bmatrix} 1_{11} & 2_{12} & -1_{13} \\ 4_{21} & 3_{22} & -5_{23} \end{bmatrix}} + \underset{2\times 3}{\begin{bmatrix} 0_{11} & -4_{12} & 2_{13} \\ -1_{21} & 6_{22} & 3_{23} \end{bmatrix}}$$

$$A + B = \underset{2\times 3}{\begin{bmatrix} (1+0)_{11} & (2+(-4))_{12} & (-1+2)_{13} \\ (4+(-1))_{21} & (3+6)_{22} & (5+3)_{23} \end{bmatrix}}$$

$$A + B = \underset{2\times 3}{\begin{bmatrix} 1 & -2 & 1 \\ 3 & 9 & 8 \end{bmatrix}}$$

Subtraction

Matrices can only be subtracted if they are of the same order. This is done by subtracting corresponding entries. The resulting matrix will also be of the same order.

$$A=\begin{bmatrix}1 & 2 & -1\\ 4 & 3 & -5\end{bmatrix}$$
$$2\times 3$$
$$B=\begin{bmatrix}0 & -4 & 2\\ -1 & 6 & 3\end{bmatrix}$$
$$2\times 3$$
$$A-B=\underset{2\times 3}{\begin{bmatrix}1_{11} & 2_{12} & -1_{13}\\ 4_{21} & 3_{22} & -5_{23}\end{bmatrix}}-\underset{2\times 3}{\begin{bmatrix}0_{11} & -4_{12} & 2_{13}\\ -1_{21} & 6_{22} & 3_{23}\end{bmatrix}}$$
$$A-B=\begin{bmatrix}(1-0)_{11} & (2-(-4))_{12} & (-1-2)_{13}\\ (4-(-1))_{21} & (3-6)_{22} & (5-3)_{23}\end{bmatrix}$$
$$2\times 3$$
$$A-B=\begin{bmatrix}1 & 6 & -3\\ 5 & -3 & 2\end{bmatrix}$$
$$2\times 3$$

Rule for addition and subtraction of matrices

If two matrices have the same order, the matrices are said to be conformable to addition and subtraction. We obtain the resulting matrix by adding or subtracting corresponding elements.

Scalar multiplication

If a matrix is multiplied by a scalar then each element of the matrix is multiplied by the scalar. The resulting matrix is of the same order.

$$A=\begin{pmatrix}a & b & c\\ d & e & f\end{pmatrix}$$

$$kA=\begin{pmatrix}ka & kb & kc\\ kd & ke & kf\end{pmatrix}, \; k \text{ is a scalar}$$

Example 1

(i) If $A=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$, then calculate $3A$

(ii) If $B=\begin{pmatrix}4 & -1 & 2\\ 0 & 7 & -3\end{pmatrix}$, then calculate $-2B$

Solution

(i) $3A=3\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}=\begin{pmatrix}6 & 3\\ 9 & 12\end{pmatrix}$

(ii) $-2B=-2\begin{pmatrix}4 & -1 & 2\\ 0 & 7 & -3\end{pmatrix}=\begin{pmatrix}-8 & 2 & -1\\ 0 & 14 & -6\end{pmatrix}$

Equal matrices

Two matrices are equal if:

1. They are of the same order
2. Their corresponding entries are equal.

Both conditions must be satisfied before we can deduce that the matrices are equal. Conversely, if matrices are equal then we can deduce that they must be of the same order and that their corresponding entries are equal. Consider the matrices S and T. They are of the same order and their corresponding entries are the same. Therefore, $S=T$.

$$S=\begin{pmatrix}1 & 3\\ 4 & -6\\ 2 & 0\end{pmatrix} \qquad T=\begin{pmatrix}1 & 3\\ 4 & -6\\ 2 & 0\end{pmatrix}$$

Example 2

Given that $A=\begin{bmatrix}4 & 2x\\ -y & -2\end{bmatrix}$ $B=\begin{bmatrix}4 & 6\\ 2 & -2\end{bmatrix}$ and $A=B$, find the value of x and of y.

Solution

If $A=B$, then A and B are of the same order and their corresponding entries are the same.
Equating corresponding entries:

$2x=6$ $\qquad$ $-y=2$

$\therefore x=3$ $\qquad$ $\therefore y=-2$

Example 3

Given that $A=\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}$ and $B=\begin{pmatrix}0 & 1\\ 4 & 3\end{pmatrix}$ and $A-2B=\begin{pmatrix}1 & 2x\\ \frac{y}{2} & -z\end{pmatrix}$, find the value of x, y and z.

Solution

We first need to obtain a single matrix for $A - 2B$.

$$A-2B=\begin{pmatrix}1 & 2x\\ \frac{y}{2} & -z\end{pmatrix}$$

$$\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}-2\begin{pmatrix}0 & 1\\ 4 & 3\end{pmatrix}=\begin{pmatrix}1 & 2x\\ \frac{y}{2} & -z\end{pmatrix}$$

$$\begin{pmatrix}1 & -2\\ 1 & 3\end{pmatrix}-\begin{pmatrix}0 & 2\\ 8 & 6\end{pmatrix}=\begin{pmatrix}1 & 2x\\ \frac{y}{2} & -z\end{pmatrix}$$

$$\begin{pmatrix}1 & -4\\ -7 & -3\end{pmatrix}=\begin{pmatrix}1 & 2x\\ \frac{y}{2} & -z\end{pmatrix}$$

Equating corresponding entries

$-4=2x$	$-7=\frac{y}{2}$	$-3=-z$
$2x=-4$	$\frac{y}{2}=-7$	$-z=-3$
$x=\frac{-4}{2}$	$y=-7(2)$	$z=3$
$x=-2$	$y=-14$	

The Identity or Unit matrix

If all the diagonal elements of a diagonal matrix are equal to one, then it is called the unit or identity matrix and is denoted by U or I only.

