16. TRANSFORMATION GEOMETRY

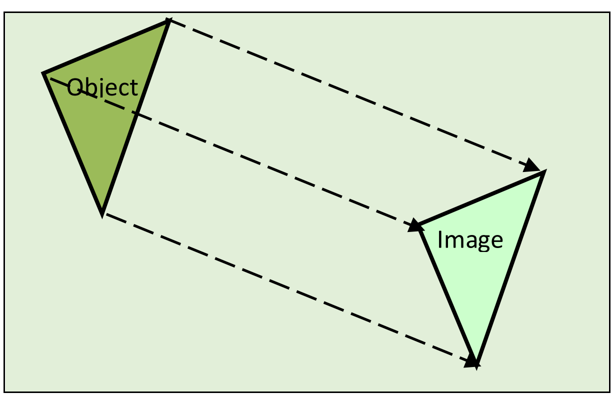
TRANSFORMATIONS

A geometric transformation involves the movement of an object from one position to another on a plane. The movement is accompanied by a change in position, orientation, shape or even size. Some examples of transformations are translation, reflection, rotation, enlargement, one-way stretch, two-way-stretch and shear.

In our study of transformations, we will be concerned mainly with movement of basic shapes (plane figures) from one position to another (image). If there is no change in size or shape, then the transformation is called an isometric transformation. If the size of the object changes then the transformation is called a size transformation. Each transformation has a unique set of characteristics or rules that define the movement.

Translation

A translation is a movement, along a straight line, in a fixed direction without any turning. It can be described informally as a glide or a slide. When an object undergoes a translation, all points on the object move the same distance and the same direction. The arrowed line represents the translation.



Describing a translation

The translation of the object in the diagram above is represented by an arrowed line. To describe it, we must know two attributes. These two attributes define a translation. A translation is defined by stating:

a) the direction of the movement
b) the distance moved by the object

In navigation and other real-life situations, we use the four Cardinal points to describe direction, but our study of transformations involves mainly movements on the Cartesian Plane and it is therefore convenient to refer to these four directions as follows:

North (parallel to the y-axis in a positive direction)
South (parallel to the y-axis in a negative direction)
East (parallel to the x-axis in a positive direction)
West (parallel to the x-axis in a negative direction)

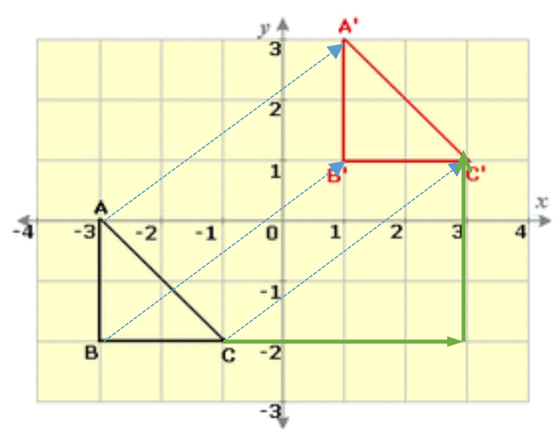
We can also use conventional units to describe distance such as metres and centimetres. However, on the Cartesian Plane we measure distance using horizontal and vertical scales on a graph.

Note that translation is used to describe any movement in a straight line. These include horizontal and vertical and diagonal movements.

Translation on the Cartesian Plane

On the Cartesian Plane, we can think of a translation as comprising two components, an x component and a y component. The x-component specifies the horizontal movement (parallel to the x-axis) and the y-component specifies the vertical component (parallel to the y-axis).

For example, in the diagram below, the translation of triangle ABC to its new position $A'B'C'$ is defined by describing the movement from A to A' or from B to B' or from C to C'. These three displacements are parallel and we refer to them as ***translation vectors***.



We define this translation using a column vector.

$AA' = BB' = CC' = \binom{4}{3}$, where 4 is the distance moved or parallel to the *x*-axis in a positive direction and 3 is the distance moved parallel to the *y*-axis in a positive direction

In this notation, the top number gives the movement along the *x* axis and the bottom number gives the movement along the *y* axis. So, in general, any translation can be described as

$\begin{pmatrix} x \\ y \end{pmatrix}$, where *x* is the movement parallel to the *x* axis and *y* the movement parallel to the *y* axis.

Direction on the Cartesian Plane

- A positive value of *x* denotes the movement is horizontal and to the right while a negative value of *x* denotes the movement is horizontal and to the left.
- A positive value of *y* denotes the movement is vertical and upwards while a negative value of *y* denotes the movement is vertical and downwards.

Example 1

$P = (3, -1)$ is mapped onto P' under a translation $T = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$. Determine the coordinates of P', the image of *P* under *T*.

Solution

For convenience, we write the coordinates of *P* as a column vector such that

$P + T = P'$

$\binom{3}{-1} + \binom{-2}{-3} = \binom{1}{-4}$

Therefore, $P' = (1, -4)$

Example 2

$A\,(3, 2)$, undergoes a translation under *T*, where *A* is mapped onto A'. If A', the image of *A*, has coordinates (7, 3). Describe the translation, *T* using a column vector.

Solution

Using the equation, $A + T = A'$, we substitute

$\binom{3}{2} + \binom{a}{b} = \binom{7}{3}$, where $T = \binom{a}{b}$.

$\binom{a}{b} = \binom{7-3}{3-2} = \binom{4}{1}$

Therefore, $T = (4, 1)$

Example 3

The point, *A* is mapped onto $A'(2, 3)$ by a translation, $T = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. Find the coordinates of *A*.

Solution

Let $A = (x, y)$

Substituting in $A + T = A'$, we obtain

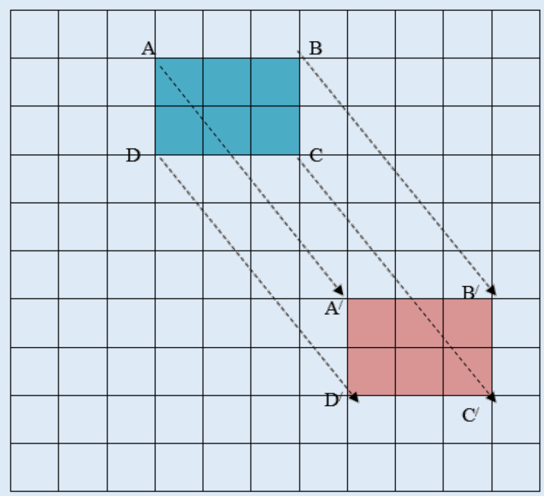
$\binom{x}{y} + \binom{-1}{4} = \binom{2}{3}$

$\binom{x}{y} = \binom{2}{3} - \binom{-1}{4} = \binom{2-(-1)}{3-4} = \binom{3}{-1}$

Therefore, $A = (3, -1)$

Example 4

The rectangle ABCD undergoes a translation to a new position A′B′C′D′. Describe the translation (a) in words (b) as a column vector



Solution

(a) The parallel and equal lines shown dotted, at each of the vertices of the rectangle represents the translation. We can look at any point, say *B* and its image B'.

The translation is a movement of 4 units parallel to the *x* axis, and –5 units parallel to the *y* axis.

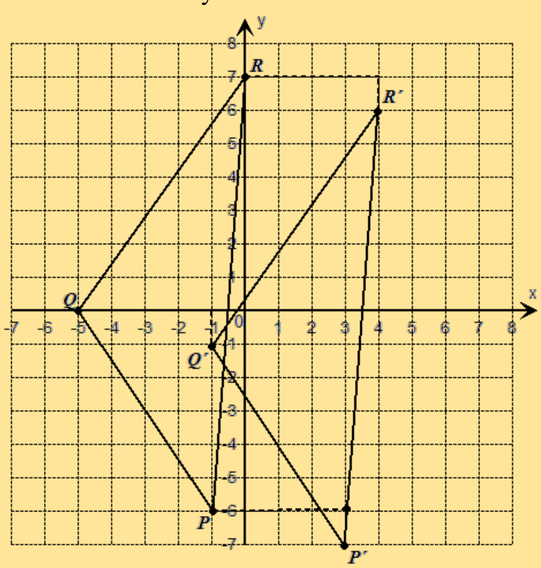
(b) Each parallel line represents the translation $\binom{4}{-5}$.

