# 10: SERIES AND SEQUENCES

## Sequence and Series

**A sequence** is a set of numbers in which consecutive terms are connected by a definite rule or pattern. The numbers in this ordered list are called elements or terms. Sequences vary depending on the pattern or rule that exist between consecutive terms. We also refer to a sequence as a progression. The **sum** of the terms of a sequence is called a **series.**

### Finite and Infinite Sequences

A finite sequence can be specified by a complete list of its elements. It consists of a countable number of terms. The set of even numbers from 2 to 10 forms the finite sequence, $\{2, 4, 6, 8, 10\}$.

An infinite sequence can be specified by an incomplete list of its elements. It has an 'infinite' number of terms and in listing the terms, we use a set of three dots after the last term in the list to indicate that it has no end. The set of even numbers forms an infinite sequence, $\{2, 4, 6, 8, \dots\}$

### The arithmetic series or progression (AP)

A series, in which each term is obtained from the preceding one by the addition of a constant quantity, is called an arithmetic progression (AP). This constant quantity is called the common difference.

The sequence
$3, 5, 7, 9, \dots$ has a common difference of 2.
The sequence
$20, 18, 16, 14, \dots$ has a common difference of $-2$
The sequence
$4, 3\frac{1}{2}, 3, 2\frac{1}{2}, \dots$ has a common difference of $-\frac{1}{2}$

Notice that the common difference can be a positive or negative or even a fractional number. Its value, though, is the same for a particular series and cannot be altered for that series.

**Notation for an Arithmetic Progression**
1st term = $a$ — Last term = $l$
Number of terms = $n$ — The $n^{th}$ term = $T_n$
The common difference, $d = T_n - T_{n-1}$

### The $n^{th}$ term of an AP

Sometimes we are required to calculate a particular term in an AP. We can do so once we know the first term and the common difference. If we require the 20th term, we need to list all the terms. The $n^{th}$ term, denoted by $T_n$ can be determined by using a formula.

Consider a sequence whose first term is denoted by $a$ and whose difference is denoted by $d$. The first five terms are shown in the table below.

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ |
|---|---|---|---|---|
| $a$ | $a+d$ | $a+2d$ | $a+3d$ | $a+4d$ |

To determine a formula for the $n^{th}$ term, we look at each consecutive term to discern a pattern. We observe that each term has $a$ and the coefficient of $d$ is always one less than the value of $n$.

**The $n^{th}$ term of an AP**
$$T_n = a + (n-1)d$$

**Example 1**

An arithmetic progression is defined by $\{4, 2, 0, -2, \dots\}$. Determine
i. the common difference, $d$
ii. the 406th term

**Solution**

i. In the AP, $\{4, 2, 0, -2, \dots\}$,
The first term $a = 4$
The common difference, $d$, can be found by subtracting any two consecutive terms in the order, $T_n - T_{n-1}$
$$d = 2 - 4 = 0 - 2 = -2$$
Hence, $d = -2$

ii. Recall: $T_n = a + (n-1)d$, where $n$ is the number of terms. Substituting for $a$, $n$ and $d$, we have
$$\therefore T_{406} = 4 + (406-1) \times (-2)$$
$$= 4 + 405 \times (-2)$$
$$= 4 - 810$$
$$= -806$$

### Sum of the first $n$ terms of an AP

For an arithmetic progression, we usually use the symbol $S_n$ to denote the sum of the first $n$ terms. For example, $S_3$ denotes the sum of the first 3 terms, $T_1 + T_2 + T_3$.

For an AP with $n$ terms, we denote $a$ and $d$ as the first term and the common difference. If the last term is denoted by $l$, then, we can calculate $S_n$ by the following series. Note, for convenience, only the first four terms and the last four terms were included.

$$S_n = a + (a+d) + (a+2d) + (a+3d) + \cdots$$
$$(l-3d) + (l-2d) + (l-d) + l$$

Reversing the order of the terms in the above equation we have:

$$S_n = l + (l-d) + (l-2d) + (l-3d) + \cdots$$
$$(a+3d) + (a+2d) + (a+d) + a$$

Adding both equations, we notice that each pair of consecutive terms add to $(a+l)$. For example:
Adding the first terms from each $S_n$, we have $(a+l)$.
Adding the second terms from each $S_n$, we have $(a+d)+(l-d) = a+l$.