A unit matrix of order 2	A unit matrix of order 3
$I=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$U=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

Matrix multiplication

Unlike addition and subtraction, the order of two matrices need not be the same for multiplication. If A and B are two matrices, for $A\times B$ to be possible, then the number of columns of A must be equal to the number of rows of B. We write this symbolically as:

$$\begin{array}{ccccc} A & \times & B & = & C\\ m\times n & & n\times p & & m\times p\end{array}$$

The table below list some examples of this principle, for 2 matrices A and B.

Order of A	Order of B	Order of C
2×3	3×2	2×2
3×4	4×2	3×2
4×2	2×3	4×3
4×3	3×2	4×2

If two matrices cannot be multiplied, they are said to be non-conformable to multiplication. For example, the product YX is not possible since the number of columns of Y (2) ≠ the number of rows of X, (4)

$$\begin{array}{ccc} Y & \times & X \Rightarrow \text{This cannot be done.}\\ 3\times 2 & & 4\times 3\end{array}$$

We will now illustrate how two matrices can be multiplied using the matrices A and B below.

$$A=\begin{pmatrix}1 & 4 & -1\\ 2 & 0 & -3\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 2 & 3 & 2\\ -1 & -1 & 1 & -1\\ 0 & 4 & 6 & 2\end{pmatrix}$$

We wish to determine the matrix AB.

First, check to see if this is possible.

$$\begin{array}{ccccc} A & \times & B & = & C\\ 2\times 3 & & 3\times 4 & & 2\times 4\end{array}$$

Based on the above rule, the matrix product AB exists and the product, C will be a 2×4 matrix.

We write out the structure of the answer, that is, what C looks like. We set up a 2×4 matrix, using the symbol e, to represent an element.

$$C=\begin{array}{cc} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4\end{array} & \\ \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14}\\ e_{21} & e_{22} & e_{23} & e_{24}\end{pmatrix} & \begin{array}{c} R_1\\ R_2\end{array}\end{array}$$

We then set up the multiplication in the correct order:

$$A\times B=\underset{2\times 3}{\begin{pmatrix}1 & 4 & -1\\ 2 & 0 & -3\end{pmatrix}}\times\underset{3\times 4}{\begin{pmatrix}1 & 2 & 3 & 2\\ -1 & -1 & 1 & -1\\ 0 & 4 & 6 & 2\end{pmatrix}}$$

Each element in the product is calculated by multiplying corresponding elements, for example, to compute e_{13}, we multiply corresponding elements of Row 1 from the first matrix, A, with elements of Column 3 from the second matrix, B.

Each element in the product is the sum of three terms. For example, e_{11} is the product of the elements of Row 1 and Column 1.

$$R_1C_1 = \begin{pmatrix}1 & 4 & -1\end{pmatrix}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Notice that there are 3 elements in each row and 3 elements in each column. Hence, there is a one-one correspondence between the elements. We simply match corresponding elements in order and multiply each pair. The sum of the three products is the value of the element. The computation is completed below.

$$e_{11} = 1\times1+4\times(-1)+(-1)\times0 = 1-4+0 = -3$$
$$e_{12} = 1\times2+4\times(-1)+(-1)\times4 = 2-4-4 = -6$$
$$e_{13} = 1\times3+4\times1+(-1)\times6 = 3+4-6 = 1$$
$$e_{21} = 2\times1+0\times(-1)+(-3)\times0 = 2+0+0 = 2$$
$$e_{22} = 2\times2+0\times(-1)+(-3)\times4 = 4+0-12 = -8$$
$$e_{23} = 2\times3+0\times1+(-3)\times6 = 6+0-18 = -12$$
$$e_{24} = 2\times2+0\times(-1)+(-3)\times(-2) = 4+0+6 = 10$$
$$\therefore A\times B = \begin{pmatrix}-3 & -6 & 1 & 0\\2 & -8 & -12 & 10\end{pmatrix}$$

Example 4

Given that $P = \begin{pmatrix}2 & -1 & 0\\1 & 4 & 1\end{pmatrix}$ and $Q = \begin{pmatrix}1 & -1\\4 & 1\\0 & 1\end{pmatrix}$,

find PQ.