Example 5

Triangle PQR with $P(-1,-6), Q(-5,0)$ and $R(0,7)$ is mapped onto triangle $P'Q'R'$ under the translation, $T=\begin{pmatrix}4\\-1\end{pmatrix}$. Determine the coordinates of P', Q' and R', the images of P, Q, and R.

Solution

We may obtain P', Q' and R' graphically by shifting each point 4 units horizontally to the right and 1 unit vertically down.



We may also obtain P', Q' and R' by calculation.

P': $\begin{pmatrix}-1\\-6\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}3\\-7\end{pmatrix}$

Q': $\begin{pmatrix}-5\\0\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}-1\\-1\end{pmatrix}$

R': $\begin{pmatrix}0\\7\end{pmatrix}+\begin{pmatrix}4\\-1\end{pmatrix}=\begin{pmatrix}4\\6\end{pmatrix}$

Hence the coordinates of P', Q' and R' are: $(-3,-7)$, $(-1,-1)$ and $(4,6)$ respectively.

Properties of translations

When an object undergoes a translation, we can observe the following properties:

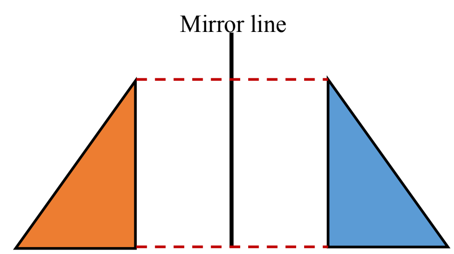
1. Each point on the object moves the same distance and in the same direction. Hence, lines joining image points to object points are parallel to each other.
2. The size, shape and orientation of the image remain the same, though the position changes.
3. A translation is an isometric or a congruent transformation, since both the object and the image are congruent.

REFLECTION

We define a reflection as a transformation in which the object turns about a line, called the mirror line. In so doing, the object actually flips, leaving the plane and turning over so that it lands on the opposite side.

In the reflection below, the triangle on the left is the object and triangle on the right is the image. The mirror line is the vertical line. The image has a different orientation to the object and is said to be flipped or laterally inverted. If we try to slide the object across the mirror line to fit on its image, it will not match, we must turn it over to fit exactly over its image.

In a reflection, the perpendicular distance between an object point and image point from the mirror line is the same. This property enables us to locate the image in a reflection.



Describing a reflection

To describe a reflection, we state the position of the mirror line. This is the straight line in which the object is to be reflected. The mirror line can be **any** straight line – vertical, horizontal or even slanted.

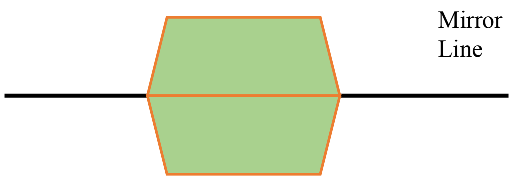
When we perform reflections on a Cartesian Plane, we usually describe the position of the mirror line by stating its equation.

Invariant Points

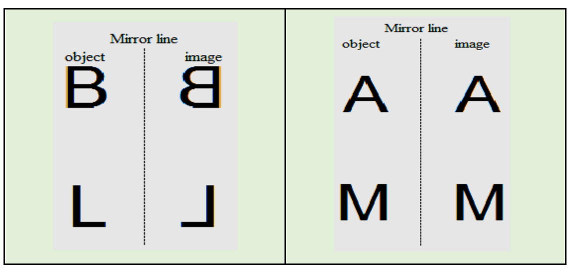
If any object point is mapped onto itself after any transformation, that point is said to be invariant. In reflection, if a figure has a point that lies on the mirror line, then the image of this point will be the same point and will coincide with the object point.

As such, only points on the mirror line are invariant points under the reflection.

In the reflection of the trapezium shown below, one of the parallel sides lie on the mirror line. The points on this line are invariant.



Under a reflection, the image is said to be laterally inverted. This property may not be obvious for some objects. For example, in reflecting the letters L and B in a vertical mirror line, lateral inversion is clearly obvious. This is because their 'flipped' images do not look the same as the original. However, for the letters, A and M, although lateral inversion takes place, the image appears unchanged. This is so because they possess an axis of symmetry which is parallel to the line of reflection.



If the same letters A and B are reflected in a horizontal mirror line, then their images will not look the same because their line of symmetry is not parallel to the line of reflection.

Lateral inversion occurs every time we perform a reflection, but it is only observed when objects do not have an axis symmetry parallel to the mirror line.

Reflection on the Cartesian plane

We can use the properties of reflection to reflect any point, line or figure on the Cartesian Plane, once we know the position of the mirror line.

Example 6

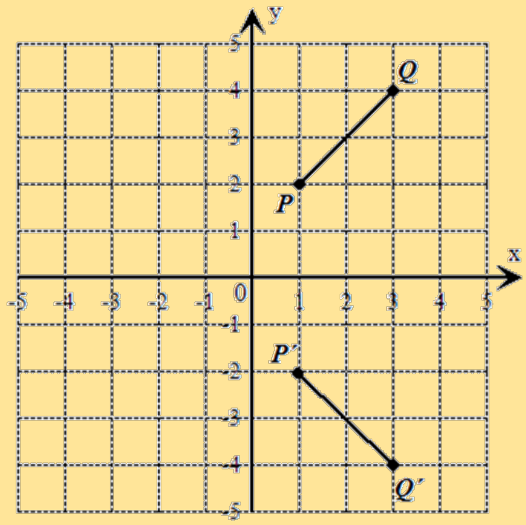
A line segment PQ with $P(1, 2)$ and $Q\ (3,4)$ is reflected in the x-axis. Perform this reflection and state the coordinates of P' and Q', the images of P and Q under the reflection.

Solution

Under a reflection in the x-axis,

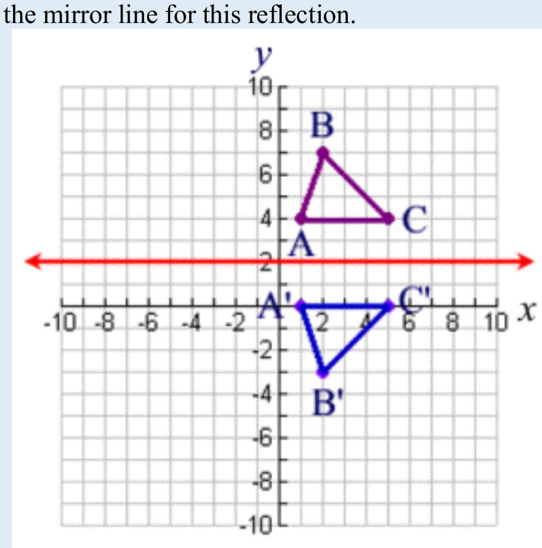
$P\ (1, 2) \rightarrow P'\ (1, -2)$

$Q\ (3, 4) \rightarrow Q'\ (3, -4)$



Example 7

Triangle $A'B'C'$ is a reflection of triangle ABC. State the mirror line for this reflection.



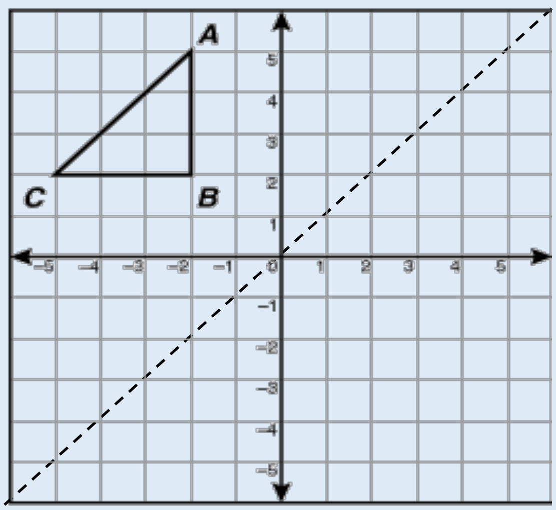
Solution

By observation, it can be seen that the line of reflection is horizontal and is half way between the two triangles.