Therefore, the sum of both sequences will look like:

$$2S_n = (a+l) + (a+l) + (a+l) + \cdots$$
$$(a+l) + (a+l) + (a+l)$$

Since there are $n$ terms in the sequence,
$2S_n = n(a+l)$
$S_n = \frac{n}{2}(a+l)$

We can replace $l$ from this formula by treating $l$ as any term. If we want to find the sum of the first $n$ terms then,
$T_n = a + (n-1)d$
So, we may take,
$\therefore l = a + (n-1)d$

$$\therefore S_n = \frac{n}{2}\{a + (a + (n-1)d)\}$$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

**The sum of the first *n* terms of an AP**

$$S_n = \frac{n}{2}(a+l)$$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

**Example 2**

Find the sum of the first 40 terms of the AP
{ 4, 6, 8, 10, …}

**Solution**

We can deduce that, $a = 4$, $d = 6 - 4 = 2$, $n = 40$

The sum of the $n$ terms is $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$$S_{40} = \frac{40}{2}\{2(4) + (40-1)2\} = 20\{8 + 2(39)\} = 1720$$

**Example 3**

Find the sum of all the terms in the A.P
10, 15, 20, …, 1000

**Solution**

The first term, $a = 10$ and $d = 15 - 10 = 5$
The last term, $l = 1000$
We do not know the number of terms, $n$.
We can determine $n$ by substituting for $a$, $l$ and $d$ in the following formula:

$$l = a + (n-1)d$$
$$1000 = 10 + (n-1)\times 5$$
$$990 = 5(n-1)$$
$$n - 1 = 198$$
$$n = 199$$

The sum of all 199 terms, $S_{199}$, can be found using

either $S_n = \frac{n}{2}(a+l)$ or $\frac{n}{2}\{2a + (n-1)d\}$.

Using $S_n = \frac{n}{2}(a+l)$

$$S_{199} = \frac{199}{2}(10 + 1000)$$
$$= 100\ 495$$

OR

Using $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$$S_{199} = \frac{199}{2}\{2(10) + (199-1)5\}$$
$$= 100495$$

**Example 4**

An arithmetic progression with first term 8 has 101 terms. If the sum of the first three terms is one-third of the sum of the last three terms, find

a. the common difference, $d$
b. the sum of the last 48 terms of the series.

**Solution**

a. Let $a = 8$, common difference = $d$, $n$ =101
The sum of the first 3 terms is:

$$T_1 + T_2 + T_3 = a + (a+d) + (a+2d) = 3a + 3d = 24 + 3d$$
$$T_{99} + T_{100} + T_{101} = (a + 98d) + (a + 99d) + (a + 100d)$$
$$T_{99} + T_{100} + T_{101} = 3a + 297d = 24 + 297d$$

The sum of the first three terms

$= \frac{1}{3}$ [sum of the last three terms]

$$(T_1 + T_2 + T_3) = \frac{(T_{99} + T_{100} + T_{101})}{3}$$

$$24 + 3d = \frac{1}{3}(24 + 297d)$$

$$16 = 99d - 3d$$

$$\therefore 96d = 16 \quad \Rightarrow \quad d = \frac{16}{96}, \quad d = \frac{1}{6}$$

b. Sum of the last 48 terms = $S_{101} - S_{53}$

Recall $S_n = \frac{n}{2}\{2a + (n-1)d\}$

$\therefore$ Sum of the last 48 terms =

$$\frac{101}{2}\left\{2(8) + (101-1)\frac{1}{6}\right\} - \frac{53}{2}\left\{2(8) + (53-1)\frac{1}{6}\right\}$$

$$= 1649\frac{2}{3} - 653\frac{2}{3}$$

$$= 996$$

**Example 5**

A marathon runner begins his first day of training by running 2 km. He then increases this distance by ½ km more each day, from the distance he ran the previous day.

i. On what day would he first cover 15 km?

ii. What is the total distance covered after 30 days?