Solution

First, we check the order of the resulting matrix:

$$\begin{matrix}P & \times & Q & = & R\\2\times3 & & 3\times2 & \Rightarrow & 2\times2\end{matrix}$$

Now set up the multiplication, equating it to the structure of the product

$$\begin{matrix}P & \times & Q & = & R\end{matrix}$$

$$\begin{pmatrix}2 & -1 & 0\\1 & 4 & 1\end{pmatrix}\times\begin{pmatrix}1 & -1\\4 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}e_{11} & e_{12}\\e_{21} & e_{22}\end{pmatrix}$$

Multiply corresponding elements as follows:

$$e_{11} = 2\times1+(-1)\times4+0\times0 = -2$$
$$e_{12} = 2\times(-1)+(-1)\times1+0\times1 = -3$$
$$e_{21} = 1\times1+4\times4+2\times0 = 17$$
$$e_{22} = 1\times(-1)+4\times1+2\times1 = 5$$

$$\therefore PQ = \begin{pmatrix}-2 & -3\\17 & 5\end{pmatrix}$$

Commutative Property

The commutative law of multiplication is not obeyed in matrices. Consider two matrices A, and B, such that:

$$\begin{matrix}A & \times & B & = & C\\4\times2 & & 2\times4 & & 4\times4\\B & \times & A & = & D\\2\times4 & & 4\times2 & & 2\times2\end{matrix}$$

The product, $AB = C$ and C is a 4×4 matrix.
The product $BA = D$ and D is a 2×2 matrix.
Since the order of C and D is not the same, C and D cannot be equal, $C \neq D$, and it follows that $AB \neq BC$.

We can conclude that matrix multiplication is not communicative.

Example 5

Given that $A = \begin{pmatrix}1 & -1\end{pmatrix}$ and $B = \begin{pmatrix}1\\2\end{pmatrix}$.

Calculate (i) AB (ii) BA
Comment on the products.

Solution

(i) $\begin{matrix}A & \times & B & = & C\\1\times2 & & 2\times1 & \Rightarrow & 1\times1\end{matrix}$

$$C = (e_{11})$$

$$\begin{matrix}A & \times & B & = & C\end{matrix}$$

$$\begin{pmatrix}1 & -1\end{pmatrix}\times\begin{pmatrix}1\\2\end{pmatrix} = (e_{11})$$

$$e_{11} = 1\times1+(-1)\times2 = 1-2 = -1$$

$$\therefore AB = (-1)$$

(ii) $\begin{matrix}B & \times & A & = & D\\2\times1 & & 1\times2 & \Rightarrow & 2\times2\end{matrix}$

$$D = \begin{pmatrix}e_{11} & e_{12}\\e_{21} & e_{22}\end{pmatrix}$$

$$\begin{pmatrix}1\\2\end{pmatrix}\times\begin{pmatrix}1 & -1\end{pmatrix} = \begin{pmatrix}e_{11} & e_{12}\\e_{21} & e_{22}\end{pmatrix}$$

$e_{11} = 1 \times 1 = 1$
$e_{12} = 1 \times -1 = -1$
$e_{21} = 2 \times 1 = 2$
$e_{22} = 2 \times -1 = -2$
$BA = \begin{pmatrix}1 & -1\\2 & -2\end{pmatrix}$

The products $AB \neq BA$. This illustrates the non-commutative property of matrix multiplication.

Example 6

Given that $A=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}$, $B=\begin{pmatrix}3 & -1\\ 0 & 4\end{pmatrix}$ and $AB=\begin{pmatrix}2a & -b\\ c & \frac{d}{3}\end{pmatrix}$, calculate the values of a, b, c and d.

Solution

$$A\times B=\begin{pmatrix}-1 & 0\\ 2 & 3\end{pmatrix}\times\begin{pmatrix}3 & -1\\ 0 & 4\end{pmatrix}$$

$$\quad 2\times 2 \qquad\qquad 2\times 2$$

$$\xleftrightarrow{2\times 2}$$

$$A\times B=\begin{pmatrix}e_{11} & e_{12}\\ e_{21} & e_{22}\end{pmatrix}$$

$$e_{11}=(-1)\times 3+0\times 0=-3$$

$$e_{12}=(-1)\times(-1)+0\times 4=1$$

$$e_{21}=2\times 3+3\times 0=6$$

$$e_{22}=2\times(-1)+3\times 4=10$$

$$\therefore AB=\begin{pmatrix}-3 & 1\\ 6 & 10\end{pmatrix}$$

$$\begin{pmatrix}2a & -b\\ c & \frac{d}{3}\end{pmatrix}=\begin{pmatrix}-3 & 1\\ 6 & 10\end{pmatrix}$$

Equating corresponding elements:

$-3=2a$	$-b=1$	By inspection	$\frac{d}{3}=10$
$2a=-3$	$b=-1$	$c=6$	$d=10\times 3$
$a=-\frac{3}{2}$			$d=30$

Identity elements and inverses - 2×2 matrices

We have encountered the terms additive identity elements and additive inverses in relation to numbers.

The additive inverse of a number N is $-N$ because

$$\underset{\text{Number}}{\underset{\uparrow}{N}} \; + \; \underset{\text{Inverse}}{\underset{\uparrow}{-N}} \; = \; \underset{\text{Identity}}{\underset{\uparrow}{\text{Zero}}}$$

Zero is called the identity element for addition because adding zero to any number leaves the number unchanged. If N represents any number, then $N+0=N$

We can apply the same concept.

The additive inverse of a matrix M is $-M$ because

$$\underset{\text{Matrix}}{\underset{\uparrow}{M}} \; + \; \underset{\text{Inverse}}{\underset{\uparrow}{-M}} \; = \; \underset{\text{Identity}}{\underset{\uparrow}{\text{Zero Matrix}}}$$

The null or zero matrix is the identity element for the addition of matrices because adding the zero matrix to any matrix leaves the matrix unchanged.

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

We will now recall the terms multiplicative identity elements and multiplicative inverses in relation to numbers.