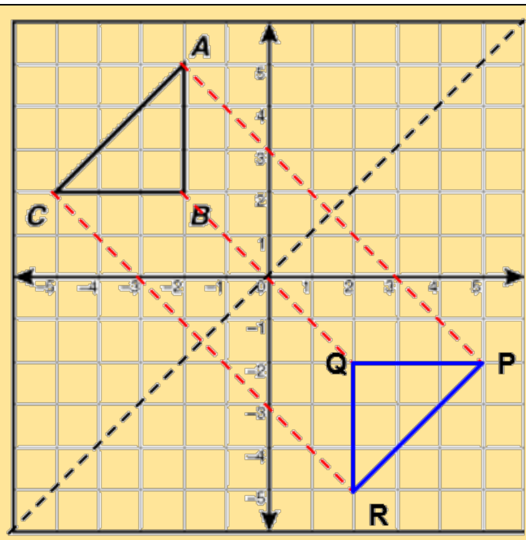
The equation of the mirror line is $y = 2$.

Example 8

Triangle ABC is reflected in the line $y = x$.
(i) Draw the image of the triangle ABC when it is reflected in the line $y = x$.
(ii) State the coordinates of the image points under the reflection.



Solution



To locate the image points, say, P, we draw the line AP, perpendicular to the mirror line with A and P equidistant from the mirror line. In a similar fashion, we draw BQ and CR. The coordinates of the image points, P, Q and R are

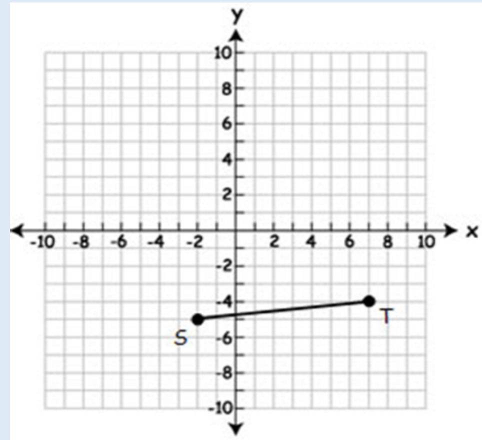
$A(-2, 5) \rightarrow P(5, -2)$

$B(-2, -2) \rightarrow Q(2, -2)$

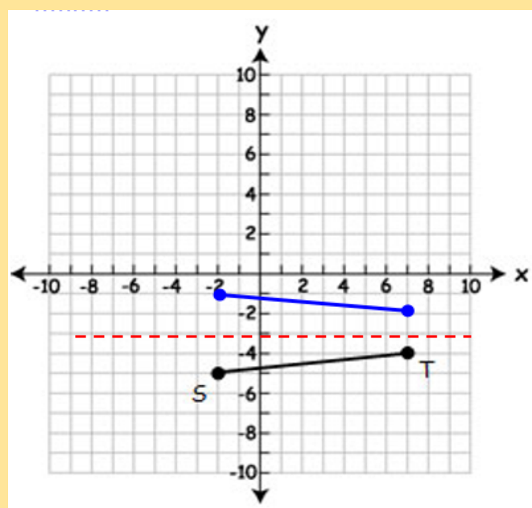
$C(-5, 2) \rightarrow R(2, -5)$

Example 9

Reflect ST in the line $y = -3$.
State the coordinates of S' and T'



Solution



Coordinates are:

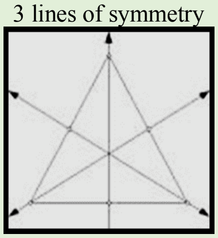
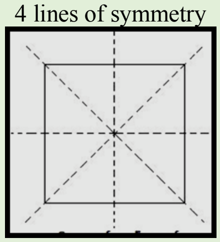
$S(-2, -5) \rightarrow S'(-2, -1)$

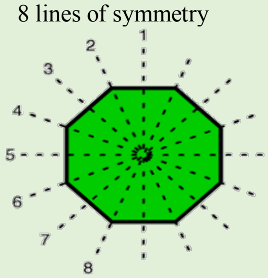
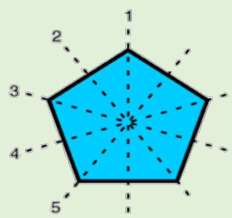
$T(7, -4) \rightarrow T'(7, -2)$

Line symmetry

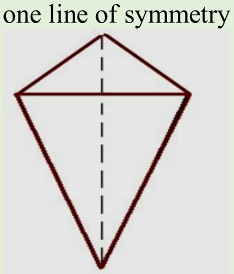
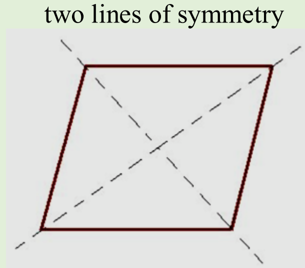
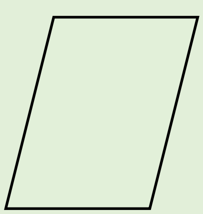
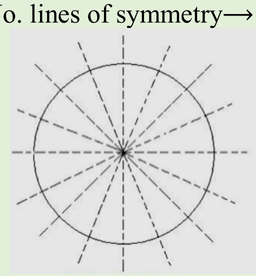
When we perform a reflection, the mirror line always represents an axis of bilateral symmetry. A figure is said to have line symmetry if, when folded about the line of symmetry, the two parts match exactly. There is absolutely no overlapping of the halves created by the folding. The line of symmetry also divides the figure into two congruent parts.

The number of lines of symmetry in some regular polygons is shown below.

Equilateral Triangle 3 lines of symmetry	**Square** 4 lines of symmetry
	

Regular Octagon 8 lines of symmetry	**Regular Hexagon** 6 lines of symmetry
	

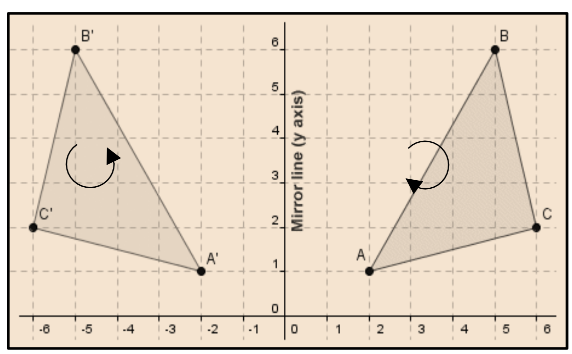
From the above drawings, it can be deduced that the number of lines of symmetry in a regular polygon is the same as the number of sides in a polygon. Other shapes can have any number of lines of symmetry.

Kite one line of symmetry	Rhombus two lines of symmetry
	
Parallelogram No lines of symmetry	Circle No. lines of symmetry⟶ ∞
	

Properties of reflection

When an object undergoes a reflection, we can observe the following properties:

1. The object and image are identical in shape and size, that is, they are congruent.
2. The image is laterally inverted (flipped). This not noticeable in shapes that have a line of symmetry parallel to the mirror line.
3. The image and object lie on opposite sides of the mirror line.
4. The line joining any object point to its corresponding image point is perpendicular to the mirror line.
5. The mirror line is the perpendicular bisector of a line joining an object point to its corresponding image point. Hence, the image and the object are the same perpendicular distance from the mirror line.
6. The mirror line is an axis of bilateral symmetry, dividing the shape into two equal parts that overlap.
7. If a point, A lies on the mirror line l, then its image, A', is in the same position as A and as such A is said to be an invariant point.
8. Image and object are always in the opposite sense. This means that if we move along the vertices of the object in the direction from $A \rightarrow B \rightarrow C$, and this is a clockwise direction, then the direction when moving from $A' \rightarrow B' \rightarrow C'$ will be anticlockwise or vice versa.