**Solution**

i The distances covered on a daily basis form the following AP:

$2, 2\frac{1}{2}, 3, 3\frac{1}{2}, 4, 4\frac{1}{2}, \ldots$

where the first term $a = 2$ and the common difference $d = \frac{1}{2}$.

We are looking for *n* such that $T_n = 15$.

Since $T_n = a + (n-1)d$, then

$$15 = 2 + (n-1) \times \frac{1}{2}$$

$$30 = 4 + (n-1)$$

$$n = 27$$

Hence, on the 27th day, he would run a distance of exactly 15 km.

ii The total distance covered after 30 days is the sum of the first 30 terms of the arithmetic progression.

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$S_{30} = \frac{30}{2}\left\{2(2) + (30-1) \times \frac{1}{2}\right\}$$

$$= 15\left\{4 + 14\frac{1}{2}\right\}$$

$$= 277.5 \text{ km}$$

**Example 6**

Find the sum of all the numbers from 1000 to 2000 which are divisible by 5.

**Solution**

The series is 1000, 1005, 1010, …, 1995, 2000.
The terms are in AP with the first term, $a = 1000$
The common difference, $d = 1005 - 1000 = 5$

Last term, $l = a + (n-1)d$, where $n$ = number of terms.

$$\therefore 2000 = 1000 + (n-1)5$$

$$1000 = (n-1)5$$

$$n = 201$$

To find the sum of all the terms from 1000 to 2000, we can use any one of the methods below.

$$S_{201} = \frac{n}{2}(a + l)$$

$$= \frac{201}{2}(1000 + 2000)$$

$$= 301500$$

OR

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

$$= \frac{201}{2}\{2(1000) + (201-1)5\}$$

$$= 301500$$

## The Geometric Progression (GP)

A series in which each term is obtained from the preceding one by the multiplication of a constant quantity is called a geometric series or progression (GP). This constant quantity is called the common ratio.
For any GP, we can find the common ratio by **dividing** any term by the term that came just before.

**The common ratio**

The geometric series {4, 12, 36, 108, …} has a common ratio of 3, note $4\times3=12$, $12\times3=36$, $36\times3=108$ and so on.

**OR** We may say 12 ÷ 4 = 3 or 36 ÷ 12 = 3 and so on.

The geometric series {16, 8, 4, 2, …} has a common ratio of ½, note

16 ×½ = 8, 8 × ½ =4, 4 × ½ = 2, and so on.

OR 8 ÷ 16 = ½, 4 ÷ 8 = ½, 2 ÷ 4 = ½

The geometric series {4, −8, 16, −32, …} has a common ratio of −2, note 4 × −2 = −8, −8 × −2 =16, 16 ×−2 = −32 and so on

OR −8 ÷ 4 = −2, and 16 ÷ −8 = −2

Notice that the common ratio may take any value. That is, it may be positive or negative or fractional.

**Notation for a Geometric Progression**

First term = $a$

Number of terms = $n$

The $n^{th}$ term = $T_n$

The common ratio, $r$ is $\dfrac{T_n}{T_{n-1}}$

**The $n^{th}$ term of a Geometric Progression**

The $n^{th}$ term of a GP, denoted by $T_n$ can be determined by using a formula. Consider a sequence whose first term is denoted by $a$ and whose ratio is denoted by $r$. The first five terms are shown in the table below.

| $T_1$ | $T_2$ | $T_3$ | $T_4$ | $T_5$ |
|---|---|---|---|---|
| $a$ | $ar$ | $ar^2$ | $ar^3$ | $ar^4$ |

From the above table, we observe that the coefficient of $r$ is $a$ and the power of $r$ is always one less than the value of $n$.

**The $n^{th}$ term of a G.P**

$$T_n = ar^{n-1}$$

**Example 7**

A geometric progression has terms {4, 8, 16,….} Determine the 20th term.

**Solution**

The first term, $a=4$, the common ratio $r=\frac{8}{4}=2,\ \ |r|>1$.