The multiplicative inverse of a number, N is N^{-1} because

$$\underset{\text{Number}}{\underset{\uparrow}{N}} \; \times \; \underset{\text{Inverse}}{\underset{\uparrow}{N^{-1}}} \; = \; \underset{\text{Identity}}{\underset{\uparrow}{1}}$$

Element

The identity element for multiplication of numbers is 1 because multiplying any number by 1 leaves the number unchanged. If N represents any number, then

$$N\times 1=N$$

We can apply the same concept to matrices, whereby,

The multiplicative inverse of a matrix, M is M^{-1} because

$$\underset{\text{Matrix}}{\underset{\uparrow}{M}} \; \times \; \underset{\text{Inverse}}{\underset{\uparrow}{M^{-1}}} \; = \; \underset{\text{Identity Matrix}}{\underset{\uparrow}{I}}$$

The identity element for multiplication of matrices is the identity matrix, I because multiplying any matrix by I leaves the matrix unchanged. If M represents any 2×2 matrix, then

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\times\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$$

To find the additive inverse of a matrix, we simply reverse the directions of all its elements. However, to find the multiplicative inverse of a matrix requires a number of steps. We will now illustrate how to carry these steps in relation to 2×2 matrices only.

Inverse of a 2×2 matrix

The inverse of a matrix requires applying a set of procedures and is not as simple as finding the inverse of a number. In addition, all matrices do not have inverses. To determine if a matrix has an inverse we must examine its determinant.

The determinant of a 2×2 matrix

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $|M|$ or $\det M = ad - bc$.

If $\det M$ or $|M| = 0$, then the matrix does not have an inverse and is singular.

After establishing that the matrix is non-singular we proceed to find its adjunct.

The adjunct of a 2×2 matrix

The adjunct of a 2x2 matrix is found by rearranging the elements on the main diagonal and changing the signs of the elements on the other diagonal.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the adjunct of M, is

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

The inverse of a 2×2 matrix

If , $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $|A| = ad - bc$, where $ad - bc \neq 0$, then

$$M^{-1} = \frac{1}{|A|} \text{ (adjoint of A)}$$

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example 7

Given that $B = \begin{pmatrix} 6 & 2 \\ 3 & 4 \end{pmatrix}$, find B^{-1}.

Solution

First, examine the value of the determinant.

$\det B = (6 \times 4) - (2 \times 3)$

$= 24 - 6$

$= 18$

Since det B is not zero, B has an inverse.

$$B^{-1} = \frac{1}{18} \begin{pmatrix} 4 & -(2) \\ -(3) & 6 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{18} & -\frac{2}{18} \\ -\frac{3}{18} & \frac{6}{18} \end{pmatrix}$$

Example 8

Show that the matrix, N is singular where $N = \begin{pmatrix} 6 & 4 \\ 3 & 2 \end{pmatrix}$.

Solution

$|N| = (6 \times 2) - (4 \times 3)$

$= 12 - 12$

$= 0$

$|N| = 0 \Rightarrow N$ is singular and has no inverse.

Example 9

Given that $A = \begin{pmatrix} 4 & a \\ 2 & -3 \end{pmatrix}$ and that A is singular, find a.

Solution

If A is singular then $|A| = 0$.

$(4 \times -3) - (a \times 2) = 0$

$-12 - 2a = 0$

$-2a = 12$

$a = -6$

Property of the inverse

Only square matrices have inverses. The product of a matrix and its inverse is the identity inverse. This is true regardless of its order.

Example 10

Given that $Q = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}$, find Q^{-1}. Show that $Q \times Q^{-1} = I$.

Solution

Det Q = $(2 \times -3) - (-1 \times 4) = -6 - (-4) = -2$

Determinant is non-zero, Q has an inverse

$$Q^{-1} = \frac{1}{-2} \begin{pmatrix} -3 & 1 \\ -4 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & -1 \end{pmatrix}$$

$$Q \times Q^{-1} = \begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 2 & -1 \end{pmatrix}$$

$$Q \times Q^{-1} = \begin{pmatrix} 3-1 & -1+1 \\ 6-6 & -2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Using matrices to solve a pair of simultaneous equations

To solve a pair of simultaneous equations using the matrix method, we must first convert the pair of equations to the matrix form.

This is done by extracting the coefficients of the variables to form a 2×2 matrix. This matrix of coefficients multiplied by the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ should produce the left-hand side of the matrix equation, as shown in the examples below.

$3x+2y=8$ $5x-6y=4$ The matrix equation is: $\begin{pmatrix} 3 & 2 \\ 5 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 8 \\ 4 \end{pmatrix}$	$4x-3y=1$ $5x+7y=12$ The matrix equation is $\begin{pmatrix} 4 & -3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ 12 \end{pmatrix}$

Using a matrix method, we will solve for x and y in:

$7x-3y=11 \quad \ldots(1)$

$4x+5y=13 \quad \ldots(2)$

1. Rewrite as a matrix equation:

$$\therefore \begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

This is of the form $AX = B$, where A, X and B are matrices with

$$A = \begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and } B = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

2. Isolate $X = \begin{pmatrix} x \\ y \end{pmatrix}$

In our study of simple equations, we used inverses to isolate the variable and solve the equation. We do likewise when solving matrix equations. To isolate the unknown matrix, X, we need to determine the inverse of the matrix, A.