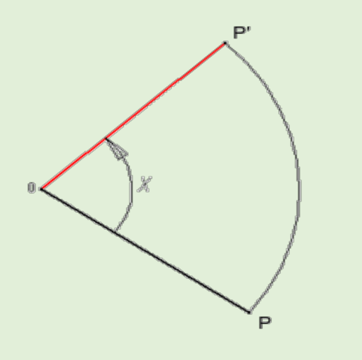


ROTATION

In our study of transformations so far, we have seen that in a translation the object moves in a straight line without turning, while in a reflection, the object turns about a line. A rotation is defined as a geometric transformation in which an object is turned or rotated about a fixed point, called the **center of rotation**. The size of the turn is specified by the **angle of rotation**. The direction of the turn can be anti-clockwise, or clockwise.

Rotation of a point

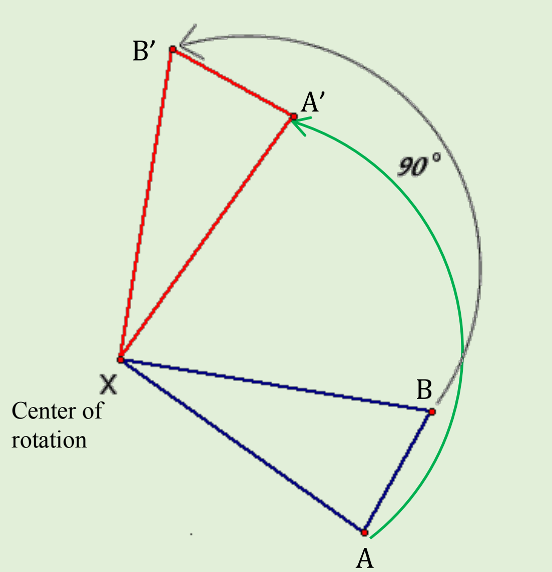
The point, P is rotated about O, through angle X in an anticlockwise direction.
The image of P is P'



In rotating a figure, all points on the object move along an arc of a circle whose center is the center of rotation. As the object turns, its orientation changes, but it returns to its original position after a complete revolution or 360 degrees.

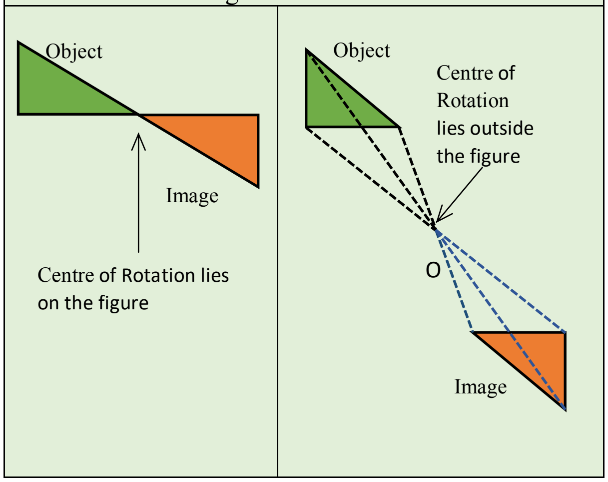
Rotation of a figure

Triangle ABX is rotated about X through 90^0 in an anticlockwise direction.



The center of rotation can be located at a point on the figure or at a point outside the figure. The diagram below shows two such rotations. The position of the center of rotation differs in each case. Both figures undergo a half-turn or a rotation of 180 degrees.

A rotation of 180^0 about the center O.
The angle of rotation is 180^0



It is not required to state the direction of a rotation of 180^0 because the position of the image is the same in both clockwise and anticlockwise turns through this angle.

Describing a rotation

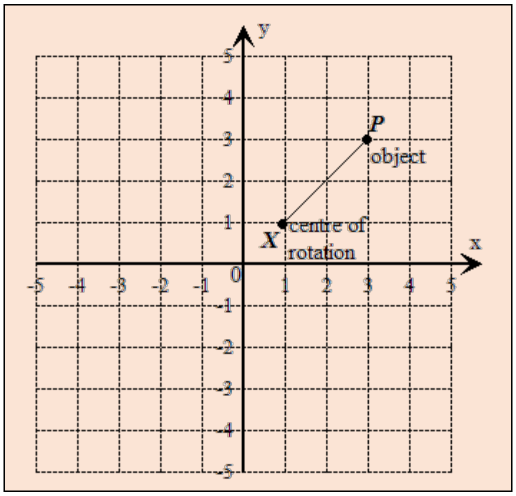
In describing a rotation, we must state:

1. The center of rotation.
2. The angle of rotation, that is the angle through which all points on the object turns. If an object P is rotated about O to image P' then the angle of rotation is POP'.
3. The direction of rotation, which is either clockwise or anti-clockwise. A positive angle is considered as an anticlockwise turn while a negative angle is a clockwise turn.

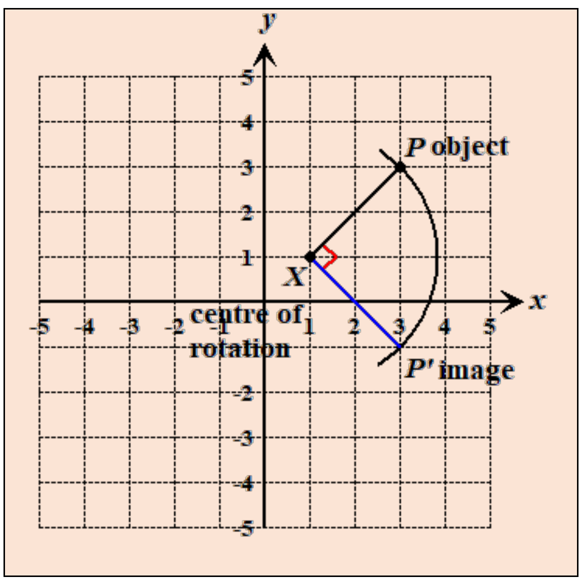
Rotation on the Cartesian Plane

To locate the image under a rotation, we need to know the position of the object, the center of rotation, the angle of rotation and the direction of the rotation. Geometrical instruments such as a protractor, ruler and a pair of compasses will be required. It is good practice to show all construction lines when performing rotations or any other transformations.
For example, to perform a clockwise rotation of the point P (3, 3) through $90°$, with center (1,1), we follow the steps shown below.

1. Plot object point P (3,3) and the center of rotation X (1,1).



2. With center X and radius XP an arc is drawn in the clockwise direction, sufficiently long enough to complete a quarter turn.



3. Place a protractor with its horizontal or zero line on XP and center point on X. Mark off a point, P' on the arc such that angle PXP' = 90^0. Join the points XP'.

4. The object point, P has been rotated through X (center of rotation) about an angle of 90^0 in a clockwise direction. Note that:

$$XP = XP'$$
$$PXP' = 90^0$$

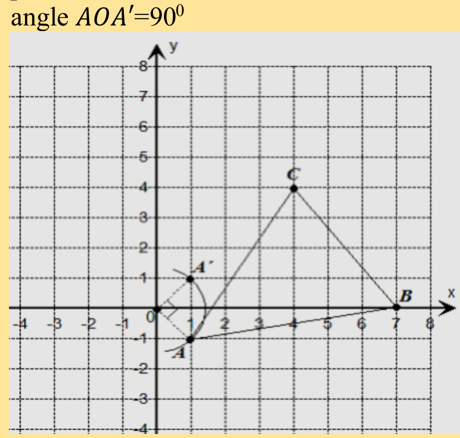
Example 10

$A=(1,-1)$, $B=(7,0)$ and $C=(4,4)$. Triangle ABC is mapped onto triangle $A'B'C'$ under an anti-clockwise rotation of 90° about O. Illustrate on a clearly labelled diagram and state the coordinates of A', B' and C'.