The 20th term is $T_{20} = 4(2)^{20-1} = 4(2)^{19} = 2097152$

**Sum of the first $n$ terms of a GP**

Let us use the symbol $S_n$ to represent the sum of the first $n$ terms of a geometric progression. So, the sum of the first three terms, $S_3$ is $T_1+T_2+T_3$. We will now derive the formula for the sum of the $n$ terms of a GP.

Let us remember that each term is $r$ times the previous term or the ratio of any term to the term before it is always $r$:1

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1} \qquad (1)$$

Equation (1) $\times r$

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^n \qquad (2)$$

Equation (1) – Equation (2)

$$S_n - rS_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} - (ar + ar^2 + ar^3 + \cdots ar^{n-1} + ar^n)$$

Notice all the terms will cancel out except the first term of equation (1) and the last term of equation (2) and which now carries a minus sign before it. Hence,

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r} \qquad |r|<1$$

When $|r|>1$, the subtraction is taken in the reverse order and produces the alternate form of the formula.

$$S_n = \frac{a(r^n-1)}{r-1} \qquad |r|>1$$

The formula for the sum of $n$ terms in a GP can now be stated.

**The sum of the first $n$ terms of a GP**

If the first term of a GP is denoted by $a$, the number of terms by $n$, the sum of the first $n$ terms by $S_n$ and the common ratio by $r$, then

$$S_n = \frac{a(r^n-1)}{r-1} \quad |r|>1 \qquad\qquad S_n = \frac{a(1-r^n)}{1-r} \quad |r|<1$$

**Example 8**

Find the sum of the first 40 terms in the geometric progression {4, 8, 16, …}.

**Solution**

The first term, $a = 4$, the common ratio,

$r = \frac{8}{4} = 2, \quad |r| > 1$

The sum of the first $n$ terms is:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{4((2)^{40} - 1)}{2 - 1}$$

$$= 4((2)^{40} - 1)$$

**Divergent and Convergent Series**

Consider the sum of the terms of an AP in which $a = 3$ and $d = 2$.

$$S_n = 3 + 5 + +7 + 9 +$$

Notice that the terms of the progression are increasing and as $n$ gets larger the sum will approach infinity. It is impossible to sum all the terms of this sequence. This is because this sum approaches infinity. All AP's have sums that approach infinity and such a series is a **divergent** series.

Consider the sum of the terms of a GP in which $|r| > 1$. For example, let $a = 3$ and $r = 2$.

$$S_n = 3 + 6 + 12 + 24 + 48 + \cdots$$

Notice that the terms of the progression are increasing and as $n$ gets larger the sum will approach infinity. It is impossible to sum all the terms of this sequence. This is because this sum approaches infinity. This type of GP is **divergent**.

Now, let us consider the sum of the terms of a GP when $|r| < 1$. For example, let $a = 243$ and $r = \frac{1}{3}$.

$$S_n = 243 + 81 + 27 + 9 + 3 + 1 + \frac{1}{3} + \frac{1}{9} + \cdots$$

Notice that the terms of the progression are decreasing and as $n$ gets larger the terms approach zero. Hence, the sum of the terms approaches a fixed value as we keep adding more and more terms. This is called the sum to infinity. This type of GP is **convergent**.

Hence, the sum to infinity of a G.P, exists only when $|r| < 1$. We can derive a simple formula to calculate this sum for any GP.

Starting with the formula for the sum of $n$ terms of any converging GP, we have

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}$$

As the value of $n \to \infty$, then, $r^n$ gets smaller and smaller, that is, it tends to 0. Hence, as $r^n \to 0$, the term $\frac{ar^n}{1 - r} \to 0$.

Therefore, $S_n$ approaches $\frac{a}{1 - r} - 0 = \frac{a}{1 - r}$.

This means that when $|r| < 1$, the more and more terms of the GP that we add, the closer and closer we would get to the value of $\frac{a}{1 - r}$. We say that the series converges with limit $\frac{a}{1 - r}$ and this value is called its sum to infinity.

**The sum to infinity of a GP**

$$S_\infty = \frac{a}{1 - r}$$

with the first term, $a$ and common ratio, $r$.

This exists only when, $|r| < 1$.