Let $A=\begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix}$, now $\begin{aligned}|A|&=(7\times5)-(-3\times4)\\&=47\end{aligned}$

So, the inverse of A is

$$A^{-1}=\frac{1}{47}\begin{pmatrix} 5 & -(-3) \\ -(4) & 7 \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{47} & \frac{3}{47} \\ -\frac{4}{47} & \frac{7}{47} \end{pmatrix}$$

Pre-multiply the matrix equation by A^{-1} on both sides to isolate the unknown matrix.

$$\begin{pmatrix} \frac{5}{47} & \frac{3}{47} \\ -\frac{4}{47} & \frac{7}{47} \end{pmatrix}\begin{pmatrix} 7 & -3 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} \frac{5}{47} & \frac{3}{47} \\ -\frac{4}{47} & \frac{7}{47} \end{pmatrix}\begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

The left-hand side simplifies to:

$$A \times A^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

3. Simplify the right-hand side:

$$\begin{pmatrix} \frac{5}{47} & \frac{3}{47} \\ -\frac{4}{47} & \frac{7}{47} \end{pmatrix}\begin{pmatrix} 11 \\ 13 \end{pmatrix}=\begin{pmatrix} e_{11} \\ e_{21} \end{pmatrix}$$

$2\times2 \qquad 2\times1 \qquad 2\times1$

$$e_{11}=\left(\frac{5}{47}\times11\right)+\left(\frac{3}{47}\times13\right)=2$$

$$e_{21}=\left(-\frac{4}{47}\times11\right)+\left(\frac{7}{47}\times13\right)=1$$

Equating LHS to RHS gives

$$\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Equating corresponding elements,

$x=2$ and $y=1$

Summary of steps in solving simultaneous equations using the matrix method.

1. Express the pair of equations as a matrix equation.
2. Find A^{-1}, the inverse of A, which is the 2×2 matrix in the matrix equation
3. Pre-multiply both sides of the equation by A^{-1}, this simplifies to: $\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}B$
4. Simplify the right-hand side to obtain a 2×1 matrix.
5. Equate corresponding elements to obtain the values of the unknowns.

It should now be noted that so far, we have encountered three methods to solve a pair of simultaneous equations. These are:

1. Algebraic methods (elimination and substitution)
2. Graphical method
3. Matrix method

In some cases, we are asked to use a specific method and so it is necessary to be familiar with all three methods. Of course, when we have a choice we select the one that is most efficient and easy for us to apply.

Example 10

Express the equations
$2x + 5y = 6$
$3x + 4y = 8$
in the form $AX = B$, where A, X and B are matrices.
Hence, solve for x and y using the matrix method.

Solution

The equations in a matrix form are:

$$\begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$AX = B$$

$\times A^{-1}$

$$A \times A^{-1} \times X = A^{-1} \times B$$

$$I \times X = A^{-1}B$$

$$X = A^{-1}B$$

Det A = $2 \times 4 - 5 \times 3 = -7$

$$A^{-1} = \begin{pmatrix} -\frac{4}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{2}{7} \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{4}{7} & \frac{5}{7} \\ \frac{3}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} \left(-\frac{4}{7} \times 6\right) + \left(\frac{5}{7} \times 8\right) \\ \left(\frac{3}{7} \times 6\right) + \left(-\frac{2}{7} \times 8\right) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{24}{7} + \frac{40}{7} \\ \frac{18}{7} - \frac{16}{7} \end{pmatrix}$$

$$= \begin{pmatrix} 2\frac{2}{7} \\ \frac{2}{7} \end{pmatrix}$$

Hence, $x = 2\frac{2}{7}, y = \frac{2}{7}$

Using matrices to solve word problems

A single matrix can capture an array of numbers and it is sometimes convenient to use such arrays to represent information. For example, a company producing three sizes of T-shirts may sell different quantities of each size on a particular day. They may record their sales in a table after 2 days as such:

	Small	Medium	Large
Day 1	8	12	3
Day 2	6	10	5

Their prices can also be recorded in another table, such as:

	Small	Medium	Large
Cost	\$15	\$20	\$25

If we wish to determine the total sales for each day, we can set up a pair of matrices whose product will produce these totals.

We can represent the matrix of quantities sold as follows: Each cell refers to the number of T-shirts sold on a particular day in a given size. Two possibilities exist, a 2×3 or a 3×1.

$$\begin{pmatrix} 8 & 12 & 3 \\ 6 & 10 & 5 \end{pmatrix} \text{ or } \begin{pmatrix} 8 & 6 \\ 12 & 10 \\ 6 & 5 \end{pmatrix}$$

We can also represent the cost of each size as a matrix, either a 3×1 or a 1×3 matrix can be used. This would look like:

$$\begin{pmatrix} 15 & 20 & 25 \end{pmatrix} \text{ or } \begin{pmatrix} 15 \\ 20 \\ 25 \end{pmatrix}$$

Assuming we wish to calculate the total sales on each day, we can set up a multiplication of two matrices such that the product yields the total.