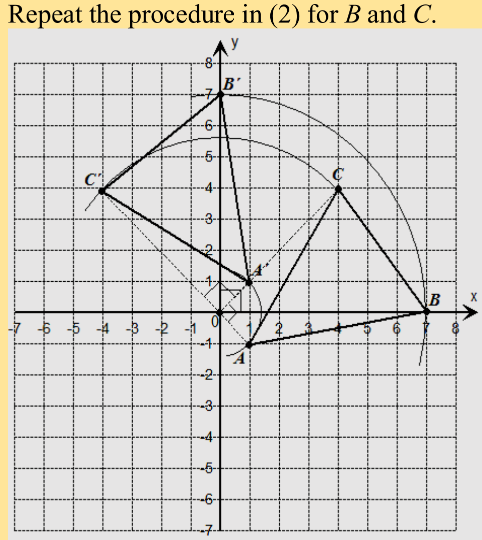
Solution

1. Draw the object. Using the same scale on both axes, say 1 cm ≡ 1 unit, on both axes we plot the points A, B and C. Join the points to obtain the object, triangle ABC.

2. Performing the rotation. Starting with point, A, we join OA. With center O and radius OA, an arc is drawn in the anti-clockwise direction. Place a protractor with its center point at O and mark off A' on the arc so that angle $AOA'=90^0$



3. Performing rotation on remaining points. Repeat the procedure in (2) for B and C.

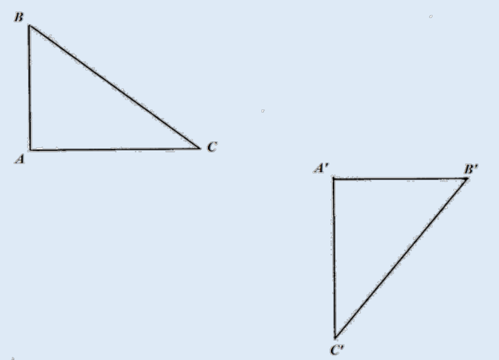


4. We should obtain the following coordinates for the image of ABC.
$A' = (1, 1), B' = (0, 7)$ and $C' = (-4, 4)$

Example 11

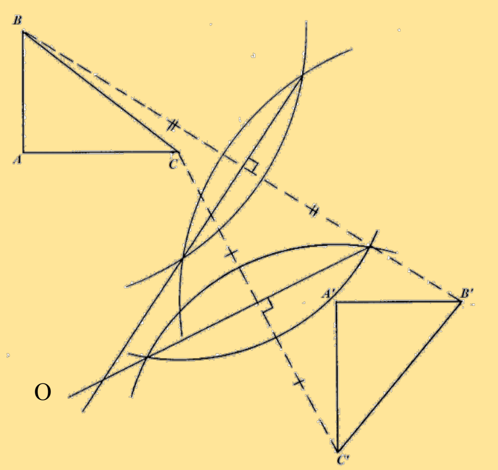
Triangle ABC is mapped onto triangle $A'B'C'$ under a rotation R.

(i) Locate the center of rotation (ii) Describe R.



Solution

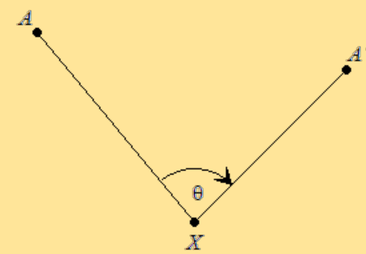
(i) We draw a straight line from any object point to its corresponding image point and construct the perpendicular bisector. For example, we join B to B' and bisect it. Then, the procedure is repeated with a second set of points, for example, C and C'. The two perpendicular bisectors are extended, if necessary, so as to meet at the center of rotation, say O.



It is not necessary to construct a third perpendicular bisector since the perpendicular bisectors are all concurrent, that is, they all pass through the same point.

(ii) To describe R, we must state the direction and magnitude of the turn. We consider any point and its image, say A and A'.

The angle $A\hat{X}A'$ or $B\hat{X}B' = \theta$ is the angle of rotation. The rotation from A to A', indicates that the angle of rotation is in a clockwise direction.
$\therefore R$ describes a rotation of θ º clockwise about X.



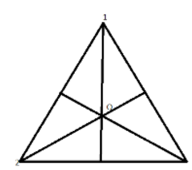
Rotational symmetry

A shape has rotational symmetry if, when rotated through 360^0 about its center point, the shape matches exactly its original position, a number of times. The number of such matches during the complete turn, is **the order of rotational symmetry**. If a shape only matches itself once in one revolution, the shape has no rotational symmetry. In fact, there is no rotational symmetry of order 1, because every figure would at least have this. Since 2 is the smallest order of rotational symmetry, 180 degrees is the largest degree of rotation that is possible.

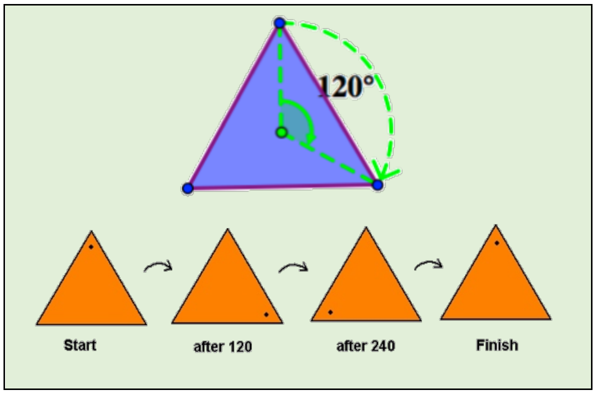
Examples of rotational symmetry

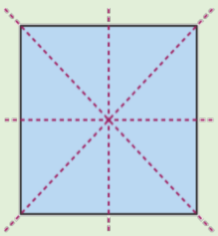
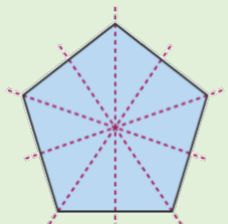
All regular polygons have rotational symmetry. The order of rotational symmetry is the same as the number of sides. The center of the figure must first be located to test for rotational symmetry.

For an equilateral triangle, the center is the point of intersection of the three lines as shown.
Each line is drawn from the vertex to the mid-point of the opposite side.



The order of rotational symmetry in an equilateral triangle is 3. The angle of rotation is $360° \div 3 = 120°$.
This means that after three turns of 120 each about the center, the triangle fits back its outline 3 times.
This is illustrated below.



A square has order 4, the angle of rotation is $360° \div 4 = 90°$. 	A regular pentagon has order 5, the angle of rotation is $360° \div 5 = 72°$ 

Properties of rotation

1. Each point on the object turns in the same direction and through the same angle about the center of rotation.
2. Image and object are congruent and sense is preserved, but orientation changes. In fact, we tend to visually identify rotation by this property.
3. An anticlockwise rotation is a positive turn while a clockwise rotation is a negative turn.
4. The perpendicular bisectors of any two straight lines, passing through a point and its image, meet at the center of rotation.

DILATION OR ENLARGEMENT

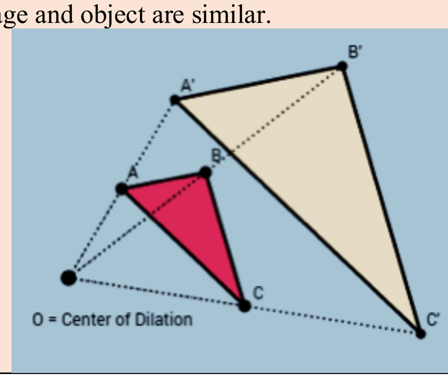
The transformations translation, reflection and rotation are all isometric transformations as they all preserve size and shape. The image and object under these transformations are congruent. A dilation, also called an enlargement, is a transformation in which an object will generally undergo a change in size, but not shape. The change in size is determined by the **scale factor** of the enlargement. We refer to enlargement as a size transformation.

Under enlargement, the image is **similar** to the original figure. We may recall that similar figures have the same shape and their corresponding angles are equal. Their corresponding sides are in a fixed ratio and this ratio is actually a measure of the scale factor of the enlargement.