**Example 9**

Find the sum to infinity of the geometric series $\{8, 4, 2, \ldots\}$

**Solution**

In the geometric progression $\{8, 4, 2, \ldots\}$ the first term, $a = 8$. The common ratio,

$r = \frac{4}{8} = \frac{1}{2}, \quad |r| < 1$

$$S_\infty = \frac{a}{1 - r} = \frac{8}{1 - \frac{1}{2}} = 16$$

**Example 10**

In a geometric progression with the common ratio, $r$, the sum to infinity is four times the first term. Find the value of $r$, $|r| < 1$.

**Solution**

We are given that $S_\infty = 4 \times$ first term

$$\frac{a}{1 - r} = a$$

$$4a(1 - r) = a$$

$$4(1 - r) = 1$$

$$1 - r = 1/4$$

$$r = 3/4$$

In some questions, we are given a sequence and the type of sequence is not stated. In such cases, we have to deduce from the data what type of sequence is involved.

**Example 11**

An item of jewellery appreciates by 10% of its value each year. If its original value is $1 000, find its estimated value at the start of the 11th year.

**Solution**

We look at the initial value of the item and calculate its new value each year. At the start, the value of the item of jewellery is $1000

At the end of year 1, the value is
$ 1000 + 10% ( $1000)
= $1000 × 1.1

At the end of year 2, the value is
($ 1000 × 1.1) + 10 % of ($ 1000 × 1.1)
= $ 1000 × $(1.1)^2$

At the end of year 3, the value is
($ 1000 × $1.1^2$) + 10 % of ($ 1000 × $1.1^2$)
= $ 1000 × $(1.1)^3$

When we observe its new value each year, a special pattern emerges.

This is a geometric progression with first term of $ 1000 and common ratio of 1.1.

Hence, the estimated value of the item of jewellery after eleven years can be obtained by using the formula for the $n$ th term of the GP.

$T_n = ar^{n-1}$
=$ 1000 × $1.1^{11-1}$ = $ 1000 × $1.1^{10}$ = $2 593.74

The value at the start of the 11th year is $2 593.74

**Example 12**

An earth mover has a job to remove 800 $m^3$ of soil over a certain period. On the first day 10 $m^3$ is moved and on each day the amount of soil moved is $\frac{6}{5}$ times the amount removed on the previous day.
This continues until the job is complete. Find the number of days taken to complete the job.

**Solution**

We calculate the amount of soil removed for each day and look for a pattern.

| Day | 1 | 2 | 3 | 4… |
|---|---|---|---|---|
| Amount of soil removed in $m^3$ | 10 | $10\times\frac{6}{5}$ | $10\times\frac{6}{5}\times\frac{6}{5}$ | $10\times\frac{6}{5}\times\frac{6}{5}\times\frac{6}{5}\ldots$ |
| | 10 | $10\times\frac{6}{5}$ | $10\times\left(\frac{6}{5}\right)^2$ | $10\times\left(\frac{6}{5}\right)^3\ldots$ |
| Term | $a$ | $ar$ | $ar^2$ | $ar^3\ldots$ |

Where $a=10$ and $r=\frac{6}{5}$ are the first term and the common ratio of a geometric progression.

To complete the job, 800 $m^3$ of soil is to be removed. Let $n$ be the number of days taken to complete the job. Then $S_n = 800$ where $S_n$ = sum of the first $n$ terms of the geometric progression.

$$S_n = \frac{a\left(r^n - 1\right)}{r-1} \qquad |r| > 1$$

Substituting in the above formula, we have:

$$\therefore \frac{10\left(\left(\frac{6}{5}\right)^n - 1\right)}{\frac{6}{5} - 1} = 800$$

$$10\left(\left(\frac{6}{5}\right)^n - 1\right) = 160$$

$$1.2^n = 17$$

To solve for $n$, take logs to the base 10

$$lg1.2^n = lg17$$
$$nlg1.2 = lg17$$
$$n = \frac{lg17}{lg1.2}$$
$$n = 15.5, n \epsilon Z^+$$

Hence the job is complete on the 16th day

## Sequences and series using Σ notation

We can describe an AP by writing a formula for the $n^{th}$ term of the sequence instead of listing the terms. Let $T_n$ represent the $n$th term, where