Using our rules for matrix multiplication, we can come up with the following product:

$$\begin{pmatrix} 8 & 12 & 3 \\ 6 & 10 & 5 \end{pmatrix}\begin{pmatrix} 15 \\ 20 \\ 25 \end{pmatrix}.$$

This product is conformable to multiplication because a 2×3 multiplied by a 3×1 results in a 2×1 matrix. We now set up the multiplication.

$$\begin{pmatrix} 8 & 12 & 3 \\ 6 & 10 & 5 \end{pmatrix}\begin{pmatrix} 15 \\ 20 \\ 25 \end{pmatrix}$$

$$= \begin{pmatrix} 8 \times 15 + 12 \times 20 + 3 \times 25 \\ 6 \times 15 + 10 \times 20 + 5 \times 25 \end{pmatrix}$$

$= \begin{pmatrix} 120+240+75 \\ 90+200+125 \end{pmatrix} = \begin{pmatrix} 435 \\ 415 \end{pmatrix}$

This is interpreted as:

The total sales of T-shirts for day 1 is \$435.
The total sales of T-shirts for day 2 is \$415

Example 11

A business makes toy trucks and toy cars. The following table is used in calculating the cost of manufacturing each toy.

	Labour (Hours)	Wood (Blocks)	Paint (Tins)
Trucks	6	8	3
Cars	3	4	2
Boats	5	7	1

Labour costs \$80 per hour, wood costs \$10 per block and paint costs \$20 per tin.
Using matrix multiplication, calculate the cost of manufacturing each toy.

Solution

The matrix for the quantities of toys manufactured is

$$\begin{pmatrix} 6 & 8 & 3 \\ 3 & 4 & 2 \\ 5 & 7 & 1 \end{pmatrix}$$

The cost matrix is written as follows: $\begin{pmatrix} 80 \\ 20 \\ 10 \end{pmatrix}$.

The required product is:

$$\underset{3\times 3}{\begin{pmatrix} 6 & 8 & 3 \\ 3 & 4 & 2 \\ 5 & 7 & 1 \end{pmatrix}} \underset{3\times 1}{\begin{pmatrix} 80 \\ 20 \\ 10 \end{pmatrix}} = \underset{3\times 1}{\begin{pmatrix} 240+160+30 \\ 240+80+20 \\ 400+140+10 \end{pmatrix}}$$

$$= \begin{pmatrix} 430 \\ 340 \\ 560 \end{pmatrix}$$

We now interpret the product as

Cost of producing trucks is \$430,
Cost of producing cars \$340 and
Cost of producing boats \$560.

MATRIX TRANSFORMATIONS

In our study of transformations so far, we examined different types of movements and their effect on plane shapes. To perform these movements, we needed to know the geometric properties of the transformations so that we can locate their images.

In this section, we will learn how to perform geometric transformations using matrices – this method relies on algebraic rather than geometric techniques and can be quite effective. In fact, we do not need to use a Cartesian Plane to locate the image, we simply calculate the coordinates of the image points using the appropriate matrix.

Matrices for Translation

When we performed a translation, we used a column matrix to locate the image. For example, to determine the coordinates of the image of $P(5,\ 4)$ under a translation of 3 units parallel to the X-axis, we represent both the object point and translation as column vectors. Then we added vectors to obtain the image point.

$$\underset{\substack{\uparrow \\ \text{Object Point}}}{\begin{pmatrix} 5 \\ 4 \end{pmatrix}} + \underset{\substack{\uparrow \\ \text{Translation vector}}}{\begin{pmatrix} 3 \\ 0 \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{Image Point}}}{\begin{pmatrix} 8 \\ 4 \end{pmatrix}}$$

Hence, the coordinates of the image of P is (8, 4).
Note that for the purpose of computation, we write the coordinates of (x, y) as a column matrix.

Translation matrices are written as 2×1 matrices, also called column vectors because a translation is really a vector. To perform any of the other geometric transformations, we use 2× 2 matrices.

Deriving Matrices for Transformations

The matrices for performing reflections, rotations and dilations are all 2× 2 matrices. To derive these matrices, we use a general principle that holds for transforming an object point (x, y) to its corresponding image point (x', y').

This general principle is stated below.

In general, the image of a point (x, y), under a transformation, defined by a 2× 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, is the point (x', y'), where

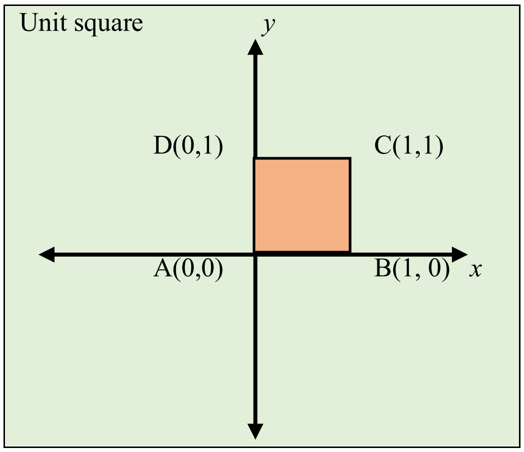
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix},$$

Our knowledge of the geometric properties of these transformations will be applied to determine the matrices for each transformation.

Consider a unit square, drawn in the first quadrant whose coordinates are (0, 0), (1, 0), (1, 1) and (0, 1). Under a given transformation each point (x, y) will move to (x', y') where,

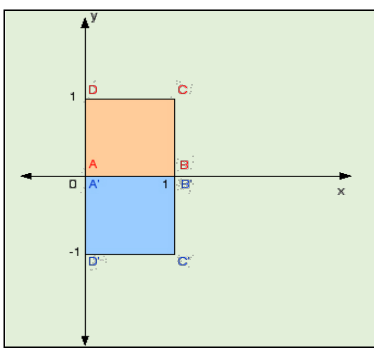
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

We can calculate the values of $a, b, c,$ and d by substituting two pairs of object and image points in the above matrix equation.