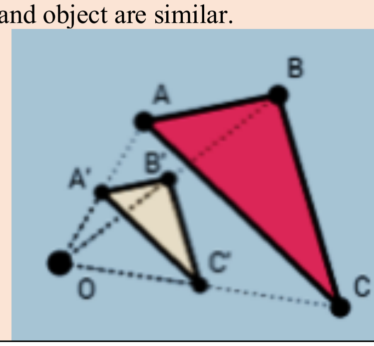
Enlargement and Reduction

Dilations can be of two major types – enlargement and reduction. When the scale factor ($\boldsymbol{k}$) is greater than one, the image is larger than the object (enlargement) and when the scale factor is less than one the image is smaller than the object (reduction).

An enlargement ($\boldsymbol{k} > \mathbf{1}$)
The image, $A'B'C'$ is larger than the object, ABC
Shape is preserved but size changes.
Image and object are similar.



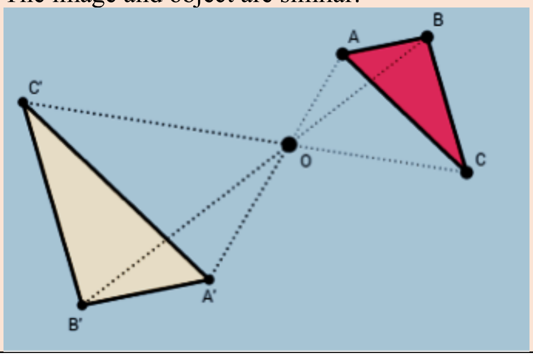
A reduction ($0 < \boldsymbol{k} < \mathbf{1}$)
The image, $A'B'C'$ is smaller than the object, ABC
Shape is preserved but size changes.
Image and object are similar.



Positive and negative scale factors

A dilation can have a positive or negative scale factors. When the scale factor is positive, the object and image lie on the same side of the center (see above). When the scale factor is negative, the object and image lie on opposite sides of the center, as illustrated below.

An enlargement with negative scale factor (**$k < 0$**).
The image $A'B'C'$ is larger than the object ABC.
Image and object are on opposite sides of the center.
Shape is preserved but size changes.
The image and object are similar.



Describing an enlargement (or dilation)

In describing an enlargement (or dilation), we must always state:

1. The position of the center of enlargement which tells us from where the enlargement is measured.
2. The scale factor, ***k***, of the enlargement which tells us by how much the object has been enlarged or reduced to produce the image.

Properties of the scale factor

The scale factor of a dilation is multiplicative in nature as it tells us how many times the image is enlarged or reduced. It is this ratio that is used in performing the enlargement. In general, for any scale factor, k.

$$\frac{\text{Image distance from the center of enlargement}}{\text{Object distance from the center of enlargement}} = \frac{k}{1}$$

The ratio, k is also the same as

$$\frac{\text{Image Length}}{\text{Object Length}} = \frac{k}{1}$$

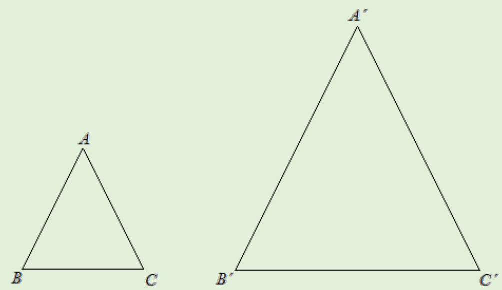
When $k > 1$, the image is larger than the object.

When $0 < k < 1$, the image is smaller than the object.

When $k = 1 or -1$, the image and object are congruent.
When the scale factor is negative, $(k < 0)$, the image and object lies on opposite of the center. It is 'flipped' and **sense** is not preserved.

Area of image under enlargement

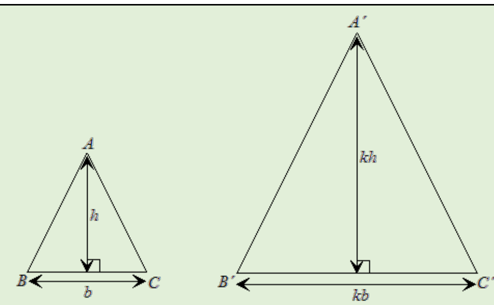
We know that the object and image are similar under an enlargement. We know also that when two figures are similar, the ratio of their corresponding sides is the same. Consider the ΔABC mapped onto $\Delta A'B'C'$ by an enlargement, scale factor, k.



Using the property of the scale factor of an enlargement:

$$B'C' = k \times BC$$
$$A'C' = k \times AC$$
$$A'B' = k \times AB$$

We wish to compare the area of ΔABC with the area of $\Delta A'B'C'$. The height of $\Delta A'B'C'$ will also be k times the height of ΔABC because of the property of similarity.



Base = b
Height = h

Base = kb
Height = kh

Area of triangle ABC $= \frac{1}{2}bh$

Area of triangle $A'B'C'$ $= \frac{1}{2}kb \times \frac{1}{2}kh$

$= \frac{1}{2}k^2bh$

$= k^2(\frac{1}{2}bh)$

$= k^2 \times$Area of ΔABC

Hence, the area of the image is k^2 times the area of the object, that is, the square of the scale factor. This property is true for any enlargement of scale factor, k,

$$\frac{\text{Area of Image}}{\text{Area of Object}} = k^2$$

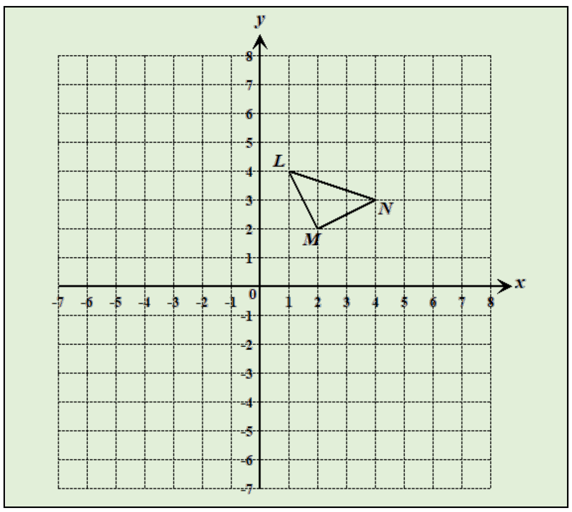
Enlargement on the Cartesian Plane

To perform an enlargement, we need to know the position of the center, the scale factor and the position of the object. A pair of compasses is also required to locate the points on the image.

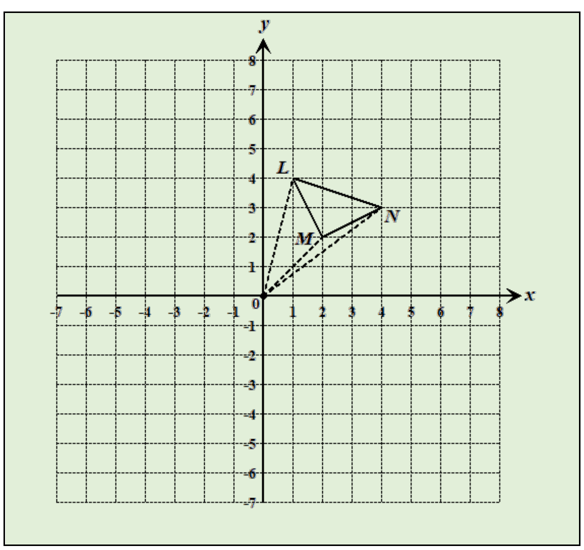
Positive scale factor

If we are given a triangle LMN whose coordinates are $L(1,4), M(2,2)$ and $N(4,3)$ and we wish to perform an enlargement with a positive scale factor of 2 and center (0, 0), we use the following steps.