$T_n = 2n - 1$ and $n = 1,2,3,\ldots$($n$ is a positive integer)

The terms of the A.P can be calculated by substituting $n$ = 1, 2, 3, … to obtain:

| | | |
|---|---|---|
| $T_1 = 2(1) - 1$ <br> $= 1$ | $T_2 = 2(2) - 1$ <br> $= 3$ | $T_3 = 2(3) - 1$ <br> $= 5$ |

And so, the AP is $\{1, 3, 5, \ldots\}$
The sum of this AP to infinity is a series and can be written using the $\Sigma$ notation as

$$\sum_{n=1}^{n=\infty}(2n-1)$$

So we can write $\sum_{n=1}^{n=\infty}(2n-1)=1+3+5+\ldots$

**Summation Notation**

A series may be denoted by $\sum_{n=a}^{b}T_n$ , where $\Sigma$ denotes *the sum of* and $\sum_{n=a}^{b}T_n$ denotes the sum of all the terms from $n = a$ to $n = b$.

**Example 13**

Find the sum $\sum_{k=1}^{50}(3k+2)$.

**Solution**

$\sum_{k=1}^{50}(3k+2)$ means the sum of the terms of the sequence from $k=1$ to $k=50$, inclusive.
To obtain the first term substitute $k=1$, so $T_1=3(1)+2=5$

The second and third terms are $T_2=3(2)+2=8$, and $T_3=3(3)+2=11$.

The sequence looks like $\{5, 8, 11, \ldots\}$

This is an AP with 1st term $a=5$ common difference, $d=8-5=3$ and the number of terms, $n=50$

The last term, $l=3(50)+2=152$

$$\therefore \sum_{k=1}^{50}(3k+2)=S_{50}=\frac{n}{2}(a+l)=\frac{50}{2}(5+152)=3925$$

We could have used:

$$S_{50}=\frac{n}{2}\{2a+(n-1)d\}$$

$$=\frac{50}{2}\{2(5)+(50-1)3\}=3925$$

**Example 14**

Evaluate $\sum_{r=1}^{40}(2r-3)$ .

**Solution**

$$\sum_{r=1}^{40}(2r-3)=2(1)-3+2(2)-3+2(3)-3+\ldots+2(40)-3$$

$$\sum_{r=1}^{40}(2r-3)=-1+1+3+\ldots+77$$

This is an AP with $a=1, d=2, n=40$ and $l=77$

$$\therefore \sum_{r=1}^{40}(2r-3)=S_{40}$$

For an AP $S_n=\frac{n}{2}(a+l)$ OR $S_n=\frac{n}{2}\{2a+(n-1)d\}$

$$\sum_{r=1}^{40}(2r-3)=\frac{40}{2}(-1+77)=20(76)=1520$$

OR

$$\sum_{r=1}^{40}(2r-3)=\frac{40}{2}\{2(-1)+(40-1)2\}$$

$$=20\{-2+(39\times 2)\}=20(76)=1520$$

**Alternative Method**

We could have solved this problem using basic laws involving the summation notation. We can distribute the summation as shown:

$$\sum_{r=1}^{40}(2r-3)\equiv 2\sum_{r=1}^{40}r-\sum_{r=1}^{40}3$$

Expressing the first term as an AP, we have

$$\sum_{r=1}^{40}r=1+2+3+\ldots+40$$

This is an AP with $a=1$, $d=1$, $n=40$ and $l=40$.

Any of the formulae for $S_n$ may be used.

$$S_{40}=\frac{40}{2}(1+40)=20(41)$$

$$S_{40}=820$$

We now simplify the second term.

$\sum_{r=1}^{40}3=3+3+3+\ldots+3$ (40 times). So, $\sum_{r=1}^{40}3=120$

$$\sum_{r=1}^{40}(2r-3)\equiv 2\sum_{r=1}^{40}r-\sum_{r=1}^{40}3$$

$$\sum_{r=1}^{40}(2r-3)=2(820)-120$$

$$\sum_{r=1}^{40}(2r-3)=1520$$