Matrices for Reflection

We will start with the matrix for reflection in the x axis. Under this reflection, the unit square will flip so that it is now in the fourth quadrant.



Unit square and its image after a reflection in the x-axis.
$(1, 0) \longrightarrow (1, 0)$
$(0, 1) \longrightarrow (0, -1)$
$(1, 1) \longrightarrow (1, -1)$
$(0, 0) \longrightarrow (0, 0)$

For convenience, we select the two points (1, 0) and (0, 1), This is repeated for each of the transformations performed below.

Reflection in the *X*-axis

$(1, 0) \longrightarrow (1, 0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Therefore $a = 1$ and $c = 0$	$(0, 1) \longrightarrow (0, -1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Therefore $b = 0$ and $d = -1$

The matrix for reflection in the *X*-axis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Reflection in the *Y*-axis

$(1, 0) \longrightarrow (-1, 0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Therefore $a = -1$ and $c = 0$	$(0, 1) \longrightarrow (0, 1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Therefore $b = 0$ and $d = 1$

The matrix for reflection in the *Y*-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection in the line $y = x$

$(1, 0) \longrightarrow (0, 1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Therefore $a = 0$ and $c = 1$	$(0, 1) \longrightarrow (1, 0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Therefore $b = 1$ and $d = 0$

The matrix for reflection in the *X*-axis is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Reflection in the line $y = -x$	
$(1,0) \longrightarrow (0,-1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Therefore $a = 0$ and $c = -1$	$(0,1) \longrightarrow (-1,0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Therefore $b = -1$ and $d = 0$
The matrix for reflection in the line $y = -x$ is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	

Matrices for Rotation

The matrices derived for rotation are defined for an anticlockwise rotation about the origin. In each case, the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represents an anticlockwise rotation about the origin. Also, we should note that an anticlockwise rotation of 90^0 is the same as a clockwise rotation of 270^0 so we need not derive matrices for clockwise rotations.

Anticlockwise rotation about the origin through 90^0	
$(1,0) \longrightarrow (0,1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Therefore $a = 0$ and $c = 1$	$(0,1) \longrightarrow (-1,0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Therefore $b = -1$ and $d = 0$
The matrix for an anticlockwise rotation about the origin through 90^0 is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	

Anticlockwise rotation about the origin through 270^0	
$(1,0) \longrightarrow (0,-1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Therefore $a = 0$ and $c = -1$	$(0,1) \longrightarrow (1,0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ Therefore $b = 1$ and $d = 0$
The matrix for an anticlockwise rotation about the origin through 270^0 is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	

A rotation about the origin through 180^0	
$(1,0) \longrightarrow (-1,0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ Therefore $a = -1$ and $c = 0$	$(0,1) \longrightarrow (0,-1)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ Therefore $b = 0$ and $d = -1$
The matrix for a rotation about the origin through 180^0 is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	

Note that for a 180^0 rotation, there is no need to specify the direction since clockwise or anticlockwise turns will produce the same image.

General matrix for rotation

The above matrices are confined to angles that are multiples of 90^0 degrees. If one has to perform a rotation through other angles, there is a general matrix for rotation of θ^0 about the origin in an anticlockwise direction where θ can be any angle. The matrix for an anticlockwise rotation through θ^0 about the origin is

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Matrix for Dilation (enlargement)

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ represent the matrix for a dilation. We can now derive the general matrix for a dilation with scale factor, k, about the origin. Recall that under a dilation, a point (x, y) is mapped onto (kx, ky). So, the unit square will have sides that are k units in length.

Enlargement with center origin, scale factor, k	
$(1,0) \longrightarrow (k,0)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$ Multiplying $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} k \\ 0 \end{pmatrix}$ Therefore $a = k$ and $c = 0$	$(0,1) \longrightarrow (0,k)$ Hence, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}$ Multiplying $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ k \end{pmatrix}$ Therefore $b = 0$ and $d = k$
The matrix for enlargement with centre origin, scale factor, k is $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$	

Example 12

The triangle PQR has coordinates $P(3, 5)$, $Q(6, 5)$ and $R(6, 7)$.
Determine the coordinates of the image of triangle PQR under a reflection in the y- axis

Solution

Matrix for reflection in the Y axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.
To calculate the image of P, Q and R under the reflection, pre-multiply as follows:

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, hence
$P(3,5) \xrightarrow{is\ mapped\ onto} P'(-3,5)$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$, hence
$P(6,5) \xrightarrow{is\ mapped\ onto} Q'(-6,5)$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \end{pmatrix}$, hence
$P(6,7) \xrightarrow{is\ mapped\ onto} R'(-6,7)$

Under the reflection in the Y-axis, triangle PQR is mapped onto triangle $P'Q'R'$ where $P'(-3,5), Q'(-6,5)$ and $R'(-6,7)$.