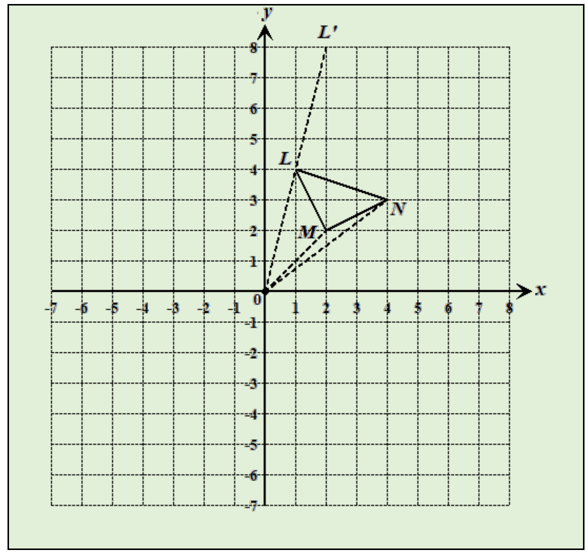
1. Plot the given points on a Cartesian Diagram using the same scales on both axes and draw the triangle *LMN*.



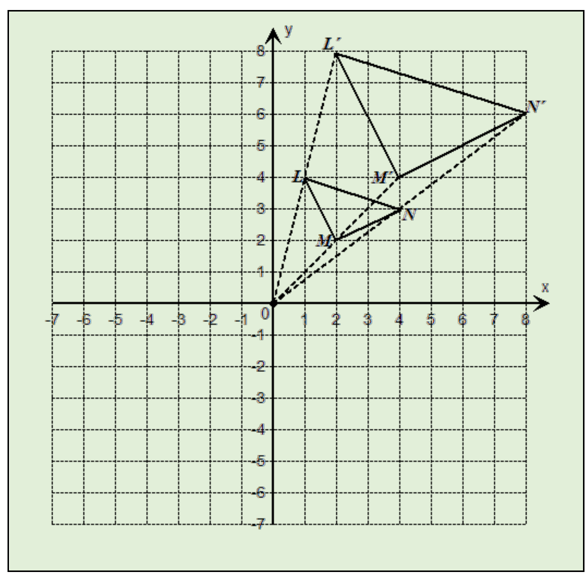
2. Connect each vertex of the object to the center of enlargement, X, as shown.



3. We now extend the line OL to a point L' such that OL' is twice OL, since the scale factor is 2. The position of L' is the image of L under the enlargement, center O and scale factor, 2.



4. The same procedure is repeated with the object points M and N to locate M'andN'. We join these three image points to obtain the image of the triangle *LMN*.



5. Triangle $L'M'N'$ is the image of object *LMN* under the enlargement, center *O* and a scale factor of 2.

Negative scale factor

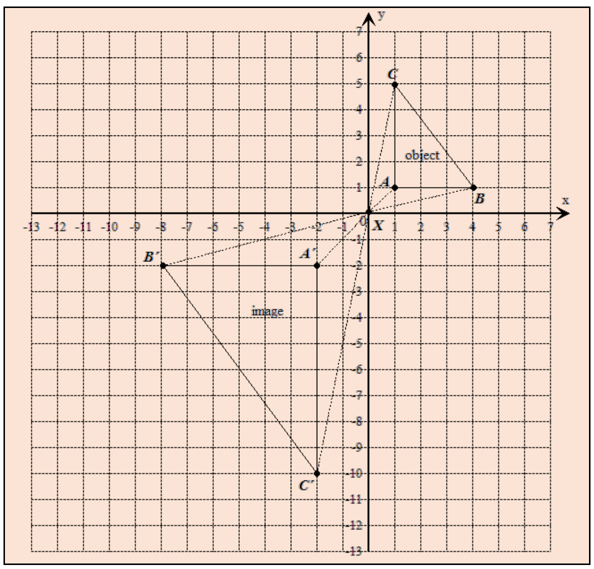
In performing a negative enlargement, we use a similar sequence of steps as we did for a positive enlargement but in this case the image and object are on opposite sides of the center.
Assume we are given a triangle ABC whose coordinates are $A(1,1), B(4,1)$ and $C(1,5)$ and we wish to perform an enlargement with a scale factor of $k=-2$ and center (0, 0).

1. Plot the given points on a Cartesian Diagram using the same scales on both axes and draw the triangle *ABC*.

2. Connect each vertex of the object to the center of the enlargement, *O*.

3. Extend the line AO so that $A'O = 2OA$. [Recall for a positive scale factor, we extended the line in the opposite direction]

4. Repeat the procedure to locate the points B' and C'. The object and its image are shown below.



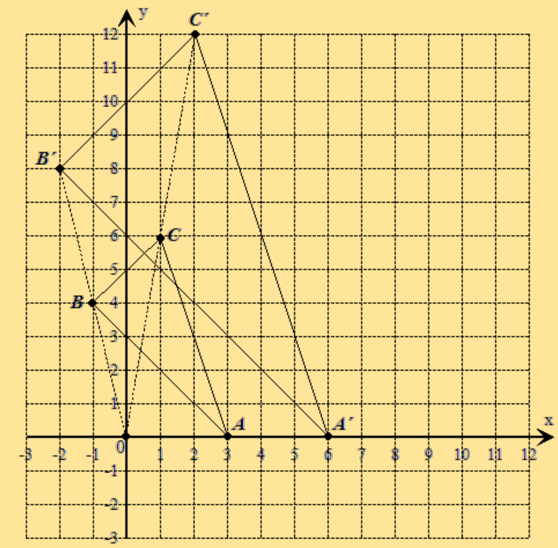
Example 12

Triangle ABC has coordinates $A(3,0), B(-1,4)$ and $C(1,6)$ respectively. Triangle *ABC* is mapped onto triangle $A'B'C'$ by an enlargement, center *O* and a scale factor where, $k=2$.

i. Determine the coordinates of $A'B'C'$.
ii. State the ratio of the area of the image to the area of the object.

Solution

The triangle ABC is drawn and the enlargement performed as described previously.



i. We note that the coordinates of the image points are $A'(6,0)$, $B'(-2,8)$ and $C'(2,12)$.
ii. To determine the ratio of the area of the image to the area of the object, we first compare the linear scale factor. Since this is 2:1, the desired ratio is $2^2:1^2$ or 4:1.

Example 13

Triangle *PQR* is mapped onto triangle $P'Q'R'$ by an enlargement with scale factor, $k=3$. If the area of triangle *PQR* = 5 cm^2, find the area of triangle, $P'Q'R'$.

Solution

Area of $\Delta P'Q'R' = (3)^2 \times 5 \quad = 45 \text{ cm}^2$

Example 14

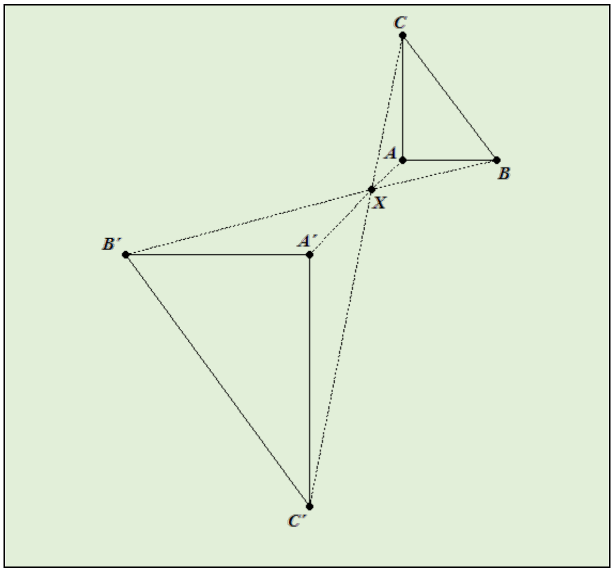
Triangle *ABC* is mapped onto triangle $A'B'C'$ by a dilation with the scale factor, $k=\frac{1}{2}$. State the relationship between the area of triangle *ABC* and triangle $A'B'C'$.

Solution

$$\text{Area of } \Delta A'B'C' = \left(\frac{1}{2}\right)^2 \times \text{Area of } \Delta ABC$$
$$= \frac{1}{4} \times \text{Area of } \Delta ABC$$

Locating the center of enlargement

If we are given an object, say ABC and its image position after an enlargement, $A'B'C'$, then we can locate the center of the enlargement. We join corresponding pairs of object and image points as shown below. The center of the enlargement is the point, X where these lines intersect.