Example 13

The triangle ABC has coordinates $A(2, 4)$, $B(7, 4)$ and $C(2, 7)$. Determine the coordinates of the image of triangle ABC under a rotation about the origin through 180^0

Solution

Matrix for rotation about the origin through 180^0 is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.
To calculate the image of P, Q and R under the rotation, pre-multiply as follows:

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$, hence
$A(3,5) \xrightarrow{is\ mapped\ onto} A'(-2,-4)$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \begin{pmatrix} -7 \\ -4 \end{pmatrix}$, hence
$B(6,5) \xrightarrow{is\ mapped\ onto} B'(-7,-5)$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -2 \\ -7 \end{pmatrix}$, hence
$C(6,7) \xrightarrow{is\ mapped\ onto} C'(-2,-7)$

Under the reflection in the Y-axis, triangle PQR is mapped onto triangle $A'B'C'$ where $A'(-2,-4), B'(-7,-5)$ and $C'(-2,-7)$.

Example 14

The triangle LMN has coordinates $L(1, 4)$, $M(3, 4)$ and $N(1, 6)$. Determine the coordinates of the image of triangle LMN under an enlargement with center (0, 0) with scale factor 3.

Solution

Matrix for enlargement with center (0, 0), k=3 is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$.
To calculate the image of P, Q and R under the enlargement, pre-multiply as follows:

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 12 \end{pmatrix}$, hence
$L(1,4) \xrightarrow{is\ mapped\ onto} L'(3,12)$

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix}$, hence
$M(3,4) \xrightarrow{is\ mapped\ onto} M'(9,12)$

$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ 18 \end{pmatrix}$, hence
$N(1,6) \xrightarrow{is\ mapped\ onto} N'(3,18)$

Under the reflection in the Y-axis, triangle LMN is mapped onto triangle $L'M'N'$ where $L'(3,12), M'(9,12)$ and $N'(3,18)$.

Matrices for combined transformations

It is possible to obtain a single matrix to describe a combination of two transformations. Once we know the 2×2 matrix for each of the transformations we can multiply them to obtain this single matrix using the following rule.

If A and B are both 2×2 matrices representing two geometric transformations, then the matrix product AB represents the combined transformation B followed by A.

Example 14

The transformation, M is a reflection in the line $y = -x$. The transformation N, is an enlargement, centre origin, k= 2.

(i) Write down the 2×2 matrices for M and N.
(ii) The matrix, P represents the combined transformation, M followed by N. Determine the matrix P.
(iii) Determine the coordinates of the image of (−3, 4) under P.

Solution

(i) M is a reflection in the line $y = -x$,

$M = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

N is an enlargement, center origin, $k = 2$,

$N = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

(ii) The combined transformation, M followed by N is NM.

$$P = NM = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$$

(iii) The image of $(-3, 4)$ under P:

$$\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} -8 \\ 6 \end{pmatrix}$$

The image of $(-3, 4)$ is $(-8, 6)$.

Example 15

Under a matrix transformation, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the points V and W are mapped onto V' and W' such that:

$$V(3,5) \longrightarrow V'(5,-3)$$
$$W(7,2) \longrightarrow V'(2,-7)$$

(i) Determine the values of $a, b, c,$ and d.

(ii) State the coordinates of Z such that $Z(x, y) \longrightarrow V'(5,1)$ under the transformation, M.

(iii) Describe fully the geometric transformation, M

Solution- parts (i) and (ii)

(i) We note that under the transformation, M a point reverses its coordinates and changes the sign of the y coordinate. By inspection, the matrix for this transformation M is derived by consideration of the following products:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$$

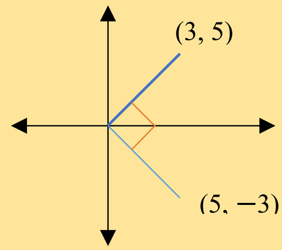
$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(ii) $(x, y) \longrightarrow V'(5,1)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$
$$y = 5, x = -1$$
$$(x, y) = (-1, 5)$$

Solution – Example 15, part (iii)

The transformation M represents a rotation of 90º clockwise about O. This is deduced by applying geometrical properties of rotation of having knowledge of the matrix for the rotation.



Alternative Method for Example 15 Part (i)

$M \times V = V'$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ $\therefore \begin{pmatrix} 3a+5b \\ 3c+5d \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$	Simultaneous equations for $V(3,5) \longrightarrow V'(5,-3)$ $3a + 5b = 5 \ldots (1)$ $3c + 5d = -3 \ldots (2)$
$M \times W = W'$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$ $\therefore \begin{pmatrix} 7a+2b \\ 7c+2d \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$	Simultaneous equations for $W(7,2) \longrightarrow V'(2,-7)$ $7a + 2b = 2 \ldots (3)$ $7c + 2d = -7 \ldots (4)$

Consider equations (1) and (3)

Equation (1) × 7 $\quad 21a + 35b = 35 \ldots (5)$

Equation (3) × −3 $\quad -21a - 6b = -6 \ldots (6)$

Equation (5) + (6) $\quad 29b = 29$

$b = 1$

Substitute $b = 1$ into equation (1)

$$3a + 5(1) = 5$$
$$3a = 0$$
$$\therefore a = 0$$

Consider equations (2) and (4)

Equation (2) × 7 $\quad 21c + 35d = -21 \ldots (7)$

Equation (4) × −3 $\quad -21c - 6d = 21 \ldots (8)$

Equation (7) + (8) $\quad 29d = 0$

$d = 0$

Substitute $d = 0$ into equation (2)

$$3c + 5(0) = -3$$
$$3c = -3$$
$$\therefore c = -1$$

Therefore,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$