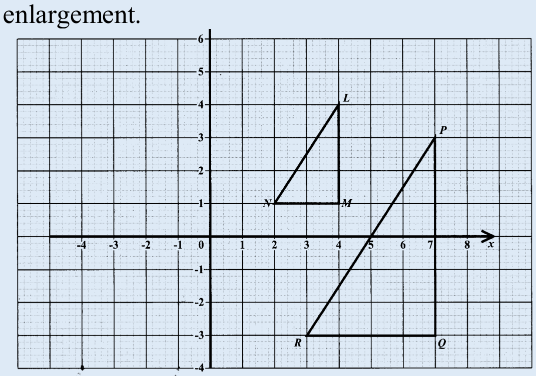
We can also determine the scale factor of the enlargement by computing the ratio of the corresponding sides. In this example, the scale factor, k is negative because the image and object lies on opposite sides of the center. $\Delta A'B'C'$ is the image of ΔABC by an enlargement, scale factor $-k$.

To determine the magnitude of k, we measure pairs of corresponding sides and compute any of the ratios:

$$A'B' : AB \text{ or } A'C' : AC \text{ or } B'C' : BC$$

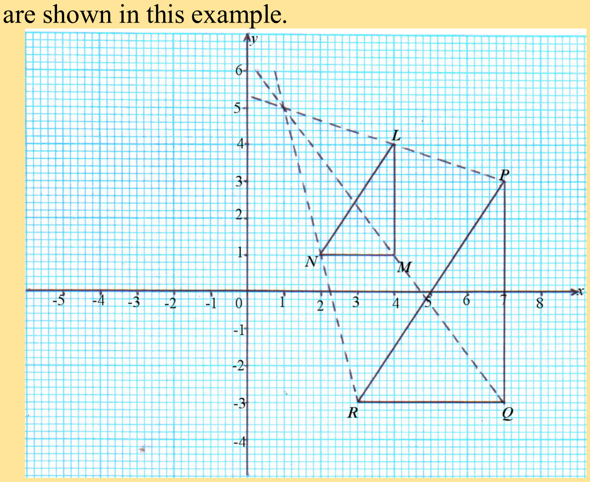
Example 15

A triangle and its image is shown below. Determine the coordinates of the center of enlargement.



Solution

To find the center of enlargement we draw lines from the image points to their corresponding object points and extend them so as to intersect. The point of intersection of these lines is the center of enlargement. We only need to do this with two sets of points since these lines are all concurrent. However, all three lines are shown in this example.



The center of enlargement is the point (1, 5).

Properties of enlargement

1. Enlargement is a non-isometric or size transformation (except if $k = \pm 1$).

2. Sense remains unchanged in an enlargement except when the scale factor, k is negative.

3. The scale factor of an enlargement is a linear scale factor, defined by the following ratios:

$$\frac{\text{Image Distance from the centre of Enlargement}}{\text{Object Distance from the centre of Enlargement}} = \frac{k}{1}$$

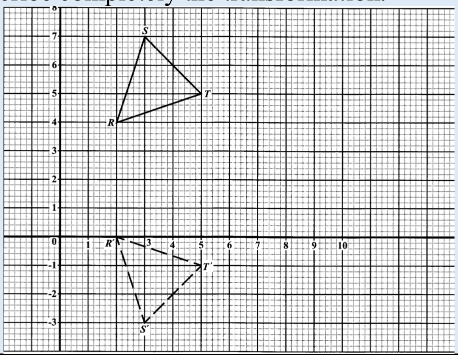
$$\frac{\text{Image Length}}{\text{Object Length}} = \frac{k}{1}$$

4. The ratio of the areas of image and object under an enlargement with scale factor, k is as follows:

$$\frac{\text{Area of Image}}{\text{Area of Object}} = k^2$$

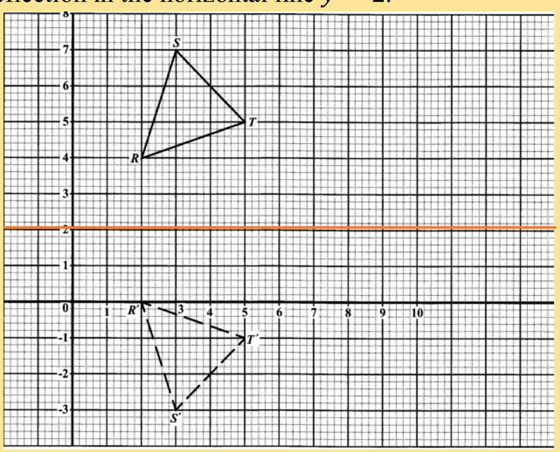
Example 16

The diagram shows triangle RST and its image $R'S'T'$ after a transformation.
Describe completely the transformation.



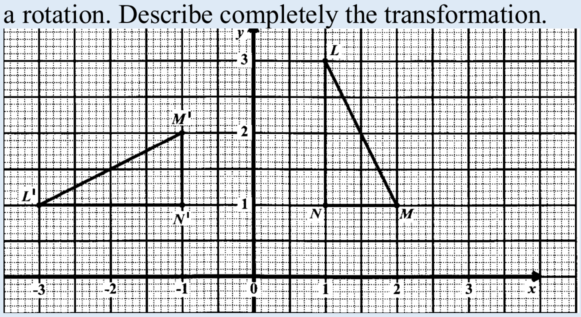
Solution

By inspection, the transformation is isometric but the image is laterally inverted. The only isometric transformation that has these properties is reflection.
To locate the mirror line, we look for the perpendicular bisector of the line joining any point to its image, say RR'. This is the line, $y = 2$.
Triangle RST is mapped onto triangle $R'S'T'$ by a reflection in the horizontal line $y = 2$.



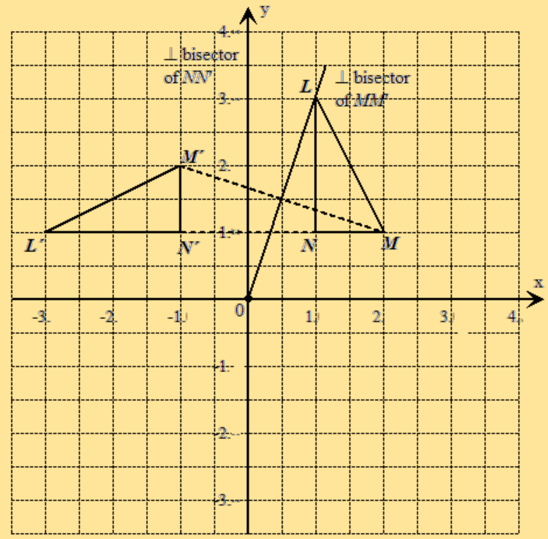
Example 17

The diagram below shows a triangle after undergoing a rotation. Describe completely the transformation.



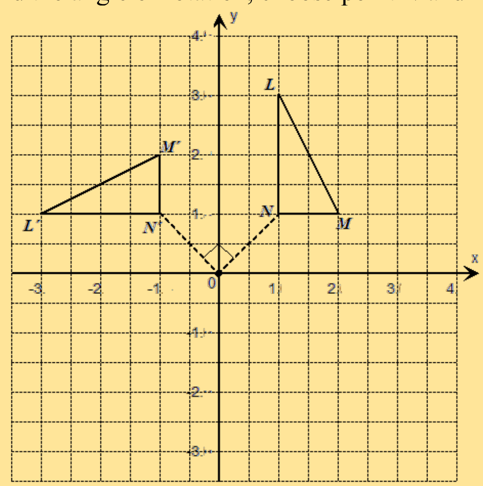
Solution

The transformation is isometric. It is neither a flip or a slide but there is a change in orientation, so it is a rotation.
To find the center of rotation, connect points M and M', and construct the perpendicular bisector of MM'.
Choose the points N and N'. The perpendicular bisector of this line is the y-axis.



The perpendicular bisectors of MM' and NN' meet at O. Therefore, O is the center of rotation.

To find the angle of rotation, choose point N and N'.



By measurement, $\angle N'ON = 90^0$. Therefore, the angle of rotation is 90°.
The direction is anti-clockwise and the center is at the origin, O.